Emergent symmetries and large N confining gauge theories

Aleksey Cherman
INT, University of Washington

with linear combinations of
G. Basar (University of Maryland, College Park),
K. Dienes (University of Arizona),
D. Dorigoni (DESY, Hamburg),
D. McGady (NBI - Copenhagen),
M. Unsal (North Carolina State University)
M. Yamazaki (IPMU, Tokyo)

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1507.08666 + to appear + ongoing

Hard versus easy

Confining theories like QCD with
N = 3 colors are notoriously hard

Strongly coupled at low energies, no analytic way
to compute spectrum or correlation functions.

Free theories are easy!

Writing down the Lagrangian = solving the theory.

$$\mathcal{L} = \sum_{i=1}^{K} \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_i^2 \phi^2\right]$$

spectrum = $\{m_1, m_2, \ldots, m_K\}$.

correlation functions taught in first
1-2 weeks of intro QFT class

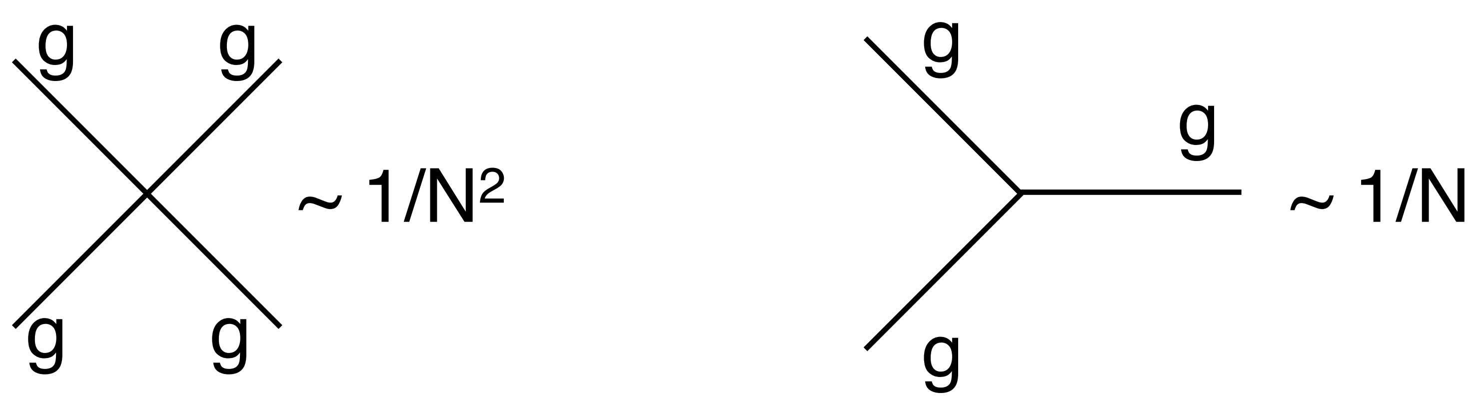
Hard versus easy

Confining theories like QCD with
N = 3 colors are notoriously hard

Free theories are easy!

Good news from 70s: QCD is a free theory at large N !

‘t Hooft 1973
Witten 1979



Free in the physical basis given by mesons and glueballs

Bad news from 2015: these free theories are so hard, no solution yet.

$$\mathcal{L}_{\text{physical}} = ???$$

Symmetries and free spectra

Typical free relativistic QFTs with K particles have ≈ K parameters

$$\{m_1, m_2, m_3, \cdots\}$$

Symmetries reduce the number of parameters.

`Symmetries' could mean e.g. $m_1 = m_2 = m_3$ (global symmetry),
or something like $m_n = 2m_{n-1}$ (spectrum-generating algebra).

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For large N confining theories $\{ m_k \}$ is an infinite set.

Only parameter in pure YM is strong scale Λ.
In 3-flavor large N QCD, only four parameters.

(+ θ angle)

Could there be some emergent (spectrum-generating) symmetries at $N = \infty$ organizing the confined-phase spectrum?

This talk: evidence that answer is yes.

Why should you care?

Nature

Understanding QCD = understanding a big already-observed chunk of nature

Understanding large N QCD would help if 1/N corrections are small…

… and there's a lot of evidence that they are.

QCD-like QFTs may be important for physics beyond the Standard Model

Formal theory

Confining gauge theories = rich class of non-supersymmetric string theories.

$g_{string} \sim 1/N$, $\boldsymbol{\alpha}' \sim \Lambda^2$

But we know very little about these particular string theories.

Better understanding of confining gauge theories $\Longrightarrow$

insights into string theory and quantum gravity.

The Plan

(1) Confining theories are hard even at large N.
Introduce calculable regime to explore possible emergent symmetries.

(2) Explain concrete reason why emergent symmetries expected even **outside** such a regime

(3) Use calculable regime to explore large N confining spectrum

Large N control parameter

To explore symmetries, need to find a control parameter “ϵ”

$$\{m_1, m_2, m_3, \cdots\}$$

$$\{m_1(\epsilon), m_2(\epsilon), m_3(\epsilon), \cdots\}$$

Desires:

Want spectrum to become calculable as $\epsilon \to 0$.

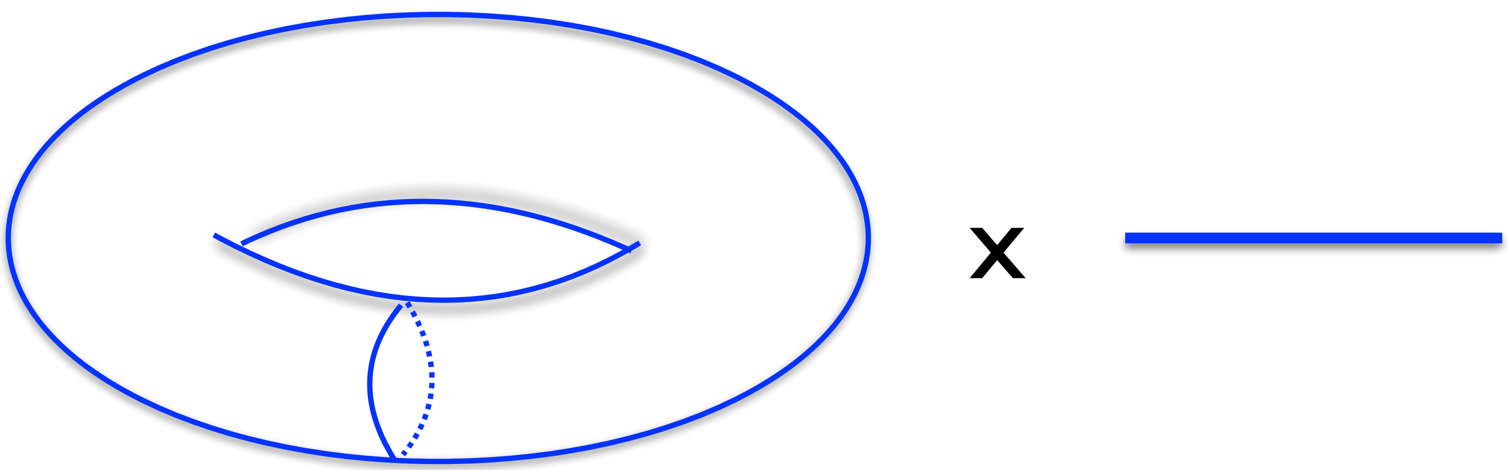
Want ϵ to be a control parameter even in the ’t Hooft limit.

Want to stay in the confined phase for all ϵ.

Control by compactification

Key idea: if we lived in a box, QCD and its cousins would be easier

Most obvious type of box: a three-torus $T^3 = S^1 \times S^1 \times S^1$



Space: T^3

Time: $\mathbb{R}$

Once T^3 is small, system becomes weakly-coupled

But in Euclidean space, finite temperature T = 1/(circle size L)

Small-volume theory = high-temperature theory

Control by compactification

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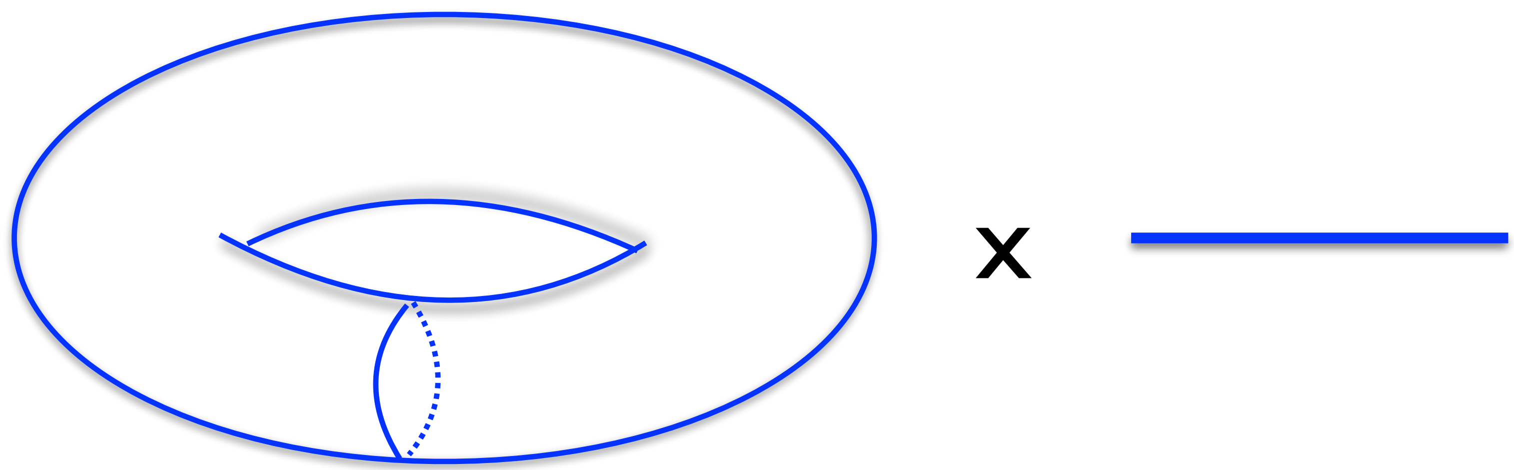
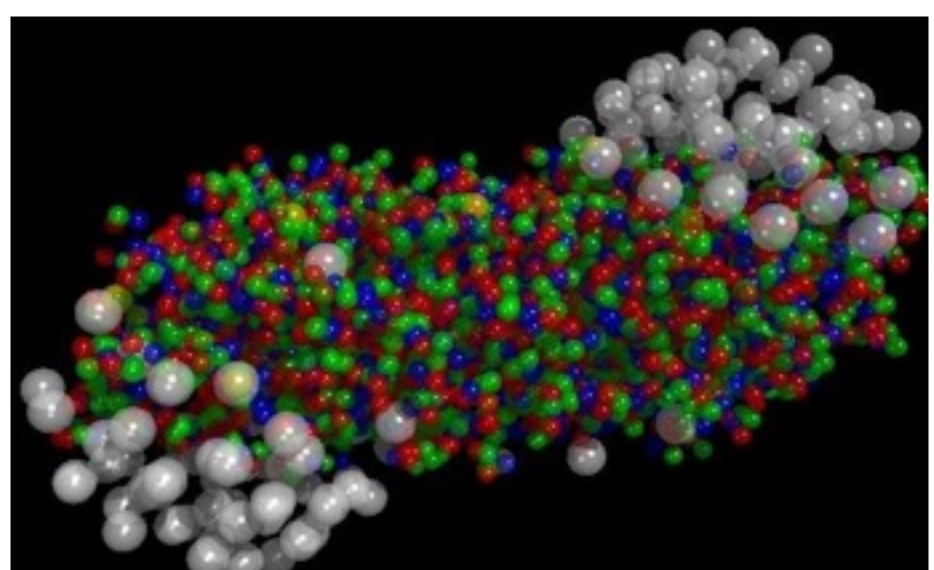


Space: T^3 Time: $\mathbb{R}$

Once T^3 is small, system becomes weakly-coupled

quark-gluon plasma



Source: CERN

Loss of confinement at small L !

Control by compactification

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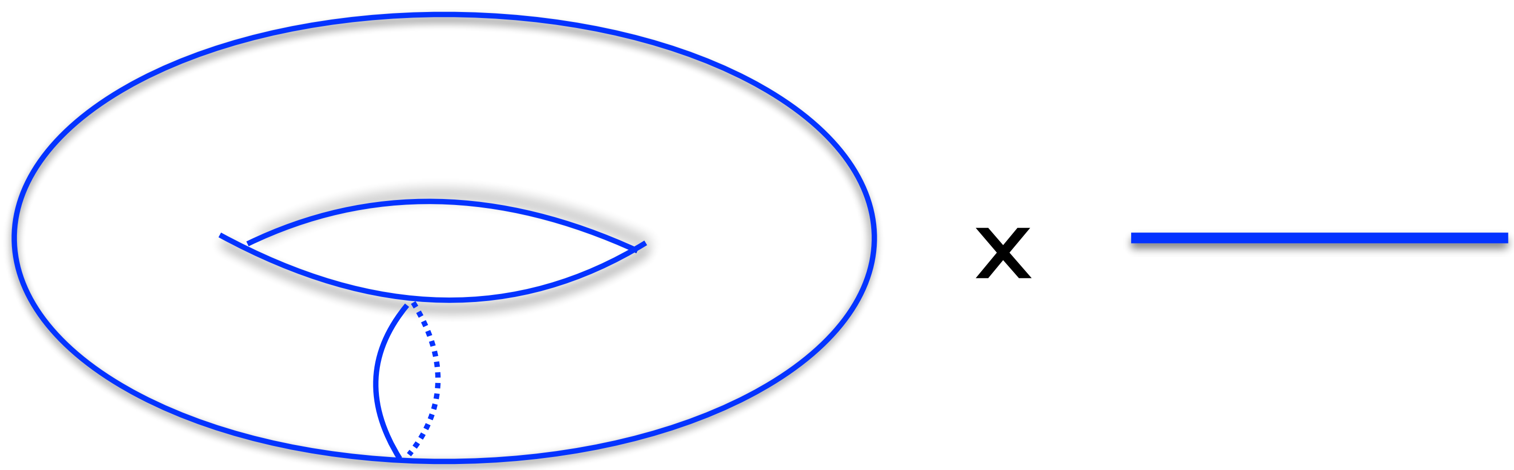
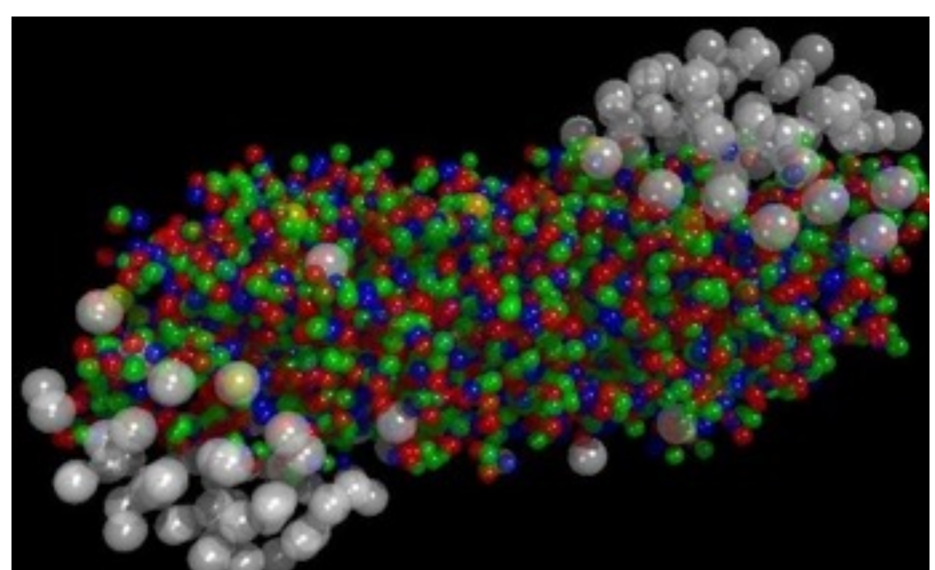


Doesn't work

Space: T^3 Time: $\mathbb{R}$

Once T^3 is small, system becomes weakly-coupled

quark-gluon plasma



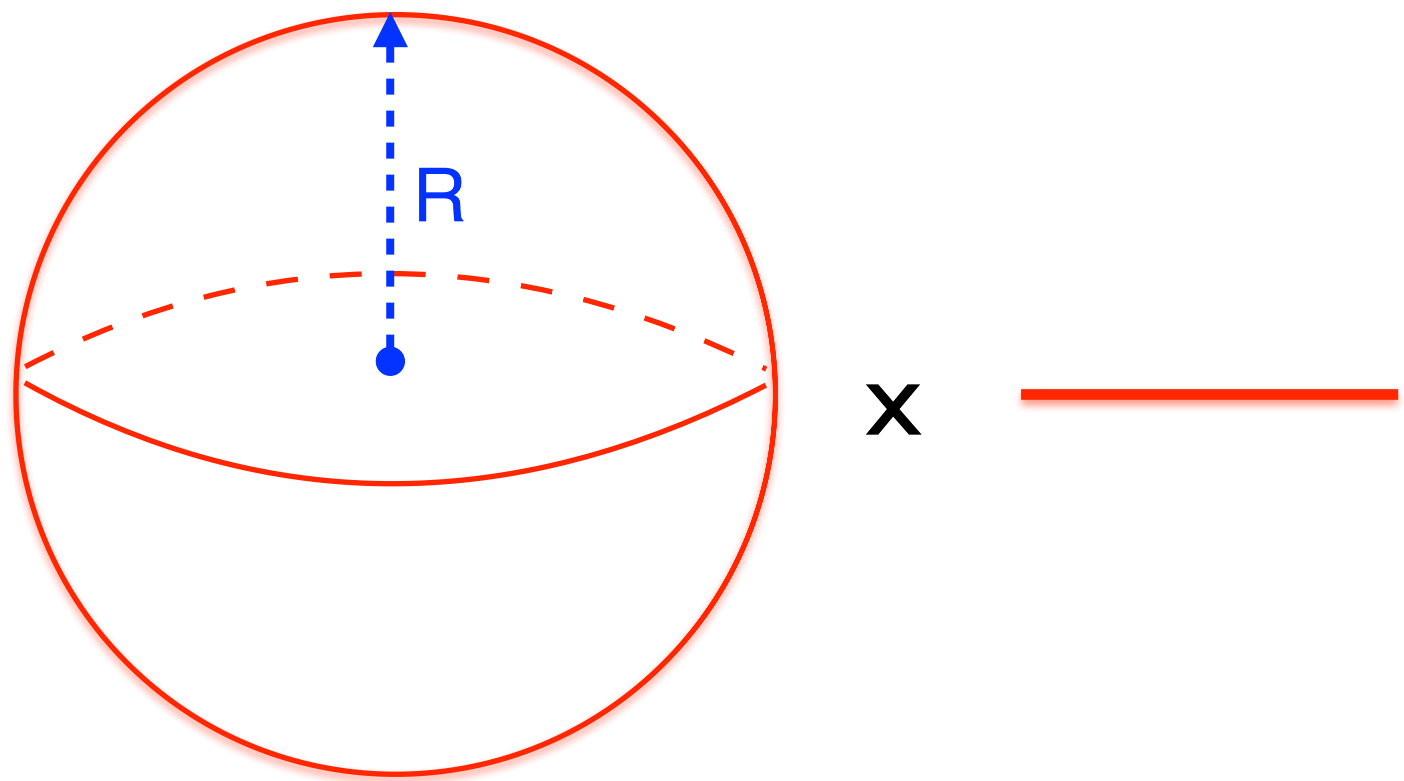
Source: CERN

Loss of confinement at small L !

$S^3_R \times \mathbb{R}$ compactification

Sundborg 1999;
Polyakov 2001;
Aharony et al, 2003

A better box for studying large N confinement!



Space: S^3

Time: $\mathbb{R}$

If $R\Lambda >> 1$, back to R^4.

$\lambda(1/R)$ is not small

If $R\Lambda << 1$, weak coupling!

$\lambda = \lambda(1/R) \to 0$

$\epsilon = R\Lambda$ is a control parameter for QCD.

$S^3_R \times \mathbb{R}$ compactification

Sundborg 1999;
Polyakov 2001;
Aharony et al, 2003

What about confinement?

Need order parameters that make sense in finite volume.

(1) Realization of center symmetry

confinement ≈ unbroken center symmetry

(2) N-scaling of the free energy

Glueball masses and numbers don't scale with N

Number of gluons scales like N^2

confinement ≈ free energy $F \sim N^0$ deconfinement ≈ free energy $F \sim N^2$

Both (1) and (2) require the large N limit to be well-defined!

Fortunately, we want to work at large N anyway…

Small RΛ behavior

Sundborg 1999;
Polyakov 2001;
Aharony et al, 2003

In principle small RΛ theories can be solved with any matter content.

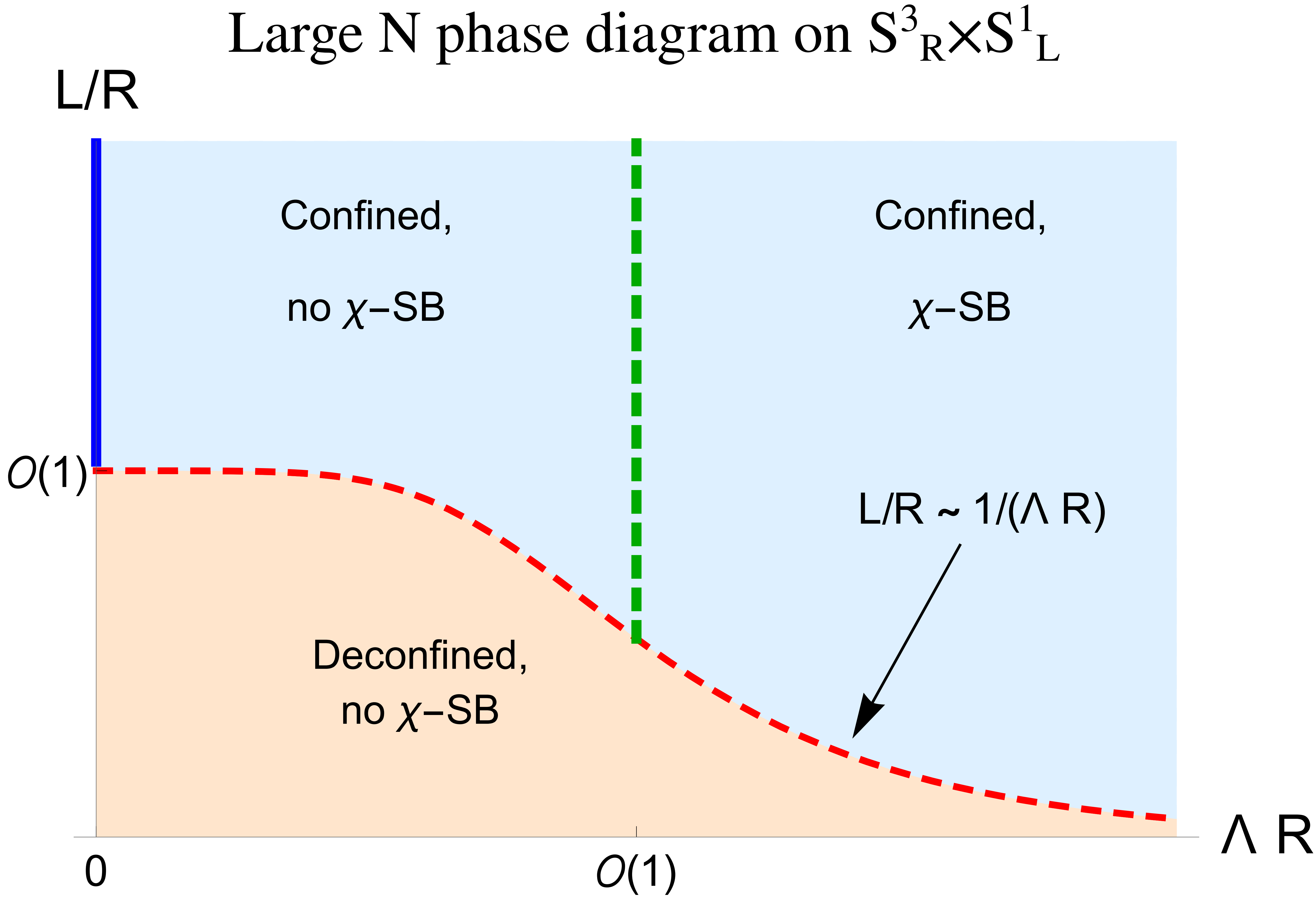
So far, analytic results for theories with massless adjoint matter only
$\Longrightarrow$ focus on theories with massless adjoint matter in this talk.

Symmetry structure:

Unbroken/broken center
symmetry for low/high T

$O(N^0)$ free energy for low T,
$O(N^2)$ free energy for high T

This is best illustrated by an illustration…



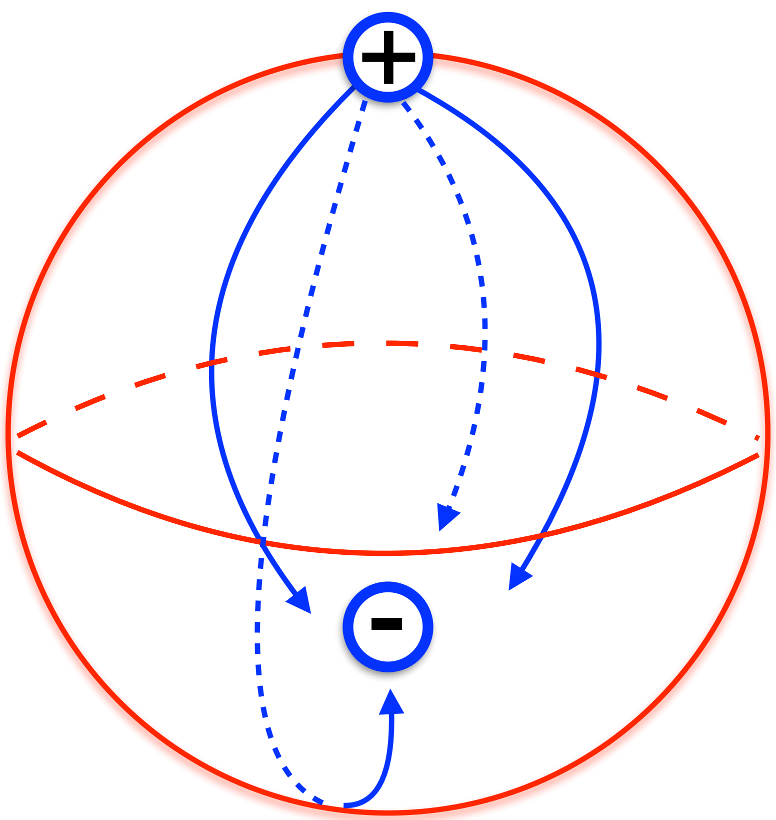
Large N phase diagram on $S^3{}_R \times S^1{}_L$
L/R
Confined,
no χ−SB
Confined,
χ−SB
O(1)
L/R ~ 1/(Λ R)
Deconfined,
no χ−SB
Λ R
0
O(1)

Large N confined-phase partition functions

Complete spectrum is exactly calculable when $R\Lambda \to 0$, $\lambda \to 0$

As $\lambda \to 0$, microscopic fields A_μ, ψ, ϕ = matrix-valued harmonic oscillators

S^3 is compact



Gauss law $\Longrightarrow$ physical states are color singlets

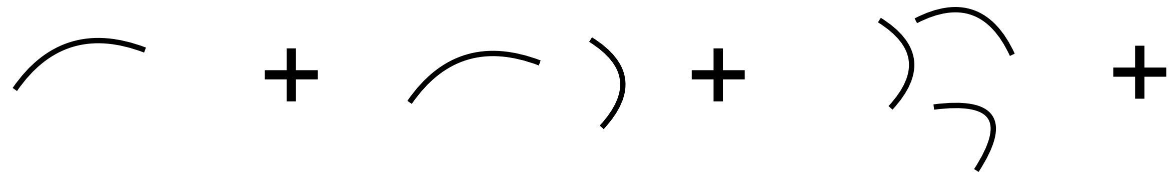
With adjoint fields, color singlets are built from color traces

Large N: $|a\rangle = \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu}|0\rangle$ $\quad$ $|a\rangle|b\rangle = \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu}\mathrm{Tr}\ F_{\alpha\beta}F^{\alpha\beta}|0\rangle$

single-particle state $\quad$ multi-particle state

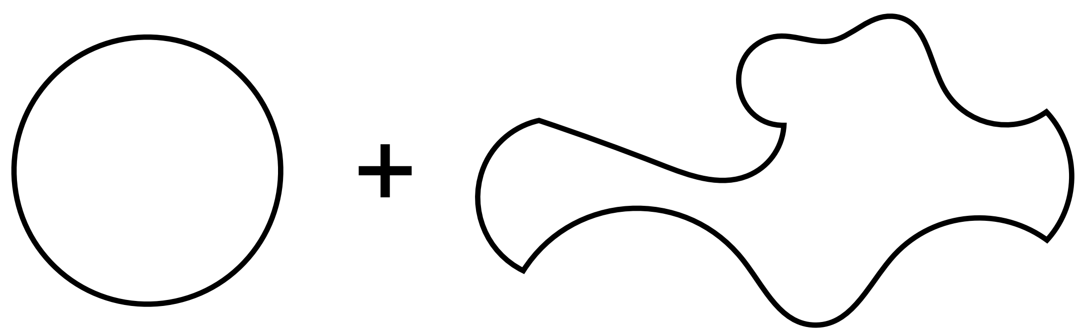
Large N confined-phase partition functions

(1) Get partition functions for microscopic fields, $z_V(q)$, $z_F(q)$, $z_S(q)$, $q = e^{-L/R}$.

$z_v, z_f, z_s =$  $+ \ldots$

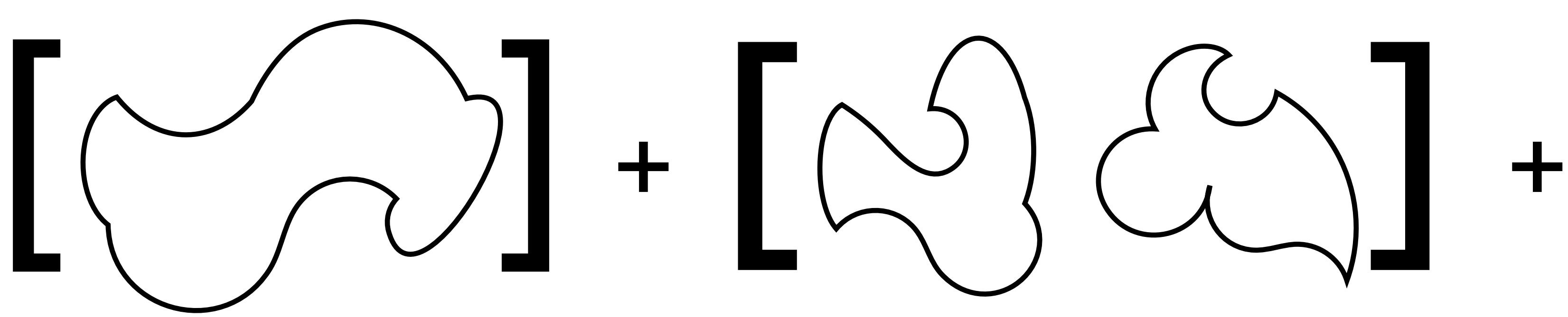
color Gauss law

(3) Build single-trace canonical partition function from $z_V(q)$, $z_F(q)$, $z_S(q)$

$Z_{ST} =$  $+ \ldots$

large N

(4) Build grand canonical partition function Z from Z_{ST}

$Z =$  $+ \ldots$

Large N confined-phase partition functions

Complete spectrum is exactly calculable when $R\Lambda \to 0$, $\lambda = 0$

The $(-1)^F$-twisted and thermal confined-phase grand-canonical partition functions are

$$Z(L) = \mathrm{Tr}\; e^{-LH}, \qquad \tilde{Z}(L) = \mathrm{Tr}\; (-1)^F e^{-LH}$$

$$Z(L) = \prod_{k=1}^{\infty} \frac{1}{1 - z_v(q^k) - n_s z_s(q^k) + (-1)^k n_f z_f(q^k)}$$

Aharony et al 2003

$$\tilde{Z}(L) = \prod_{k=1}^{\infty} \frac{1}{1 - z_v(q^k) - n_s z_s(q^k) + n_f z_f(q^k)}$$

These are the objects we'll study in most of the talk.

Emergent symmetries

Will squeeze a lot of juice from $R\Lambda \to 0, \lambda \to 0$ expressions

But free QFTs have more symmetries than interacting ones.

Can we really expect any large N emergent symmetries we find at small $R\Lambda$ to extrapolate to large $R\Lambda$?

Consideration of large N volume independence and Hagedorn instabilities suggests that answer is yes.

Basar, AC, Dorigoni, Unsal, 2013

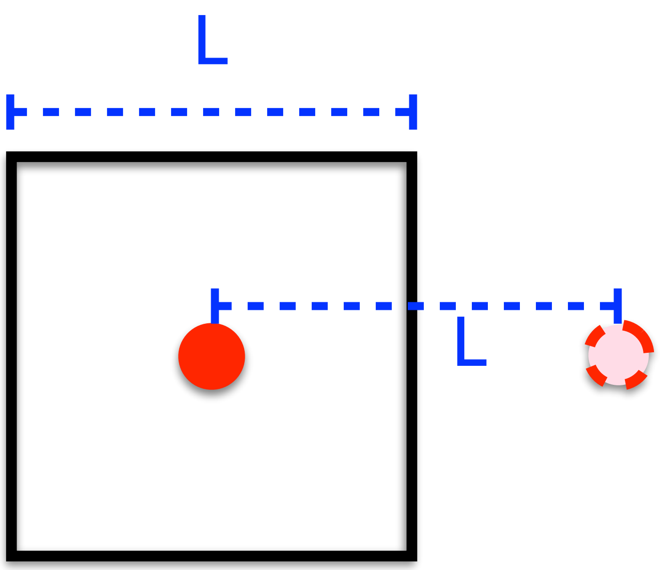
First, a reminder of what large N volume independence is…

Large N volume independence

Finite volume corrections to hadronic correlators vanish at N = ∞.

Discovered by Eguchi and Kawai in 1982, in YM theory

Cartoon picture: How would glueballs find out they're in a toroidal box?



$$\text{box effect} \sim \frac{1}{N^2} e^{-L\Lambda_{QCD}}$$

Technical argument uses center symmetry as proxy for confinement

Implication: within the confined phase at large N, no phase transitions.

Large N volume independence

EK's Jan. 1982 dream was for volume independence for **all** L…

… but the dream instantly got in trouble.

Bhanot, Heller, Neuberger Feb. 1982

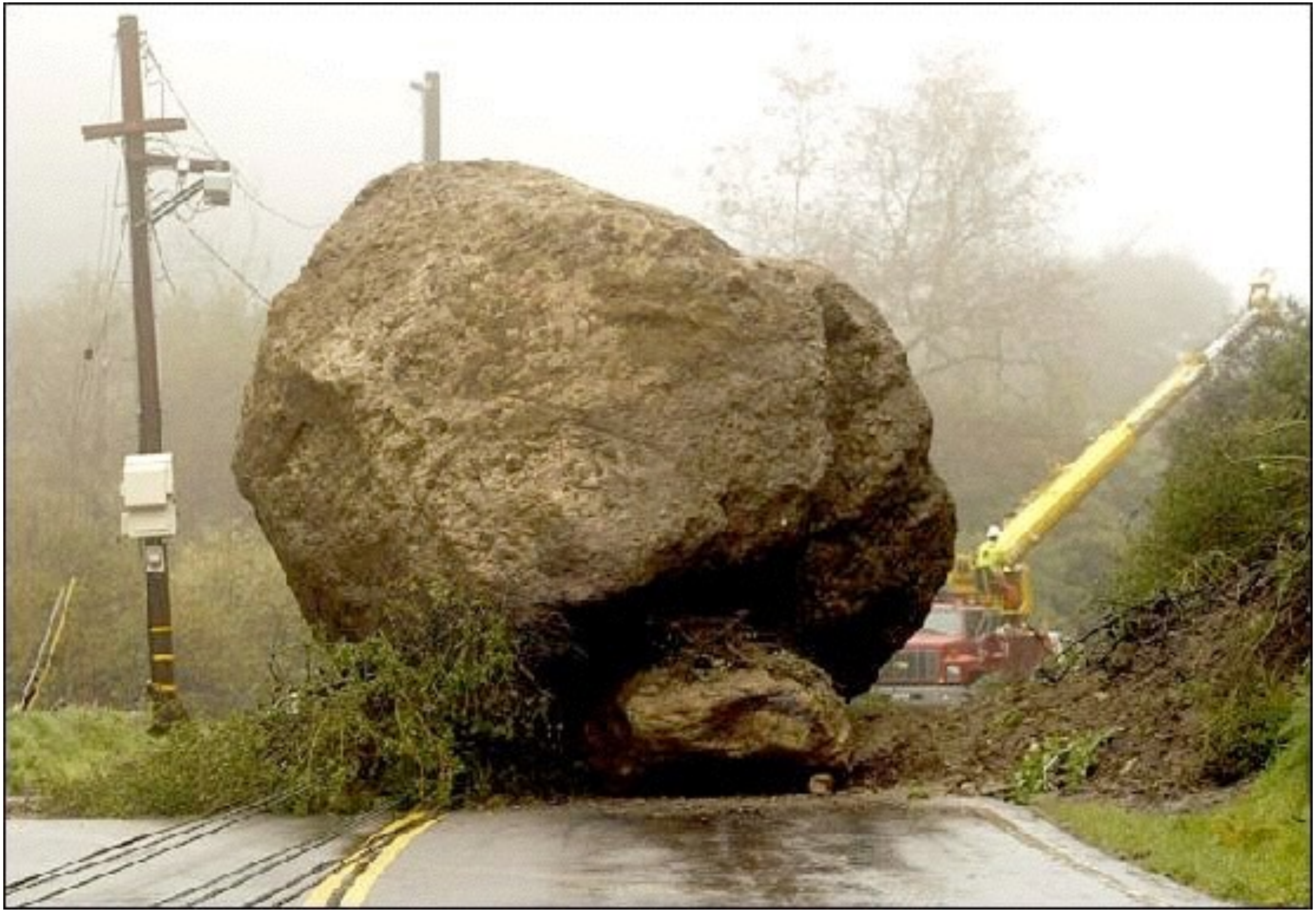
Euclidean QFT on $R^3 \times S^1_L \Longleftrightarrow$ system at temperature $T = 1/L$.

At $L \lesssim 1/\Lambda_{QCD}$ YM goes into a quark-gluon plasma phase!

Deconfinement transition kills volume independence

Volume independence only holds when $L > L_c \approx 1/\Lambda_{QCD}$

Large N volume independence



Roadblock for ~25 years...

Volume independence for any L

Kovtun, Unsal, Yaffe, 2007

Consider adjoint QCD on $R^3 \times S^1$, with periodic BCs on S^1

QCD(Adj) = SU(N) YM theory + N_F massless adjoint Majorana fermions

KUY's observation: gluons drive center-symmetry breaking, while adjoint fermions try to prevent it.

In QCD(Adj) center symmetry doesn't break at small enough L; suggests volume independence may be valid at all $L \sim N^0$.

Kovtun, Unsal, Yaffe, 2007

Lattice simulations consistent with confinement for all $L \sim N^0$.

Bringoltz+Sharpe
Hietanen, Narayanan,
Azeyanagi et al…

On $S^3 \times S^1$ with $R\Lambda \ll 1$ and $N_F > 0$, center symmetry in QCD(Adj) never breaks with periodic BCs

Unsal 2007

All evidence so far: volume independence and confinement for all $L \sim N^0$

Hagedorn instability

$$Z(L) = \mathrm{Tr}\, e^{-LH} = \int dE\, \rho(E) e^{-LE}$$

Hagedorn scaling

$$\rho(E) \sim e^{+L_H E}, L_H \sim \Delta^{-1}$$

mass gap

Signature of a string theory

Expected for any confining large N theory, and can be verified explicitly for RΛ << 1.

Sundborg 1999; Aharony et al, 2003

Once $L < L_H = 1/T_H$, partition function become singular!

Must have phase transition at or below T_H — deconfinement transition

Hagedorn instability and volume independence seem to conflict.

How can we have confinement for all L ?

Volume independence vs Hagedorn

Basar, AC, Dorigoni, Unsal, 2013

There isn't necessarily a conflict - but to avoid it we need a miracle.

Periodic BCs for fermions $\Longrightarrow$ working with twisted partition function

$$\tilde{Z}(L) = \mathrm{Tr}\ (-1)^F e^{-LH}$$
$$= \int dE\ [\rho_B(E) - \rho_F(E)]\, e^{-LE}$$

Compare this to the thermal partition function

$$Z(\beta) = \int dE\ [\rho_B(E) + \rho_F(E)]\, e^{-\beta E}$$

'All' we need is enough cancellation between ρ_B and ρ_F

How much cancellation do we need?

Basar, AC, Dorigoni, Unsal, 2013

Expect Hagedorn scaling for both ρ_B and ρ_F. More precisely:

$$\rho_B(E) \to e^{+\beta_{B,1}E} \sum_{n=n_1}^{\infty} p_{B,n,1} E^{-n} + e^{+\beta_{B,2}E} \sum_{n=n_1}^{\infty} p_{B,n,2} E^{-n} + \cdots$$

$$+ \sum_{n=\tilde{n}}^{\infty} \tilde{p}_{B,n} E^{-n} + \cdots$$

$$\rho_F(E) \to e^{+\beta_{F,1}E} \sum_{n=n_1}^{\infty} p_{F,n,1} E^{-n} + e^{+\beta_{F,2}E} \sum_{n=n_2}^{\infty} p_{F,n,2} E^{-n} + \cdots$$

$$+ \sum_{n=\tilde{n}}^{\infty} \tilde{p}_{F,n} E^{-n} + \cdots$$

ALL red terms must cancel EXACTLY to avoid an instability!

This is a wildly-over-optimistic thing to expect…

Emergent fermionic symmetries

Basar, AC, Dorigoni, Unsal, 2013

Unless, of course, cancellations enforced by some sort of symmetry!

Is there some sort of emergent fermionic symmetry at large N?

Thinking along these lines, at $N_F = 1$, we rediscover supersymmetry.

$N_F = 1$ fermionic symmetry happens to work away from large N as well.

If $N_F > 1$, emergent symmetry **can't** be supersymmetry!

N_F $(N^2 - 1)$ microscopic fermions, only $(N^2 - 1)$ microscopic bosons.

NB: At large N the QFT is free and S matrix is trivial.
No conflict with Coleman-Mandula-type no-go theorems.

Let's see what happens at $R\Lambda \ll 1$!

Level degeneracies

Basar, AC, McGady, 2014

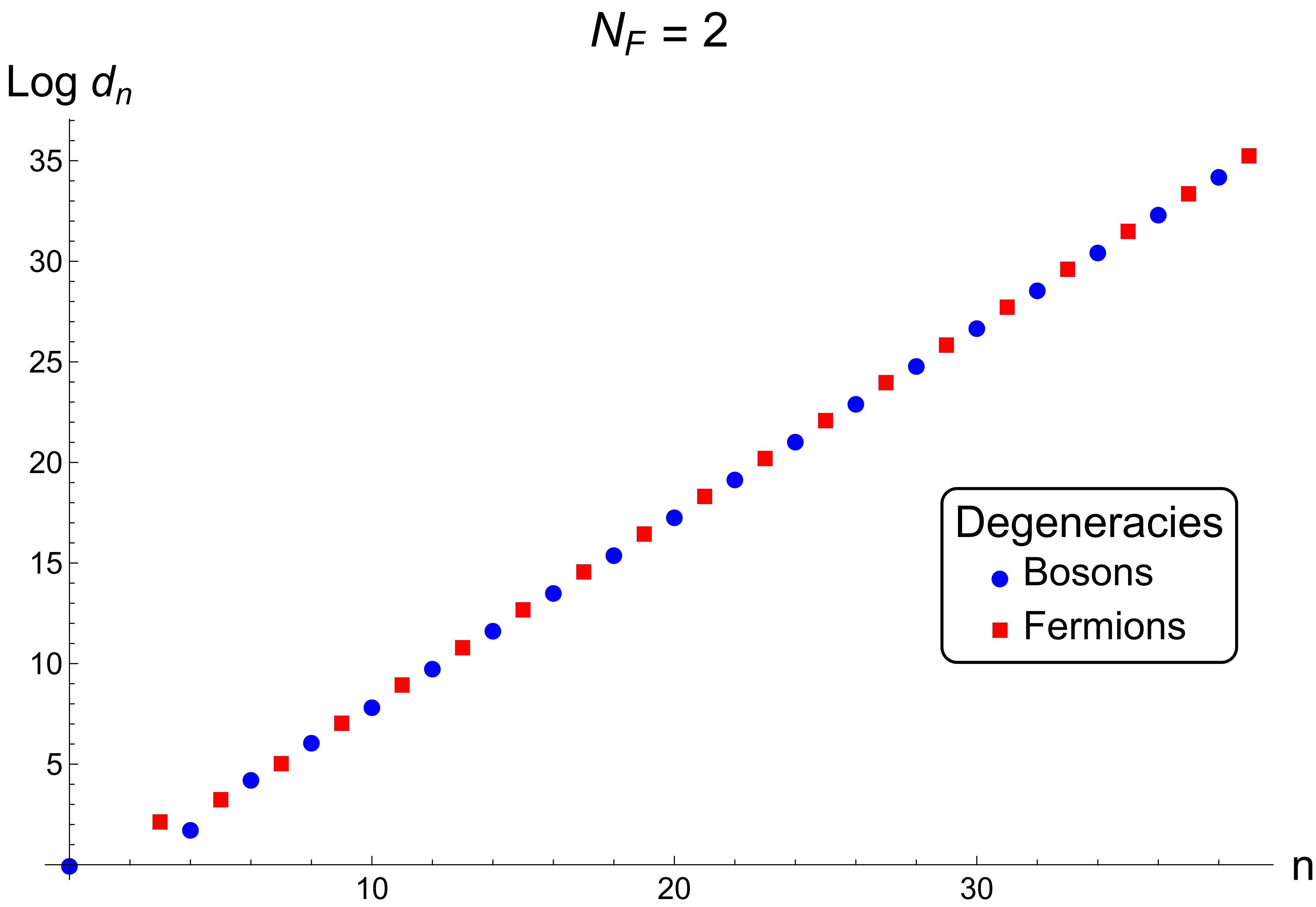
We have the full partition functions for all adjoint-matter theories:

$$Z = \sum_{n=0}^{\infty} d_n q^n$$

Hagedorn phenomenon: size of level-degeneracies d_n grow exponentially!

Can verify it by plotting $\log(d_n)$ versus n

Level degeneracies in adjoint QCD



Eyeball ⇒ leading exponential growth of B and F states <u>identical</u>

(Half-integral Bose-Fermi splitting due to S^3 curvature couplings)

Cancellation of Hagedorn instabilities

Expect the asymptotics of density of states to be described by an **infinite** series of exponentials, one for each 'Regge trajectory'

$$\rho_B(E) \to e^{+\beta_{B,1}E} \sum_n p_{B,n,1} E^{-n} + e^{+\beta_{B,2}E} \sum_n p_{B,n,2} E^{-n} + \cdots$$

$$\rho_F(E) \to e^{+\beta_{F,1}E} \sum_n p_{F,n,1} E^{-n} + e^{+\beta_{F,2}E} \sum_n p_{F,n,2} E^{-n} + \cdots$$

Can't tell whether enough cancellations happen by eyeballing d_n

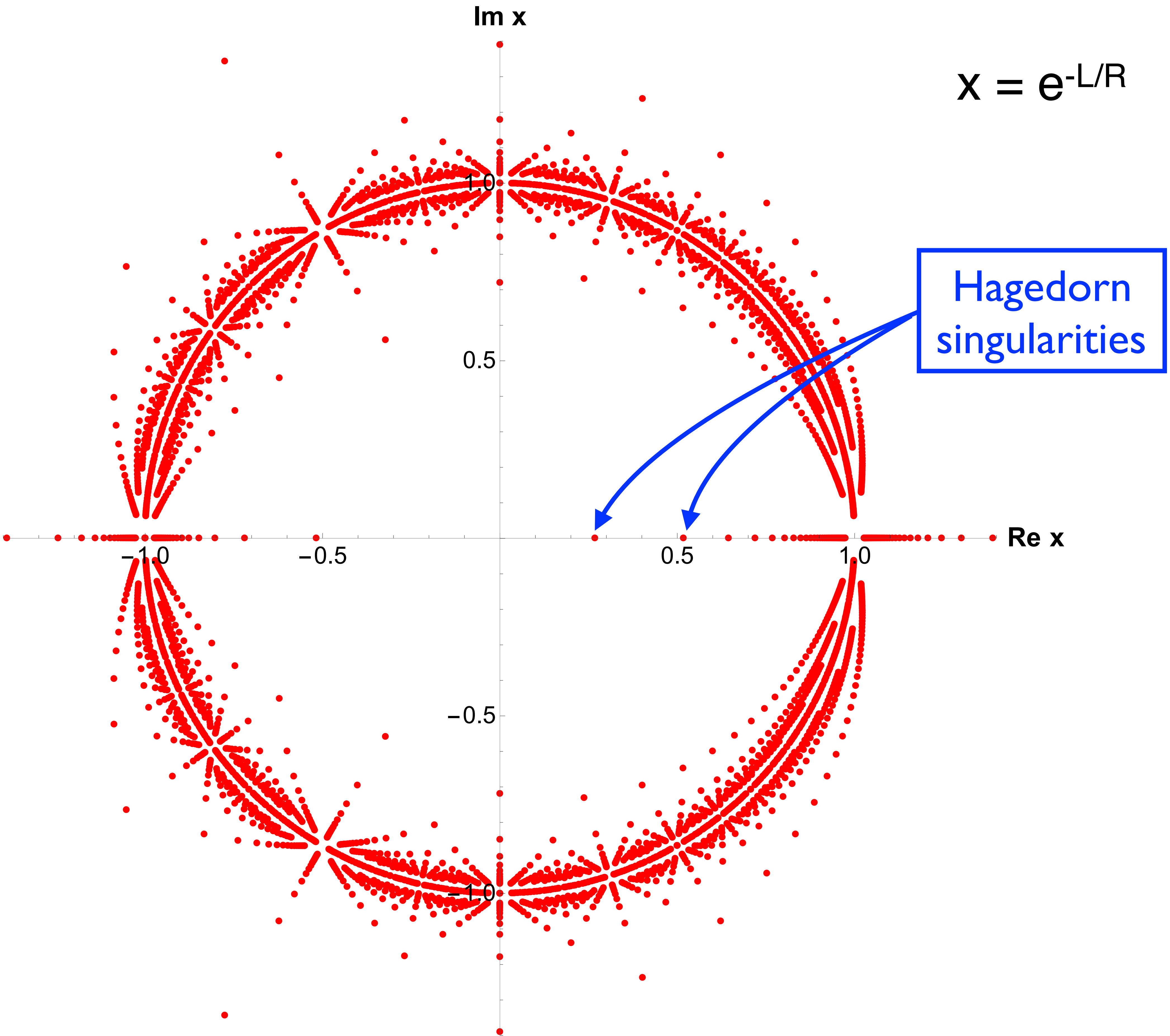
Instead, look for poles of partition functions Z[q] in $q \in [0,1]$

Reason: if $d_n \sim a^n$, then

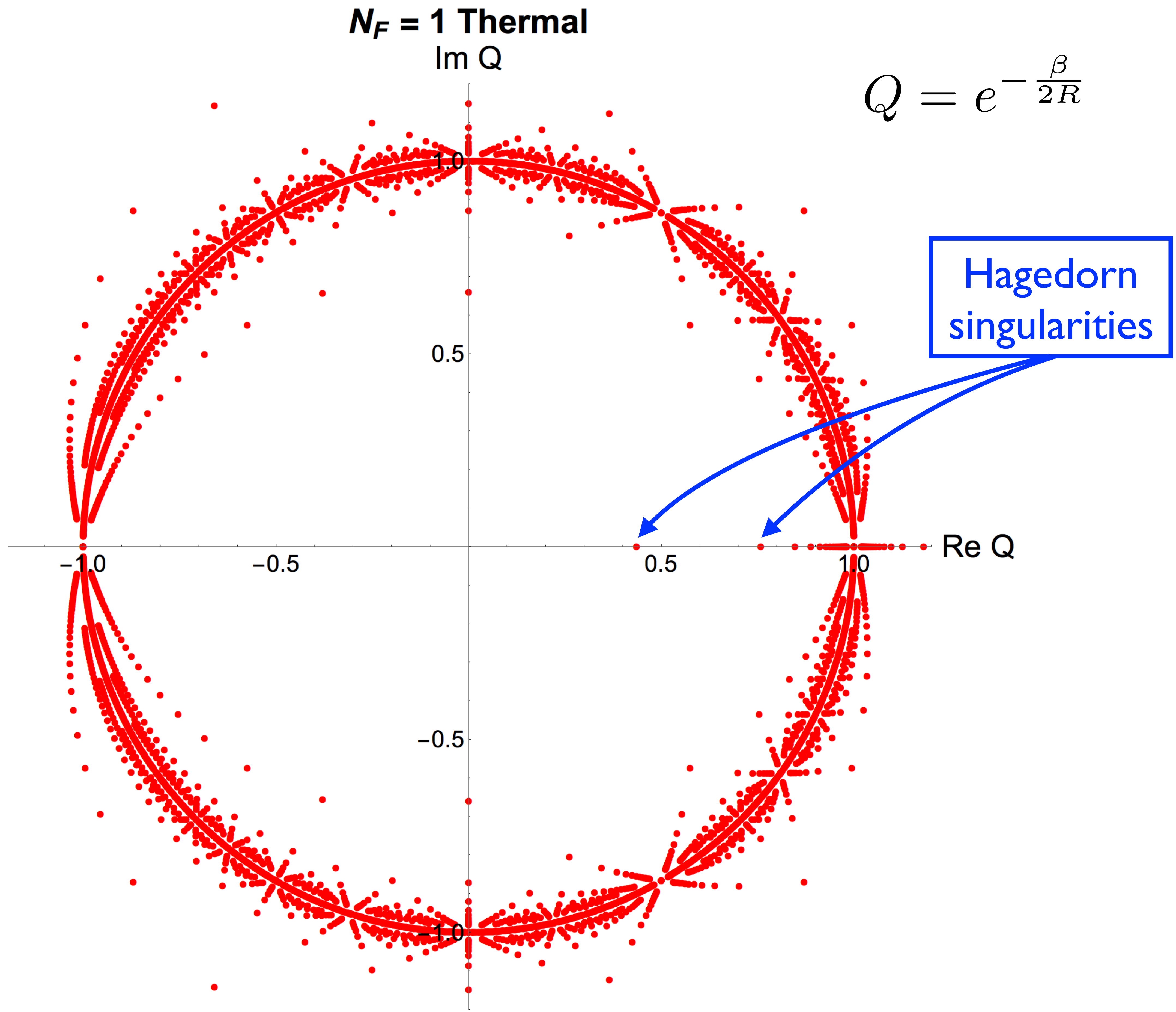
$$\sum_n d_n q^n \sim \frac{1}{1 - aq}$$

No singularities in [0,1] $\Longrightarrow$ complete cancellation of Hagedorn.

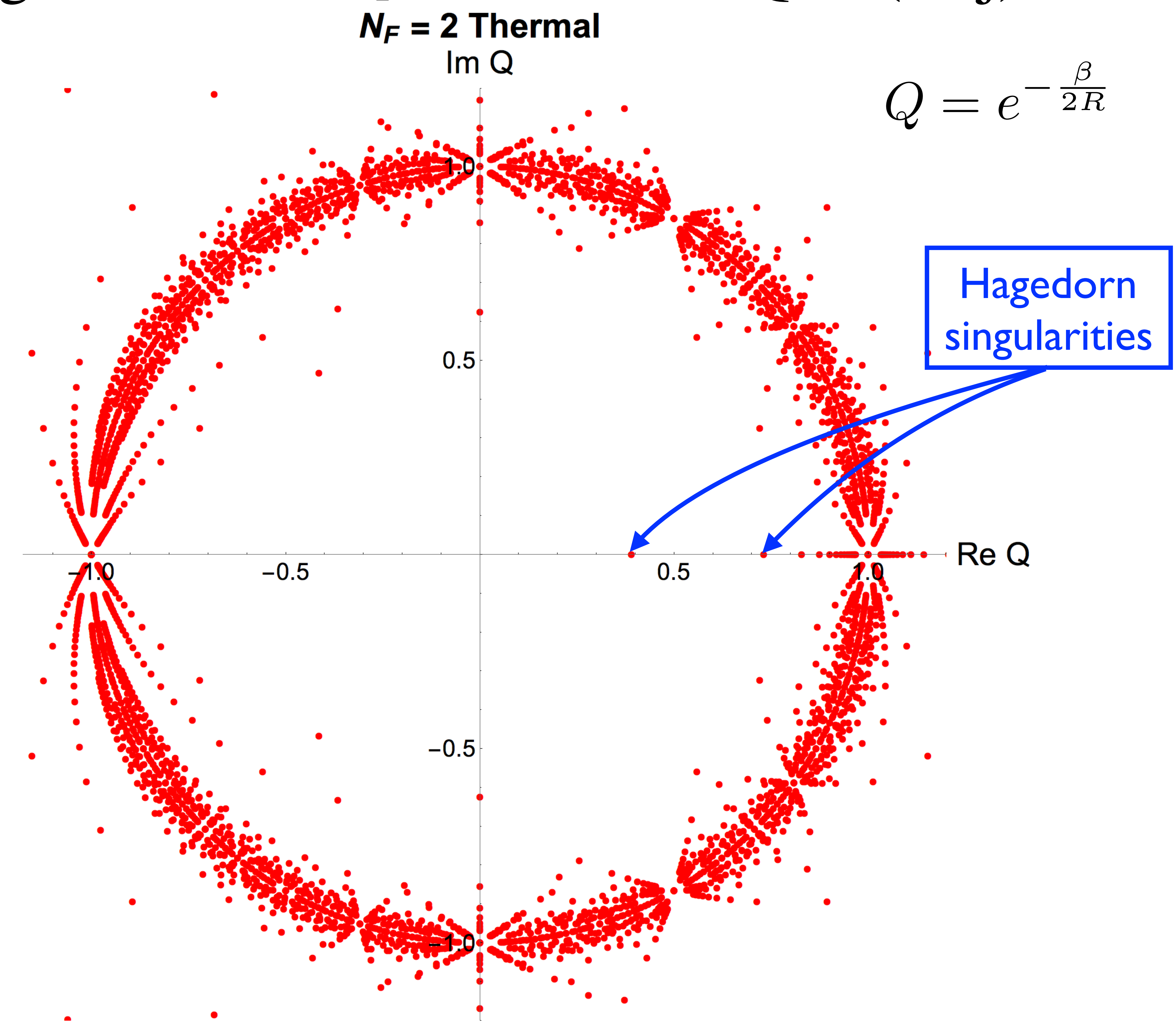
Singularities of pure YM partition function



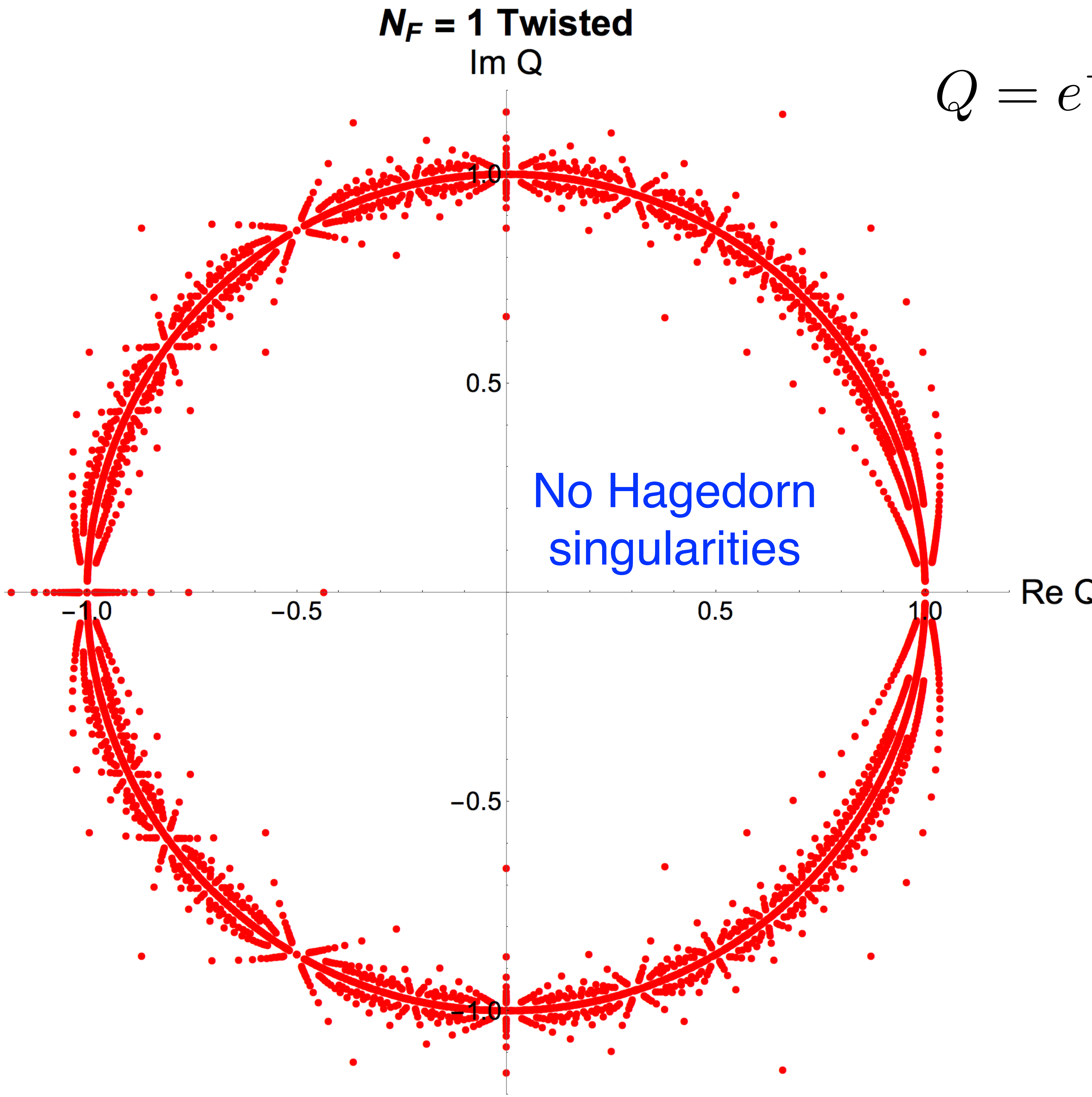
Singularities of $N_F = 1$ thermal QCD(Adj)



Singularities of $N_F = 2$ thermal QCD(Adj)

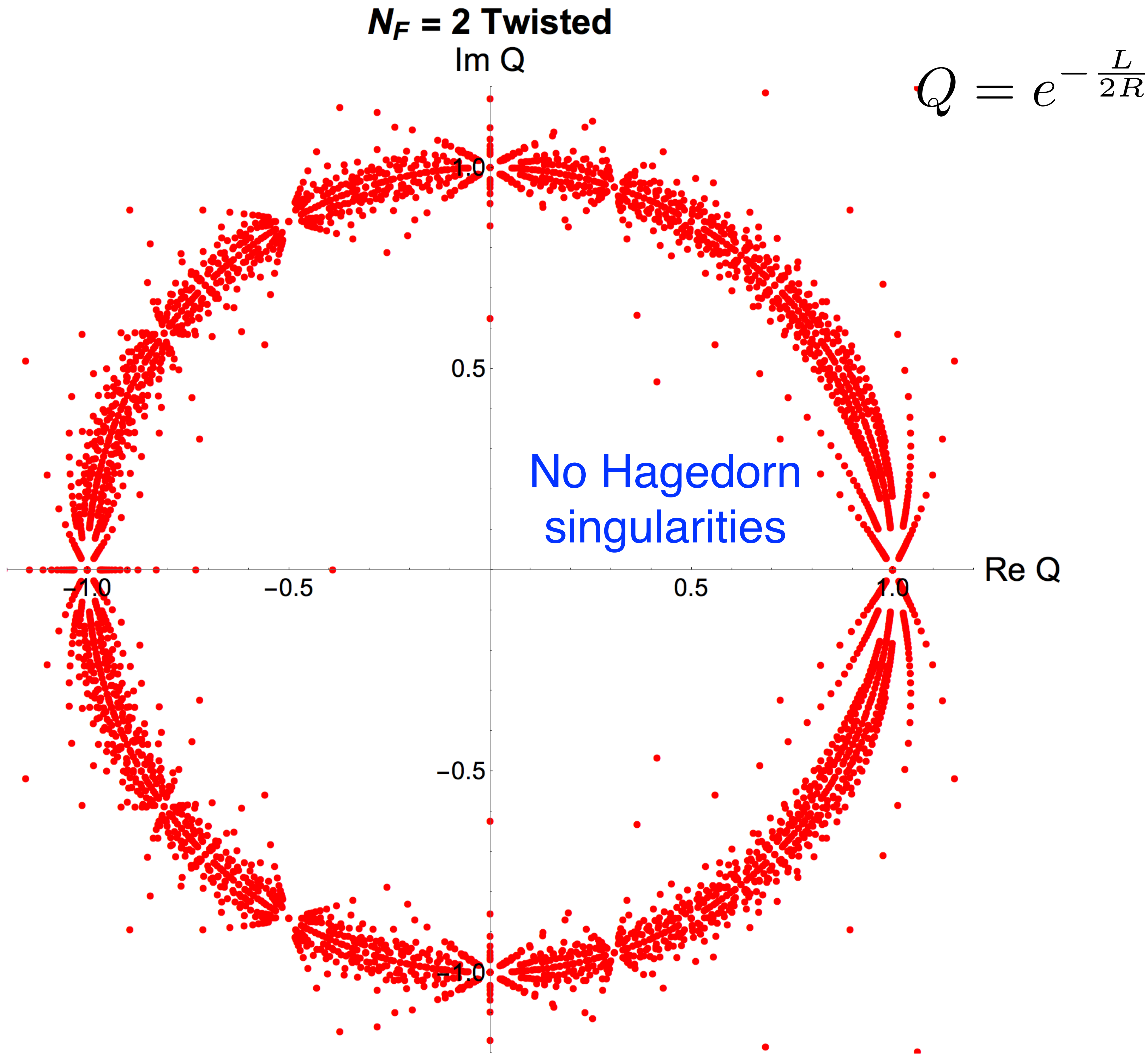


Singularities of $N_F = 1$ twisted QCD(Adj)



$$Q = e^{-\frac{L}{2R}}$$

Singularities of $N_F = 2$ twisted QCD(Adj)



Cancellation of Hagedorn instabilities

In a theory with volume independence for all L, all Hagedorn instabilities cancel for $N_F \geq 1$, as expected from general arguments.

Enormous cancellations in twisted partition function of QCD(Adj) testify to very tight relations between B and F states.

And it's happening in QFTs which are manifestly not supersymmetric.

$$\rho_B(E) \to e^{+\beta_{B,1} E} \sum_{n=n_1}^{\infty} p_{B,n,1} E^{-n} + e^{+\beta_{B,2} E} \sum_{n=n_1}^{\infty} p_{B,n,2} E^{-n} + \cdots$$

$$+ \sum_{n=\tilde{n}}^{\infty} \tilde{p}_{B,n} E^{-n} + \cdots$$

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$$+ \sum_{n=\tilde{n}}^{\infty} \tilde{p}_{F,n} E^{-n} + \cdots$$

Who ordered that?

Look for inspiration in the string theory literature…

Similar Hagedorn cancellations seen in non-supersymmetric string spectra; called ‘asymptotic/misaligned supersymmetry’

Kutasov, Seiberg; Dienes, 1990s

Implication of modular symmetries of the 2D worldsheet CFT.

Confining gauge theories are believed to have a dual string description.

Are there 2D modular symmetries controlling the spectrum of QCD(Adj) and other confining theories?

At least in the $R\Lambda \to 0$ limit, yes.

Basar, AC, McGady, Dienes, 1507.08666

Confession: despite our initial inspiration, relation of result to string theory expectations is not yet clear!

Modular structure in 4D large N YM

Basar, AC, McGady, Dienes, 1507.08666

For simplicity, consider YM first.

$$Z_{\text{YM}} = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{1-3q^n-3q^{2n}+q^{3n}} = \prod_{n=1}^{\infty} \frac{(1-q^n)^3}{(1+q^n)(1-q^n z)(1-q^n z^{-1})}$$

$$z = 2+\sqrt{3}$$

Basar, AC, McGady, Yamazaki arXiv:1406.6329

Pairing of roots {z, 1/z} related to “T-reflection symmetry”

Analytically continue confined-phase partition function in L

Insiration: Polchinski 1992

$$q = e^{-L/R} \to e^{2\pi i \tau}$$

Then Z_{YM} is a finite product of modular forms in τ.

$$\text{Im}\,\tau = L/(2\pi R) = C_{S^1}/C_{S^3}$$

Turning on Re τ may be related to twisting by total angular momentum

Modular structure in 4D large N YM

Basar, AC, McGady, Dienes, 1507.08666

$$Z_{\rm YM}(\tau) = \eta(\tau)^3 \left(\frac{-\sqrt{2} e^{-i\pi b}\eta(\tau)}{\vartheta\left[{1/2 \atop b+1/2}\right](\tau)} \right) \sqrt{\frac{2\,\eta(\tau)}{\vartheta\left[{1/2 \atop 0}\right](\tau)}}$$

where $b = i \log(2+\sqrt{3})/2\pi \approx 0.21i$

$$\vartheta\left[{\alpha \atop \beta}\right](\tau) \equiv q^{\alpha^2/2} \prod_{n=1}^{\infty} \left[(1-q^n) \times (1 + q^{n-\frac{1}{2}+\alpha} e^{2i\pi\beta})(1 + q^{n-\frac{1}{2}-\alpha} e^{-2i\pi\beta}) \right]$$

inverses

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

Fact that b is imaginary builds in the Hagedorn singularities

$Z_{\rm YM}$ is a (vector-valued, meromorphic) modular form of weight +3/2

Irrationality of b means $Z_{\rm YM}$ lives in an infinite-dimensional vector space

Modular structure with adjoint matter

Basar, AC, McGady, Dienes, to appear

$$Z(\tau; n_s) = \eta(\tau)^3 \left(\frac{-\sqrt{2} e^{-i\pi b(n_s)} \eta(\tau)}{\vartheta\left[{1/2 \atop b(n_s)+1/2}\right](\tau)} \right) \sqrt{\frac{2\,\eta(\tau)}{\vartheta\left[{1/2 \atop 0}\right](\tau)}}$$

where $b(n_s) = \arccos(2 + n_s/2)/2\pi$

$$Z(\tau; n_s, n_f) = \prod_{i=1}^{3} \frac{2\cos(\pi b_i)}{e^{+i\pi b_i}} \frac{\eta(\tau)^3}{\vartheta\left[{1/2 \atop b_i}\right](\tau)} \frac{1}{\vartheta\left[{0 \atop b_i + 1/2}\right](\tau)} .$$

$$\tilde{Z}(\tau; n_s, n_f) = \prod_{i=1}^{3} \frac{2\cos(\pi b_i)}{e^{+i\pi b_i}} \frac{\eta(\tau)^3}{\vartheta\left[{1/2 \atop b_i}\right](\tau)} \frac{1}{\vartheta\left[{0 \atop b_i}\right](\tau)} .$$

where $b_i = b_i(n_s, n_f)$

(Valid for all n_s, n_f except for SUSY cases $n_s = 2n_f - 2$.
SUSY theory Z's are also modular, but look slightly different.)

So what?

Basar, AC, McGady, Dienes, 1507.08666

Z_{YM}, $Z_{QCD(Adj)}$, … = nasty-looking functions, which turn out to be writable in terms of some special functions.

Specialness of these special functions implies some remarkable things

Reason: extremely unusual Z_{4D} be modular in a non-SUSY theory, for at least three reasons

(1) 4D-2D spectral equivalence

Implies 4D spectrum organized by symmetries of 2D theory

Focus here for time reasons

(2) Small Im τ behavior

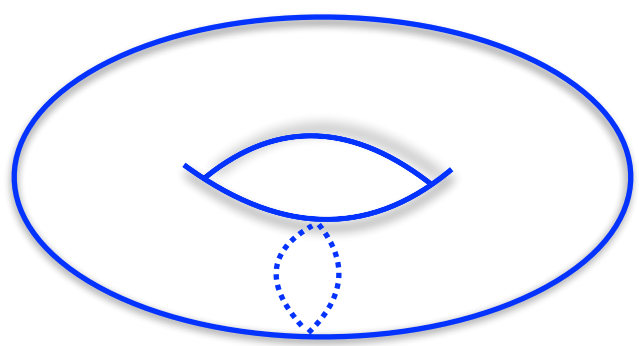
(3) Large Im τ behavior

4D-2D spectral equivalence

confined-phase large-N 4D partition function $\quad Z_{\rm 4D}(\tau) = Z_{\rm 2D}(\tau) \quad$ a chiral partition function (character) of a 2D CFT

Spectrum of certain 4D QFTs = spectrum of certain 2D QFTs, in their respective relevant sectors.

Reason: modular forms f(τ) are building blocks of 2D CFT partition functions on a torus $S^1_{C_1} \times S^1_{C_2}, \; \tau \sim C_2/C_1$.

$Z_{\text{2D CFT}}$ [] = f(τ)

\`Modular properties' of f(τ) $\Longleftrightarrow$ large coordinate transforms in $Z_{\rm 2D}$

$\Longleftrightarrow$ constraints of 2D conformal symmetry

No time to explain the modular group and action on f(τ)

4D-2D spectral equivalence

confined-phase large-N 4D partition function

$$Z_{4\mathrm{D}}(\tau) = Z_{2\mathrm{D}}(\tau)$$

a chiral partition function (character) of a 2D CFT

Spectrum of a 4D QFTs = spectrum of a 2D QFT, in their respective relevant sectors.

Reason: modular forms f(τ) are building blocks of 2D CFT partition functions on a torus $S^1_{C_1} \times S^1_{C_2}, \ \tau \sim C_2/C_1$.

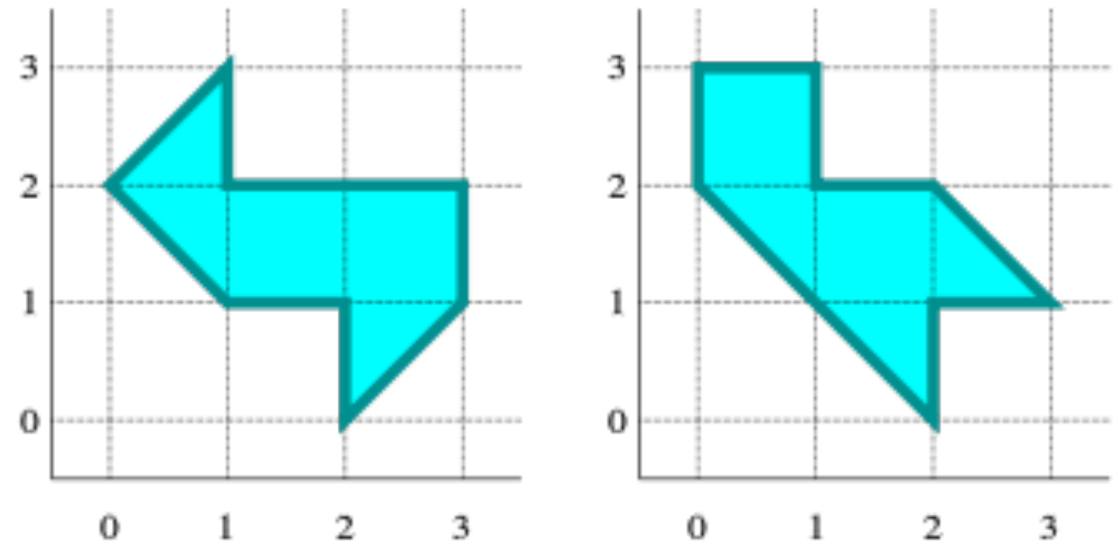
Important in view of original motivation concerning emergent symmetries!

Spectrum of 2D CFTs controlled by infinite-dimensional spectrum-generating algebras, which include Virasoro

So large N 4D spectrum should also be controlled by these symmetries!

So what are these magic 2D CFTs?

You can't hear the shape of a drum



Wikipedia

Spectra of two QFTs can be the same while correlators differ, so identification of 2D CFT based just on Z_{4D} can't be unique.

The remarkable thing is that concrete 2D CFTs with chiral partition functions coinciding with the large N Z_{4D}'s **exist**.

This miracle definitely does not happen for generic 4D theories.

4D-2D spectral equivalence for YM

$$Z_{\mathrm{YM}}(\tau) = \eta(\tau)^4 \frac{1}{\eta(\tau)} \left(\frac{-\sqrt{2} e^{-i\pi b} \eta(\tau)}{\vartheta\left[{}^{\;1/2}_{b+1/2}\right](\tau)} \right) \sqrt{\frac{2\,\eta(\tau)}{\vartheta\left[{}^{1/2}_{\;0}\right](\tau)}}$$

2 copies of chiral partition function of c = -26 bc-ghost CFT (with necessary zero mode insertions)

chiral (e.g. left-mover) partition function of c = 1 scalar CFT

chiral partition function of a c = 2 βγ-ghost CFT

chiral partition function of c = 1 scalar with R-NS boundary conditions

Ridout, Wood 2014

b sets fugacity $z = e^{2 i \pi b}$ for U(1) conserved charge in c = 2 βγ-ghost CFT

The c = 2 βγ-ghost CFT is irrational and logarithmic

Large-N 4D YM is (spectrum) equivalent to a 2D direct-product irrational CFT.

Similar story goes for theories with matter.

4D-2D spectral equivalence for YM

$$Z_{\mathrm{YM}}(\tau) = \eta(\tau)^4 \frac{1}{\eta(\tau)} \left(\frac{-\sqrt{2} e^{-i\pi b} \eta(\tau)}{\vartheta\left[{1/2 \atop b+1/2}\right](\tau)} \right) \sqrt{\frac{2\,\eta(\tau)}{\vartheta\left[{1/2 \atop 0}\right](\tau)}}$$

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Small |τ| behavior

Partition functions of generic 4D QFTs on $M_3 x S^1_L$ behave like

$$\log Z(L \to 0) \to \sigma \mathrm{Vol}_{M_3} L^{-3}$$

Free massless scalar on S^3

$$Z_{\mathrm{scalar}} = q^{1/240} \prod_{n \geq 1} (1 - q^n)^{-n^2}$$

$$\log Z_{\mathrm{scalar}}(L \to 0) \;\to\; \frac{\pi^4 R^3}{45} L^{-3} \quad \text{as } L \to 0$$

Vanishing of L^{-3} requires a symmetry; for instance, SUSY does the job.

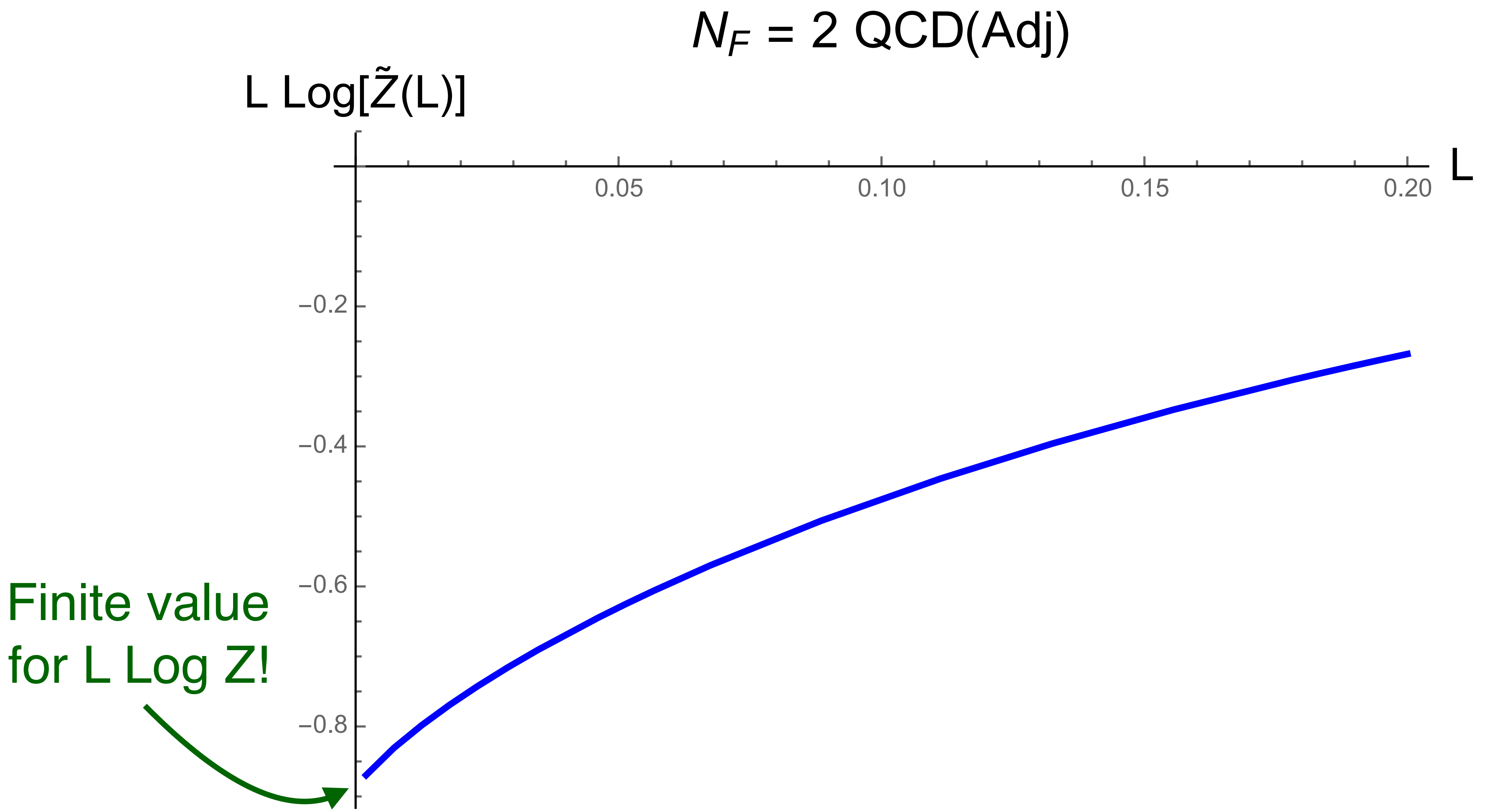
But most of our examples manifestly lack SUSY.

Small |τ| behavior of QCD(Adj)

Partition functions of generic 4D QFTs on $M_3 x S^1_L$ behave like

$$\log Z(L \to 0) \to \sigma \mathrm{Vol}_{M_3} L^{-3}$$

But large N confining-phase gauge theories are not generic QFTs.



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But large N confining-phase gauge theories are not generic QFTs.

$$\log Z(L \to 0) \to \sigma R L^{-1}$$

Infinite sum over the particle species conspires to kills the L^{-3} term.

QCD(Adj) behaves as if it were a 2D QFT, for any N_F

Vanishing of the L^{-3} term is due to modular symmetries.

$$f(|\tau| \gg 1) = e^{2\pi i \tau \Delta}(1 + \ldots) \quad \Longrightarrow \quad f(|\tau| \ll 1) = e^{-2\pi i \Delta/\tau}(1 + \ldots)$$

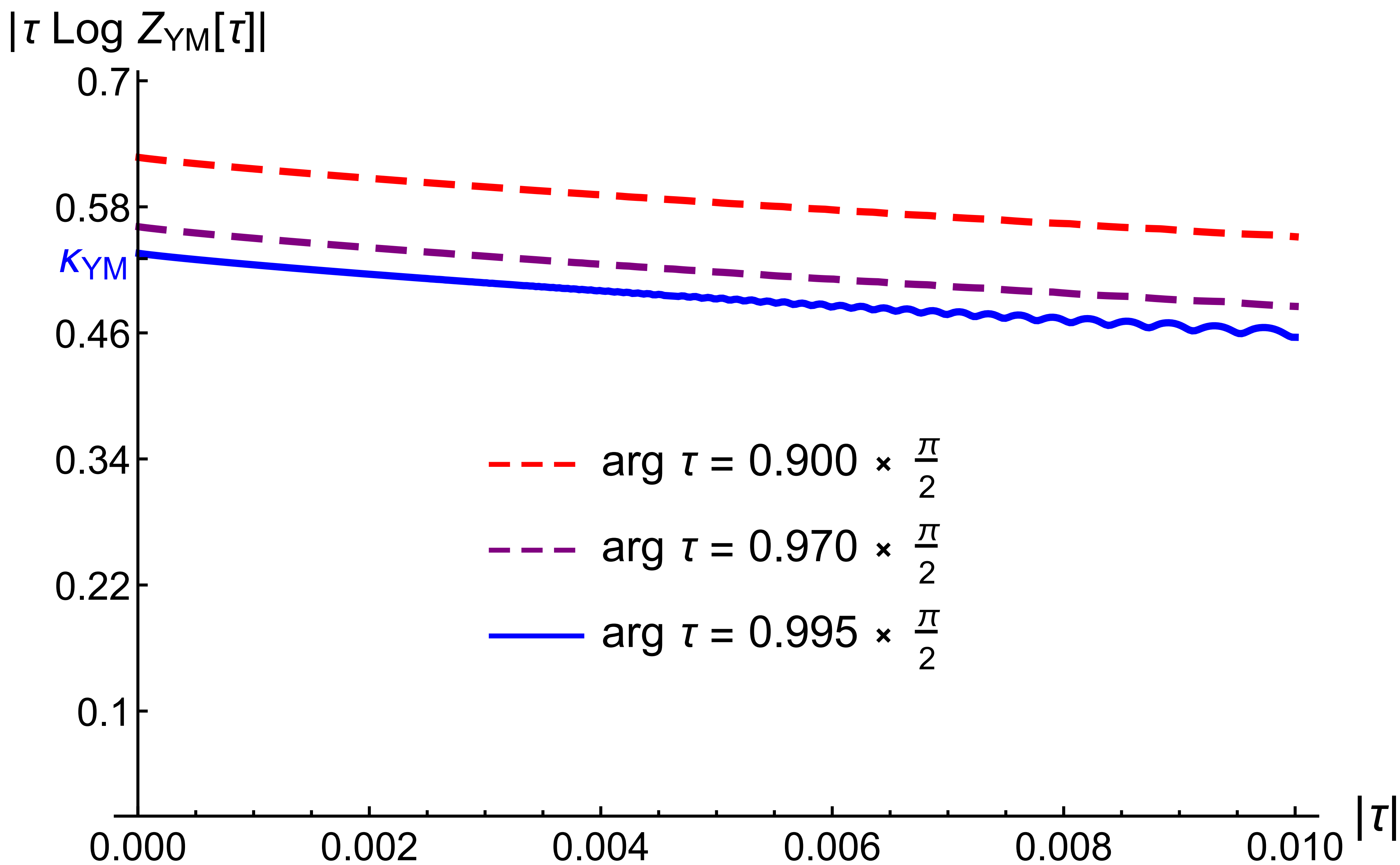
(Can be generalized for θ functions)

Small Im τ behavior of YM

For thermal partition functions like Z_{YM}, correct statement is

$$\lim_{\arg\tau\to\pi/2}\left[\lim_{|\tau|\to 0}\log Z_{\rm YM}(\tau)\right] \;\to\; \kappa_{\rm YM} R/L$$

Note: opposite order governed by deconfined phase, gives log Z ~ L^{-3}



Large Im τ behavior: the vacuum energy

'vacuum energy' of modular forms fixed by modular symmetries in terms of the spectral data $\{c_n\}$

$$\chi(\tau) = q^{\Delta} \sum_n c_n q^n$$

Modularity of Z_{4D} then fixes E_{vac}, with result independent of N_S, N_F

$$E_{\text{vac}} = 0 = \begin{cases} 3 \times \frac{1}{24} + \left(\frac{1}{24} - \frac{1}{8}\right) + \frac{1}{2}\left(\frac{1}{24} - \frac{1}{8}\right), & \text{pure YM} \\ 3 \times \left[\frac{1}{24} + \left(\frac{1}{24} - \frac{1}{8}\right) + \left(\frac{1}{24} - 0\right)\right], & \text{QCD(Adj) with } N_f = 2 \\ \ldots & \ldots \end{cases}$$

Modularity forbids finite counter-terms which could otherwise shift E_{vac}.

Matches recent direct evaluations of large N spectral sums

Basar, AC, McGady, Yamazaki, 1408.3120

NB: Assumes $N \to \infty$ before $\mu_{uv} \to \infty$.

Given $\Lambda = \mu_{\text{uv}} e^{-\frac{8\pi^2}{\beta_0 \lambda(\mu_{\text{uv}})}}$, natural order for confining theories.

In literature on e.g. $\mathcal{N} = 4$ SYM, opposite order used, giving a different result.

The role of large N

To see why large N is vital, consider YM at finite N on $S^3 \times S^1$.

QFT in finite volume with finite-rank fields $\Longrightarrow$ *smooth* partition function Z(L)

At high temperature, $\log Z \sim L^{-3}$

But any finite product of modular forms necessarily gives $\log Z \sim L^{-1}$ thanks to modular S transform

So for thermal partition functions, 4D-2D relations like ours are only conceivable in the large N limit.

In SUSY QFTs, $\log Z \sim L^{-1}$, so 4D-2D equivalences akin to ours conceivable — and sometimes exist! — at finite N

Beem et al 2013

The importance of large N and confinement $\Longrightarrow$

hope that results not just an accident of working at $\lambda \to 0$

Summary

Might expect some emergent symmetries at large N in confining theories

Interplay of volume independence and Hagedorn behavior suggests emergent fermionic symmetries

We explored these issues using $R\Lambda \to 0$ as a control parameter

Saw behavior consistent with emergent fermionic symmetries

Found that spectrum of confining 4D large N coincides with spectrum of certain 2D CFTs, and hence is controlled by their symmetries

Many consequences for behavior of 4D partition functions

Open questions

What happens at finite λ (finite $R\Lambda$)? Mapping for correlation functions?

What do the other sectors of 2D CFT mean in the 4D theory?

What's the explicit realization of e.g. Virasoro symmetry on 4D side?

Expect conserved higher spin currents at $R\Lambda \to 0$ in 4D theories;
do associated 2D CFTs have a W symmetry?

Relation to other 4D-2D relations?

Why is our 4D-2D relation possible at all? Origin in string theory?

There is a lot to do!

Thanks for listening!

Backup: modularity in SUSY QFTs

$$\tilde{Z}_{\kappa<3}(\tau) = \eta(\tau) \left(\frac{\eta(\tau)}{\vartheta\left[\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right](\tau)} \right) \prod_{\pm} \frac{2\cos(\pi b_\pm) e^{-i\pi b_\pm} \; \eta(\tau)^2}{\vartheta\left[\begin{smallmatrix} 1/2 \\ b_\pm \end{smallmatrix}\right](\tau) \; \vartheta\left[\begin{smallmatrix} 0 \\ b_\pm \end{smallmatrix}\right](\tau)} \, ,$$

$$\tilde{Z}_{\kappa=3}(\tau) = \frac{1}{\eta(\tau)} \left(\frac{\eta(\tau)}{\vartheta\left[\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right](\tau)} \right)^2 \frac{2\cos(\pi b_{\kappa=3}) e^{-i\pi b_{\kappa=3}} \; \eta(\tau)^2}{\vartheta\left[\begin{smallmatrix} 1/2 \\ b_{\kappa=3} \end{smallmatrix}\right](\tau) \; \vartheta\left[\begin{smallmatrix} 0 \\ b_{\kappa=3} \end{smallmatrix}\right](\tau)}$$

κ is number of adjoint $\mathcal{N} = 1$ matter multiplets, so κ = 3 is $\mathcal{N} = 4$ SYM

Modular weight is +1/2 and -1/2, compared to +3/2 for non-SUSY cases

$$\lim_{b_\alpha \to 0} \frac{\cos(\pi b_\alpha)}{\theta_1(b_\alpha - \frac{1}{2}, \tau)} = \frac{1}{2\eta(\tau)^3}$$

Backup: 4D-2D spectral equivalence for YM

$$Z_{\mathrm{YM}}(\tau) = \eta(\tau)^4 \frac{1}{\eta(\tau)} \left(\frac{-\sqrt{2} e^{-i\pi b} \eta(\tau)}{\vartheta\left[{}^{\;\;1/2}_{b\,+\,1/2}\right](\tau)} \right) \sqrt{\frac{2\,\eta(\tau)}{\vartheta\left[{}^{1/2}_{\;0}\right](\tau)}}$$

Large-N 4D YM is (spectrum) equivalent to a 2D direct-product irrational CFT.

To get info on primary spectrum of 2D CFT, calculated the modular orbit of Z_{YM} and the diagonal modular invariant

Eigenvalues of modular T: τ → τ + 1 operator give

$$h^{(\mathrm{eff})} = h - c/24 \pmod 1$$

$$h^{(\mathrm{eff})}_{m,\ell,\alpha} = \frac{1}{2}\left[\frac{1-\frac{1}{2}(1+(-1)^m)}{8} + m^2|b|^2\right] - \alpha$$

m, l are integers, $\alpha \in [0,1)$, consistent with irrational CFT interpretation

Anderson+Moore, Vafa 1988

Backup: Simple QM example

Basar, AC, Dorigoni, Unsal, 2013

An instructive ultra-simplified toy QM model

Finite number of degrees of freedom

$$H = \omega\, b^\dagger b + \sum_{i=1}^{N_f} \omega\, f_i^\dagger f_i$$

'gluon' bosonic oscillator

'gluino' fermionic oscillators

$N_F = 1$ $\quad Q = b^\dagger f, \ [H, Q] = 0$ SUSY!

general N_F $\quad Q_i = b^\dagger f_i, \ [H, Q_i] = 0$ Not SUSY

How does the twisted partition function behave?

Twisted partition function

Compute N_F = 1 twisted partition function of the toy model:

$$\tilde{Z} = \underset{|0,0\rangle}{1} + \left[\underset{|1,0\rangle}{e^{-L\omega}} - \underset{|0,1\rangle}{e^{-L\omega}}\right] + \left[\underset{|2,0\rangle}{e^{-2L\omega}} - \underset{|1,1\rangle}{e^{-2L\omega}}\right] + \cdots$$

$$= 1$$

SUSY!

Twisted partition function

Compute $N_F = 1$ twisted partition function of the toy model:

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$$= 1$$ SUSY!

Compute $N_F = 2$ twisted partition function:

$$\tilde{Z} = \underset{|0,0,0\rangle}{1} + \left[\underset{|1,0,0\rangle}{e^{-L\omega}} - \underset{|0,1,0\rangle,|0,0,1\rangle}{2\, e^{-L\omega}} \right]$$

$$+ \left[\underset{|2,0,0\rangle}{e^{-2L\omega}} - \underset{|1,1,0\rangle,|1,0,1\rangle}{2\, e^{-2L\omega}} + \underset{|0,1,1\rangle}{1\, e^{-2L\omega}} \right]$$

$$+ \left[\underset{|3,0,0\rangle}{e^{-3L\omega}} - \underset{|2,1,0\rangle,|2,0,1\rangle}{2\, e^{-3L\omega}} + \underset{|1,1,1\rangle}{1\, e^{-3L\omega}} \right] + \cdots$$

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$$= 1$$

SUSY!

Compute N_F = 2 twisted partition function:

$$\tilde{Z} = \underset{|0,0,0\rangle}{1} + \left[\underset{|1,0,0\rangle}{e^{-L\omega}} - \underset{|0,1,0\rangle,|0,0,1\rangle}{2\,e^{-L\omega}}\right]$$

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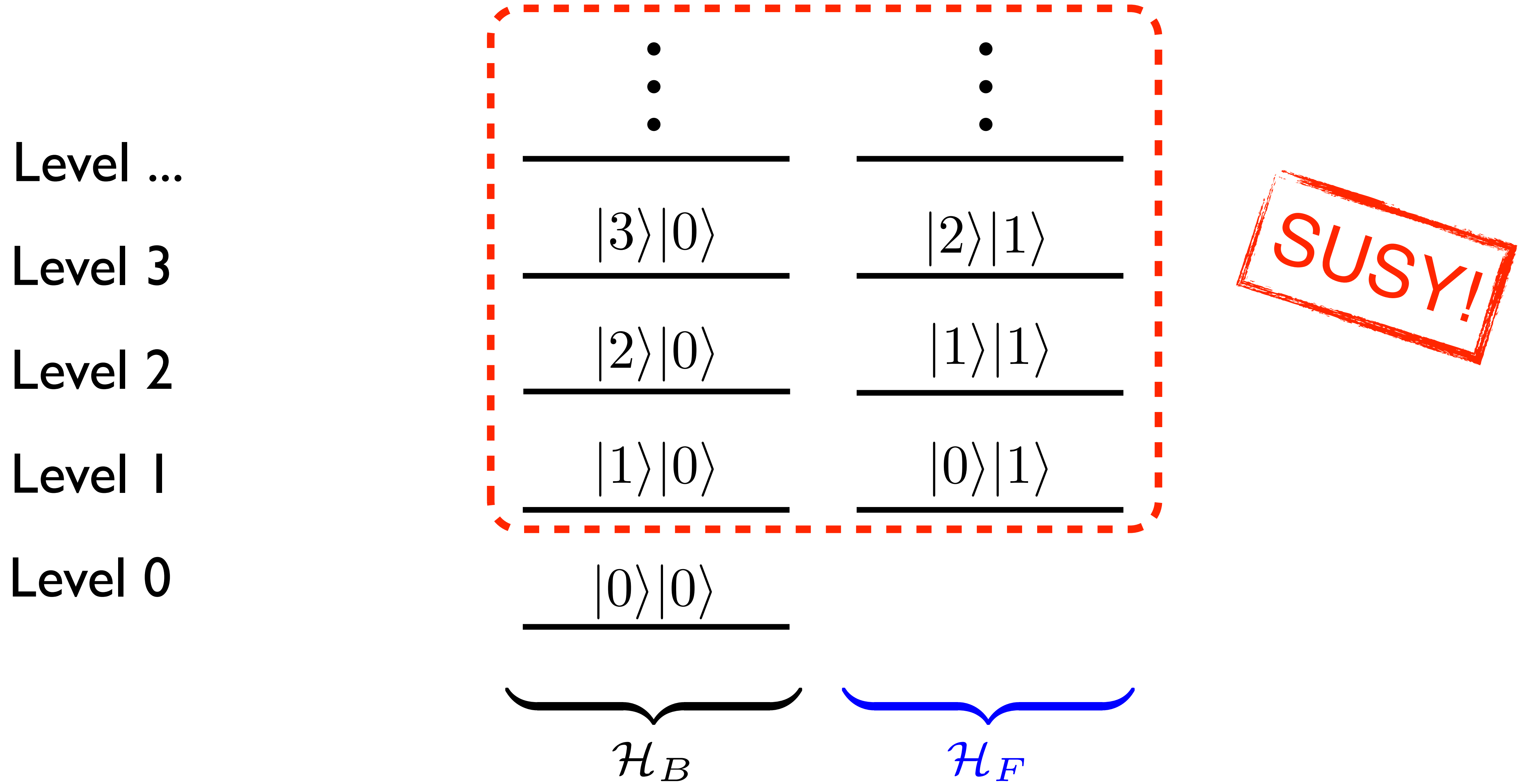
Not SUSY

$$= 1 + e^{-L\omega} - 2e^{-L\omega}$$

Cancellations at $N_F = 1$

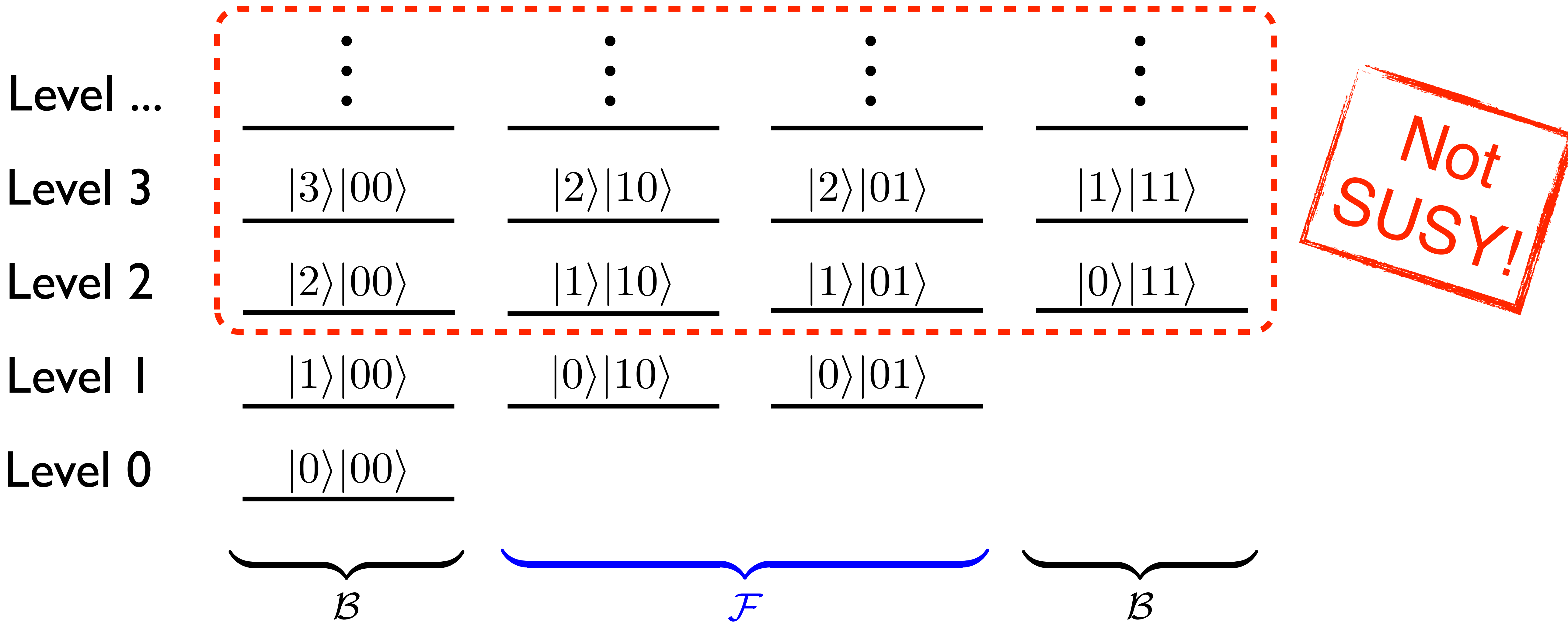


All states contribute to thermal partition function

In twisted partition function, states in the box all cancel each other

Only states annihilated by Q (outside box) contribute to $\tilde{Z}$

Cancellations at $N_F = 2$



All states contribute to thermal Z, and are related by Q_i and J_i

In twisted partition function, states in the box all cancel each other

Only states annihilated by **all** Q_i (outside box) contribute to $\tilde{Z}$

Cancellations start at level N_F