$\mathcal{N} = 4$ Yang-Mills on a lattice: an update

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Outline:

- Brief review: constructing lattice actions with exact supersymmetry
- $\mathcal{N} = 4$ Yang-Mills on the lattice (S.C, Poul Damgaard, Tom DeGrand, Joel Giedt, David Schaich, arXiv:1410.6971, arXiv:1411.0166, arXiv:1405.0644, arXiv:1505.03135)
- Recent results: Konishi anomalous dimension and static potential
- Applications: black hole thermodynamics (S.C Toby Wiseman, Anosh Joseph, arXiv:1008.4964,arXiv:0803.4964)
- Generalizations: lattice quivers and 2d super QCD. Dynamical susy breaking (S.C, Aarti Veernala, arXiv:1505.00467)

Motivations and difficulties of lattice supersymmetry

- Lots of interesting physics in supersymmetric gauge theories:
  dualities, holography, conformality, ...

- Lattice promises non-perturbative insights from first principles

Problem: Discrete spacetime breaks supersymmetry algebra

$$\left\{ Q_\alpha^{\mathrm{I}}, \overline{Q}_{\dot\alpha}^{\mathrm{J}} \right\} = 2\delta^{\mathrm{IJ}}\sigma^\mu_{\alpha\dot\alpha} P_\mu \text{ where } \mathrm{I}, \mathrm{J} = 1, \cdots, \mathcal{N}$$

$\Longrightarrow$ Impractical fine-tuning generally required to restore susy,
especially for scalar fields (from matter multiplets or $\mathcal{N} > 1$)

Solution: preserve subset of susy algebra on lattice

Possible for $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM)

Brief review of $\mathcal{N} = 4$ SYM

$\mathcal{N} = 4$ SYM is a particularly interesting theory

—AdS/CFT correspondence

—Testing ground for reformulations of scattering amplitudes

—Arguably *simplest* non-trivial field theory in four dimensions

Basic features:

- SU(N) gauge theory with four fermions Ψ^{I} and six scalars Φ^{IJ}, all massless and in adjoint rep.
- Action consists of kinetic, Yukawa and four-scalar terms with coefficients related by symmetries
- Supersymmetric: 16 supercharges Q^{I}_{α} and $\overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ with $\mathrm{I} = 1, \cdots, 4$
  Fields and Q's transform under global SU(4) $\simeq$ SO(6) R symmetry
- Conformal: β function is zero for any 't Hooft coupling λ

Topological twisting $\longrightarrow$ exact susy on the lattice

What is special about $\mathcal{N} = 4$ SYM ?

Global symmetries admit a “twisted rotation group":

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{Lorentz}} \otimes \mathrm{SO}(4)_R\right] \qquad \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R$$

eg. $\Psi^{\mathrm{I}}_{\alpha} \to L_{\alpha\beta}\Psi^{\mathrm{J}}_{\beta}(R^T)^{\mathrm{JI}}$ *When* $R = L$ *transforms like matrix...*

$$\Psi = \psi I + \psi_\mu \gamma_\mu + \psi_{\mu\nu}\gamma_\mu\gamma_\nu + \ldots \overset{repackage}{=} \eta I + \psi_a \gamma_a + \chi_{ab}\gamma_a\gamma_b$$

with $a, b = 1 \ldots 5$

Fermions appear as p-forms in twisted theory!

This change of variables gives a susy subalgebra $\{\mathcal{Q}, \mathcal{Q}\} = 2\mathcal{Q}^2 = 0$

This subalgebra can be exactly preserved on the lattice

Twisted $\mathcal{N} = 4$ SYM fields and $\mathcal{Q}$

Everything transforms with **integer spin** under $\text{SO}(4)_{tw}$ — **no spinors**

$$Q^{\mathrm{I}}_{\alpha} \;\text{ and }\; \overline{Q}^{\mathrm{I}}_{\dot\alpha} \longrightarrow \mathcal{Q},\; \mathcal{Q}_a \;\text{ and }\; \mathcal{Q}_{ab}$$

$$\Psi^{\mathrm{I}} \;\text{ and }\; \overline{\Psi}^{\mathrm{I}} \longrightarrow \eta,\; \psi_a \;\text{ and }\; \chi_{ab}$$

$$A_\mu \;\text{ and }\; \Phi^{\mathrm{IJ}} \longrightarrow \mathcal{A}_a = (A_\mu, \phi) + i(B_\mu, \overline{\phi}) \;\text{ and }\; \overline{\mathcal{A}}_a$$

The twisted-scalar supersymmetry $\mathcal{Q}$ acts as

$$\mathcal{Q}\,\mathcal{A}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

bosonic auxiliary field with e.o.m. $d = \overline{\mathcal{D}}_a \mathcal{A}_a$

1. Scalars $\rightarrow$ vectors under twisted group. Combine with gauge fields
2. The susy subalgebra $\mathcal{Q}^2 \cdot = 0$ is manifest

Twisted $\mathcal{N} = 4$ action

Obtain by dimensional reduction of twisted $\mathcal{N} = 2$ Yang-Mills in *five* dimensions:

$$
\begin{aligned}
S &= \frac{N}{2\lambda}\mathcal{Q}\int_{M^4\times S^1} \mathrm{Tr}\left(\chi_{ab}\mathcal{F}_{ab} + \eta[\overline{\mathcal{D}}_a, \mathcal{D}_a] - \frac{1}{2}\eta d\right) \\
. &\quad - \frac{N}{8\lambda}\int_{M^4\times S^1} \epsilon_{abcde}\,\mathrm{Tr}\,\chi_{ab}\overline{\mathcal{D}}_c\chi_{de}
\end{aligned}
$$

- $\mathcal{Q}^2 = 0$ and Bianchi guarantee supersymmetry *independent* of metric of M^4
- Marcus/GL twist of $\mathcal{N} = 4$.
- First obtained via orbifolding/deconstruction methods by Kaplan and Unsal.

Lattice $\mathcal{N} = 4$ SYM fields and $\mathcal{Q}$

The lattice theory is very nearly a direct transcription

- Covariant derivatives $\longrightarrow$ finite difference operators eg.

$$\mathcal{D}_a\psi_b \to \mathcal{U}_a(x)\psi_b(x+a) - \psi_b(x)\mathcal{U}_a(x+b)$$

- Gauge fields $\mathcal{A}_a \longrightarrow$ gauge links $\mathcal{U}_a$

$$\mathcal{Q}\,\mathcal{A}_a \longrightarrow \mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad\qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad\qquad \mathcal{Q}\,\overline{\mathcal{A}}_a \longrightarrow \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad\qquad \mathcal{Q}\,d = 0$$

Geometrical formulation facilitates discretization

η live on lattice sites

ψ_a live on links

χ_{ab} face links

Twisted fermions on lattice

- Twisted fermions satisfy Kähler-Dirac equation
$(D - D^\dagger)\Psi = 0, \quad \Psi = (\eta, \psi_\mu, \ldots)$ - collection of p-forms
- Can be discretized *without* inducing fermion doubling – map to staggered quarks where 4 tastes yield 4 Majorana fermions of $\mathcal{N} = 4$ - no rooting ...
- Link fermions + lattice gauge invariance protects lattice theory from fermion masses ...

A_4^* lattice with five links in four dimensions

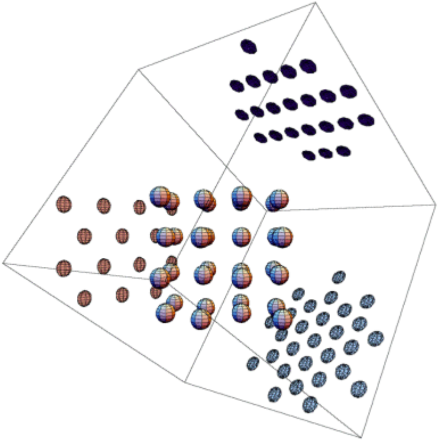
Maximize global symmetries of lattice theory if treat all five $\mathcal{U}_a$ symmetrically (S^5 symmetry)

Requirement selects lattice!

—Start with hypercubic lattice
in 5d momentum space

—**Symmetric** constraint $\sum_a \partial_a = 0$
projects to 4d momentum space

—Result is A_4 lattice
$\longrightarrow$ dual A_4^* lattice in real space



Novel features of lattice construction

- Fermions live on links not sites
- To keep $\mathcal{Q}$-susy (complex) gauge links $\mathcal{U}_a$ must also live in *algebra* like the fermions.
- Employ *flat* not Haar measure $D\mathcal{U}D\overline{\mathcal{U}}$. Still gauge invariant!
- Correct naive continuum limit *forces* use of complexified $U(N)$ theory. Allows for expansion $\mathcal{U}_a = I + \mathcal{A}_a + \ldots$

Exact lattice symmetries strongly constrain renormalization

Single marginal coupling remains to be tuned to restore all SUSYs in continuum limit

Not quite suitable for numerical calculations

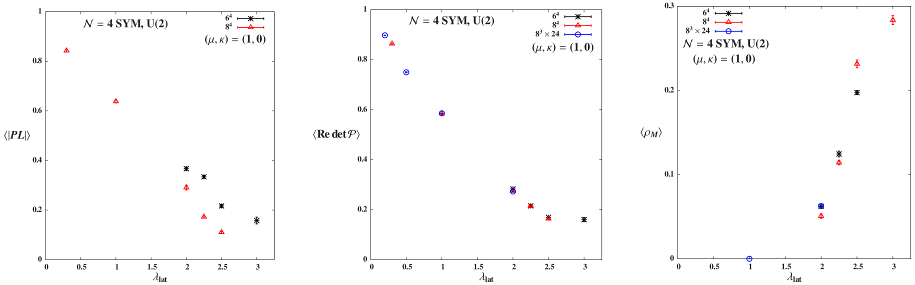
Exact 0 modes/flat directions must be regulated especially the U(1)

Need soft scalar mass term coeff μ^2 to regulate U(1) scalars but this is not enough ....

Lattice monopole instabilities

Flat directions in U(1) gauge field sector can induce transition to confined phase at strong coupling

This lattice artifact is not present in continuum $\mathcal{N} = 4$ SYM



Around $\lambda_{\text{lat}} \approx 2\ldots$

Left: Polyakov loop falls towards zero

Center: Plaquette determinant falls towards zero

Right: Density of U(1) monopole world lines becomes non-zero

Supersymmetric lifting of the U(1) flat directions

arXiv:1505.03135

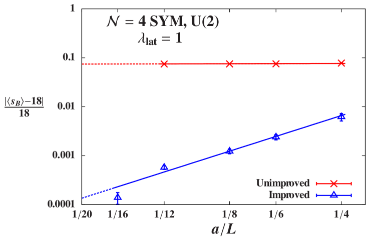
Better: modify e.o.m for auxiliary field d to add new moduli space condition $det\, P_{ab} = 1$ $GL(N,C) \rightarrow SL(N,C)$

$$S = \frac{N}{2\lambda_{\rm lat}}\mathcal{Q}\left(\chi_{ab}\mathcal{F}_{ab} + \quad \downarrow \quad -\frac{1}{2}\eta d\right) - \frac{N}{8\lambda_{\rm lat}}\epsilon_{abcde}\,\chi_{ab}\overline{\mathcal{D}}_c\,\chi_{de} + \mu^2 V$$

$$\eta\left(\overline{\mathcal{D}}_a\mathcal{U}_a + G\sum_{\mathcal{P}}\left[\det\mathcal{P} - 1\right]\mathbb{I}_N\right)$$

Scalar potential breaks $\mathcal{Q}$ softly. Ward identity restored as $1/L \rightarrow 0$

Improved action $\mathcal{O}(a)$ improved since $\mathcal{Q}$ forbids all dim-5 operators



Simulation results - preliminary

Use RHMC algorithm. Like lattice QCD. Public code available at

github.com/daschaich/susy

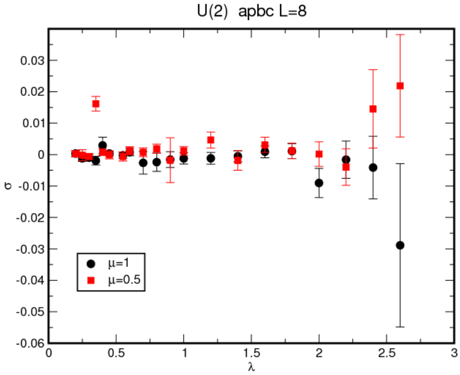
Evolved from MILC code, presented in arXiv:1410.6971

Results

- Static potential.
- Anomalous dimensions.
- Latter rely in part on a recently formulated real space RG which respects the lattice $\mathcal{Q}$-symmetry (arXiv:1408.7067)

Static potential

Fit to $V(r) = C/r + \sigma r$ using Wilson loops.

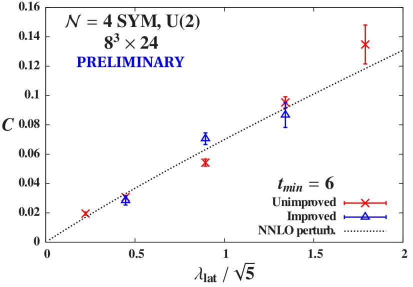


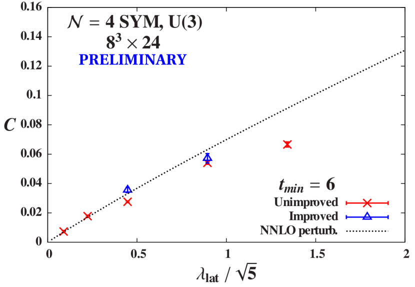
String tension $\sigma \sim 0$ all λ

Single, deconfined phase !

Coulomb fits

Coulomb coefficient





Left: Agreement with perturbation theory for $N = 2$, $\lambda \lesssim 2$

Right: Tantalizing $\sqrt{\lambda}$-like behavior for $N = 3$, $\lambda \gtrsim 1$,
possibly approaching large-N AdS/CFT prediction $C(\lambda) \propto \sqrt{\lambda}$

Konishi operator scaling dimension

$\mathcal{N} = 4$ SYM is conformal at all λ
$\longrightarrow$ power-law decay for all correlation functions

The Konishi operator is the simplest conformal primary operator

$$\mathcal{O}_K = \sum_{\mathrm{I}} \mathrm{Tr}\left[\Phi^{\mathrm{I}}\Phi^{\mathrm{I}}\right] \qquad C_K(r) \equiv \mathcal{O}_K(x+r)\mathcal{O}_K(x) \propto r^{-2\Delta_K}$$

There are many predictions for the scaling dim. $\Delta_K(\lambda) = 2 + \gamma_K(\lambda)$

- From weak-coupling perturbation theory (2-4 loops). Planar limit.
- From holography for $N \to \infty$ and $\lambda \to \infty$ but $\lambda \ll N$
- Upper bounds from the conformal bootstrap
- S duality: $\frac{4\pi N}{\lambda} \longleftrightarrow \frac{\lambda}{4\pi N}$

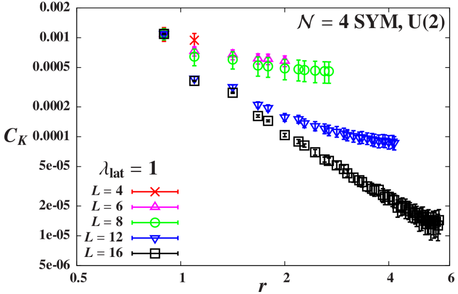
Only lattice gauge theory can access nonperturbative λ at moderate N

Konishi correlator

Extract scalar fields from polar decomposition of complexified links

$$\mathcal{U}_a \simeq U_a \left(\mathbb{I}_N + \varphi_a\right) \qquad \widehat{\mathcal{O}}_K = \sum_a \mathrm{Tr}\left[\varphi_a \varphi_a\right] \qquad \overline{\mathcal{O}}_K = \widehat{\mathcal{O}}_K - \left\langle \widehat{\mathcal{O}}_K \right\rangle$$

$$\overline{C}_K(r) = \overline{\mathcal{O}}_K(x+r)\overline{\mathcal{O}}_K(x) \propto r^{-2\Delta_K}$$



Konishi scaling dimension from Monte Carlo RG

Δ_K obtained by linearizing RG transformation at fixed point

$\mathcal{Q}$ preserving RG/blocking transformation can be used on Monte Carlo ensemble.

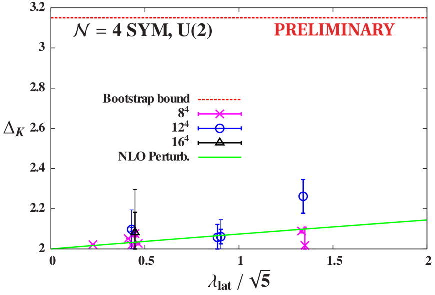
Correlators of appropriate basis set of operators yield estimate of linearized RG matrix

Eigenvalues yield scaling dimensions ..

RG blocking parameter ξ set by matching plaquettes for L vs. $L/2$

Horizontally displaced points use different auxiliary couplings μ & G

Currently running larger λ_{lat} and larger $N = 3, 4$



Uncertainties reflect RG steps, volumes, number of operators

Summary so far

- Class of supersymmetric theories that may be discretized while preserving some SUSY. Includes $\mathcal{N} = 4$ Yang-Mills in 4D.
- Lattice symmetries strongly constrain quantum effective action: moduli space survives quantum correction, only a single marginal coupling (may) need to be tuned to restore all SUSYs in continuum limit.
- Lattice simulations reveal single deconfined phase at all couplings. Measurements of anomalous dims match perturbation theory at weak coupling. Hint of large N scaling for Coulomb coeff ?

Much more remains to be done:

- Go to stronger coupling, test bootstrap bounds, signs of S duality ?
- Increase N - connect to holographic predictions
- Check for restoration of full supersymmetry in continuum limit - increased lattice volumes
- Holographic applications, generalizations to eg super QCD

Holographic applications

(with Toby Wiseman, Anosh Joseph)

Original AdSCFT correspondence:

$\mathcal{N} = 4$ YM has dual description as strings in AdS_5
SUGRA limit requires $\lambda, N \to \infty$ with $\lambda << N$

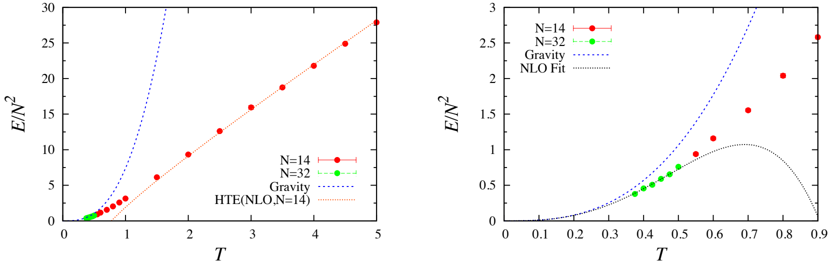
Applications: general holographic dualities

Maximally superymmetric YM in $p + 1$ dim dual to Dp-branes
At finite temperature T and in decoupling limit described by black holes type II SUGRA
Decoupling limit: $N \to \infty$ and $t = T/\lambda^{\frac{1}{3-p}} << 1$

Thermodynamics of D0 branes

YM on circle $p = 0$. Duality implies energy of black hole in IIA SUGRA equal to energy of YM quantum mechanics:

Kadoh et al arXiv:1503.08499



Yang-Mills on 2-torus: D1 branes

Two dimensionless parameters $r_x = R\sqrt{\lambda}$, $r_\tau = \beta\sqrt{\lambda} = 1/t$
SUGRA requires large N and $r_x, r_\tau \to \infty$

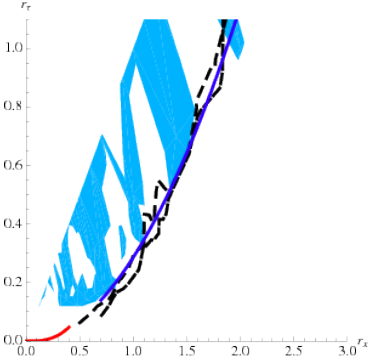
Depending on r_x, r_τ two types of black hole solution:

- Black string: wraps the spatial direction uniformly
- Black hole: localized on spatial circle

SUGRA predicts black string unstable $r_\tau < c r_x^2$

Gregory LaFlamme transition
Dual gauge theory: thermal deconfining phase transition
Order parameter: spatial Polyakov loop

Black string black hole phase transition



- Good agreement with SUGRA - blue curve $r_\tau = cr_x^2$ predict $c \sim 3.5$
- Good agreement with high T reduction bosonic QM : red curve $r_\tau = ar_x^3$

Further generalizations ...

(with Aarti Veernala)

Lattice quivers

Can construct (twisted) lattice quiver theories in D dims using same tricks - again preserving (some) SUSY.

General idea:

- Start from a theory in $(D+1)$ dims.
- Gauge each slice in “extra" dimension independently.
- Restore gauge invariance by replacing gauge fields in extra dimension by scalar fields transforming as bi-fundamentals under gauge groups on adjacent slices.
- Remarkably the prescription for replacing derivatives by difference operators generalizes simply to this case
- Scalar SUSYs in the $(D+1)$ theory survive.

Example: 2d quiver theory with (2, 2) SUSY

Start from $\mathcal{Q} = 8$ Yang-Mills in 3d.

Twisted lattice super YM action:

$$S = \mathcal{Q} \sum_x \mathrm{Tr}\left(\chi_{ab}\mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a + \frac{1}{2}\eta d\right) - \sum_x \mathrm{Tr}\, \theta_{abc} \overline{\mathcal{D}}_{[a}\chi_{bc]}$$

with $(a, b = 1 \ldots 3)$ and cubic lattice with face/body diagonals

$$\mathcal{Q}\,\mathcal{U}_a = \psi_a \qquad \mathcal{Q}\,\psi_a = 0$$

$$\mathcal{Q}\,\chi_{ab} = -\overline{\mathcal{F}}_{ab} \qquad \mathcal{Q}\,\overline{\mathcal{U}}_a = 0$$

$$\mathcal{Q}\,\eta = d \qquad \mathcal{Q}\,d = 0$$

$$\mathcal{Q}\,\theta_{abc} = 0$$

Some more details ..

Take just 2 slices in the 3 direction with $U(N_c)$ and $U(N_f)$ symmetries and relabel fields

N_c-lattice x	bifundamental fields $(x,\overline{x})$, $(\overline{x},x)$	N_f-lattice $\overline{x}$
$\mathcal{U}_\mu(x)$	$\mathcal{U}_3 \to \phi(x,\overline{x})$	$\hat{\mathcal{U}}_\mu(\overline{x})$
$\eta(x)$	$\psi_3 \to \lambda(x,\overline{x})$	$\hat{\eta}(\overline{x})$
$\psi_\mu(x)$	$\chi_{3\mu} \to \lambda_\mu(\overline{x}+\mu,x)$	$\hat{\psi}_\mu(\overline{x})$
$\chi_{\mu\nu}(x)$	$\theta_{3\mu\nu} \to \lambda_{\mu\nu}(x,\overline{x}+\mu+\nu)$	$\hat{\chi}_{\mu\nu}(\overline{x})$

Prescription for lattice derivatives generalizes:

$$\mathcal{D}_a\psi_b(x) \overset{3d}{=} \chi_{ab}(x)\left(\mathcal{U}_a(x)\psi_b(x+a)-\psi_b(x)\mathcal{U}_a(x+b)\right)$$

$$\xrightarrow{b=3,a=\mu} \lambda_\mu(x)\left(\mathcal{U}_\mu(x)\lambda(x+\mu)-\lambda(x)\hat{U}_\mu(\overline{x})\right)$$

Super QCD and F.I terms

Take limit $g_{Nf} \to 0$

- $\hat{U} \to I_{N_f \times N_f}$
- Consistent to set $\hat{\eta}, \hat{\chi}_{\mu\nu}, \hat{\psi}_\mu = 0$
- Fields $\lambda, \lambda_\mu, \phi, \ldots$ transform in fund of $U(N_c)$ gauge symmetry
- Carry additional $U(N_f)$ indices - global flavor symmetry

Can add a new $\mathcal{Q}$ exact term

$$\Delta S = r\mathcal{Q} \sum_x \mathrm{Tr}\, \eta(x) I_{N_c \times N_c}$$

Yields new e.o.m for auxiliary d-field

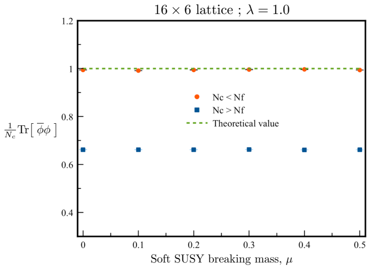
$$d = \overline{\mathcal{D}}_\mu \mathcal{U}_\mu + \phi\overline{\phi} - rI_{N_c \times N_c}$$

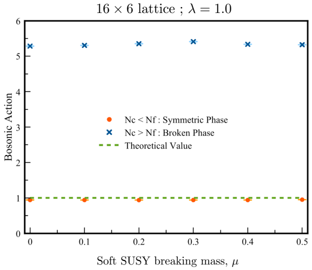
after integrating d yields new term in action: $\frac{1}{2}\left(\phi\overline{\phi} - rI\right)^2$

Dynamical $\mathcal{Q}$ breaking

- Spontaneous SUSY breaking indicated by $< d > \neq 0$. Because of F.I term depends on N_c, N_f.
- Consider $\sum_x \mathrm{Tr}\, d(x) = \sum_x \mathrm{Tr}\, \left(\phi(x)\overline{\phi}(x) - rI_{N_c}\right)$
- Setting $r = 1$ this depends on rank of $N_c \times N_c$ matrix $\sum_{f=1}^{N_f} \phi^f \overline{\phi}^f$.

$$N_f \geq N_c \quad \text{supersymmetric vacuum eg } N_f = 3, N_c = 2$$
$$N_f < N_c \quad \text{supersymmetry broken eg } N_f = 2, N_c = 3$$





Goldstino

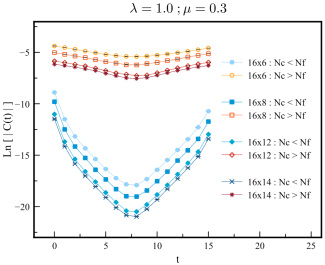
If susy breaks expect a massless fermion

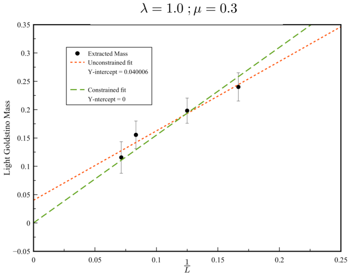
Measure

$$C(t) = \sum_{x,y} \left\langle O'(y,t) O(x,0) \right\rangle$$

where

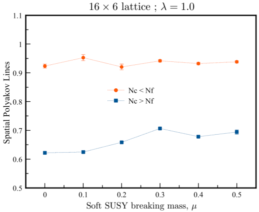
$$\begin{aligned} O(x,0) &= \psi_\mu(x,0)\mathcal{U}_\mu(x,0)\left[\phi(x,0)\overline{\phi}(x,0) - rI_{N_c}\right] \\ O'(y,t) &= \eta(y,t)\left[\phi(y,t)\overline{\phi}(y,t) - rI_{N_c}\right] \end{aligned}$$





Conclusions

- First numerical simulations of super QCD
- Can include lattice $\mathcal{Q}$ invariant F.I term.
- See clear signals for spontaneous susy breaking depending on N_c/N_f in accord with expectations.



- Also $<\phi\overline{\phi}>\neq 0$ implies Higgsing of gauge symmetries - see signals in Polyakov lines - order parameter for confinement
- Generalizations to models with antifundamentals and d=3 possible and underway ...

Overall summary

Great progress in last decade in construction and study of supersymmetric theories on lattice
Most of the work so far focused on formulations
Now emphasis turning to applications
Large scale simulations are possible using same tools/techniques as lattice QCD

Lattice offers new tool to investigate strong coupling behavior away from planar limit.
Prospects exciting !

Thank you!

Backup: Failure of Leibnitz rule in discrete space-time

Given that $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$ is problematic,

why not try $\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} = 2i\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu$ for a discrete translation?

Here $\nabla_\mu\phi(x) = \frac{1}{a}\left[\phi(x + a\widehat{\mu}) - \phi(x)\right] = \partial_\mu\phi(x) + \frac{a}{2}\partial_\mu^2\phi(x) + \mathcal{O}(a^2)$

Essential difference between ∂_μ and ∇_μ on the lattice, $a > 0$

$$\begin{aligned}\nabla_\mu\left[\phi(x)\chi(x)\right] &= a^{-1}\left[\phi(x + a\widehat{\mu})\chi(x + a\widehat{\mu}) - \phi(x)\chi(x)\right] \\ &= \left[\nabla_\mu\phi(x)\right]\chi(x) + \phi(x)\nabla_\mu\chi(x) + a\left[\nabla_\mu\phi(x)\right]\nabla_\mu\chi(x)\end{aligned}$$

We only recover the Leibnitz rule $\partial_\mu(fg) = (\partial_\mu f)g + f\partial_\mu g$ when $a \to 0$
$\Longrightarrow$ “Discrete supersymmetry” breaks down on the lattice
(Dondi & Nicolai, “Lattice Supersymmetry”, 1977)

Backup: Twisting $\longleftrightarrow$ Kähler–Dirac fermions

The Kähler–Dirac representation is related to the spinor $Q^{\mathrm{I}}_{\alpha}, \overline{Q}^{\mathrm{I}}_{\dot{\alpha}}$ by

$$\begin{pmatrix} Q^1_\alpha & Q^2_\alpha & Q^3_\alpha & Q^4_\alpha \\ \overline{Q}^1_{\dot\alpha} & \overline{Q}^2_{\dot\alpha} & \overline{Q}^3_{\dot\alpha} & \overline{Q}^4_{\dot\alpha} \end{pmatrix} = \mathcal{Q} + \mathcal{Q}_\mu\gamma_\mu + \mathcal{Q}_{\mu\nu}\gamma_\mu\gamma_\nu + \overline{\mathcal{Q}}_\mu\gamma_\mu\gamma_5 + \overline{\mathcal{Q}}\gamma_5$$

$$\longrightarrow \mathcal{Q} + \gamma_a\mathcal{Q}_a + \gamma_a\gamma_b\mathcal{Q}_{ab}$$

with $a, b = 1, \cdots, 5$

> The 4×4 matrix involves R symmetry transformations along each row and (euclidean) Lorentz transformations along each column

$\Longrightarrow$ Kähler–Dirac components transform under "twisted rotation group"

$$\mathrm{SO}(4)_{tw} \equiv \mathrm{diag}\left[\mathrm{SO}(4)_{\mathrm{euc}} \otimes \underset{\substack{\big\uparrow \\ \text{only } \mathrm{SO}(4)_R \subset \mathrm{SO}(6)_R}}{\mathrm{SO}(4)_R}\right]$$

Backup: Revisiting the sign problem

Pfaffian can be complex for lattice $\mathcal{N} = 4$ SYM, $\quad \text{pf}\,\mathcal{D} = |\text{pf}\,\mathcal{D}|e^{i\alpha}$

Previously found $1 - \langle\cos(\alpha)\rangle \ll 1$, independent of lattice volume

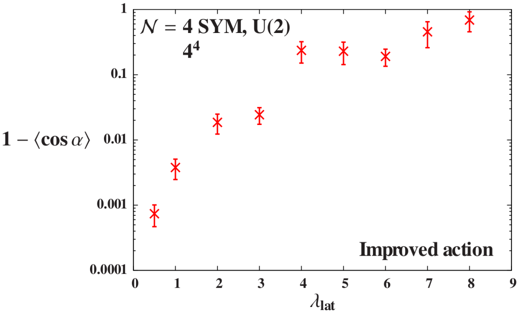
Now extending with improved action, which allows access to larger λ

Finding much larger phase fluctuations at stronger couplings

Parallel $\mathcal{O}(n^3)$ algorithm

Typical 4^4 measurement requires $\sim$60 hours, $\sim$4GB memory

Filling in more volumes & N



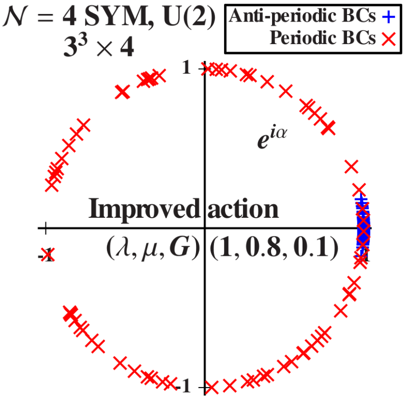
Backup: Two puzzles posed by the sign problem

- With periodic temporal boundary conditions for the fermions we have an obvious sign problem, $\langle e^{i\alpha} \rangle$ consistent with zero
- With anti-periodic BCs and all else the same $\langle e^{i\alpha} \rangle \approx 1$, phase reweighting not even necessary

Why such sensitivity to the BCs?

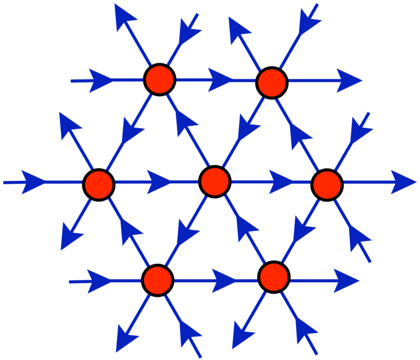
Also, other observables are nearly identical for these two ensembles

Why doesn't the sign problem have observable effects?



Backup: Hypercubic representation of A_4^* lattice

In the code it is very convenient to represent the A_4^* lattice as a hypercube with a backwards diagonal



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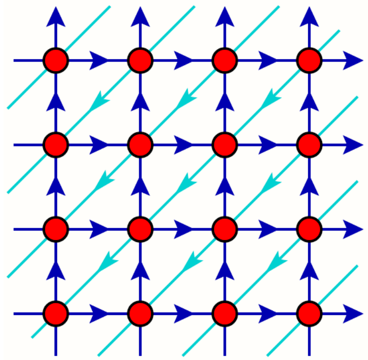


Backup: More on flat directions

1. Complex gauge field $\Longrightarrow$ U(N) = SU(N) $\otimes$ U(1) gauge invariance
   U(1) sector decouples only in continuum limit

2. $\mathcal{Q}\,\mathcal{U}_a = \psi_a \Longrightarrow$ gauge links must be elements of algebra
   Resulting flat directions required by supersymmetric construction
   but must be lifted to ensure $\mathcal{U}_a = \mathbb{I}_N + \mathcal{A}_a$ in continuum limit

We need to add two deformations to regulate flat directions

SU(N) scalar potential $\propto \mu^2 \sum_a \left(\text{Tr}\left[\mathcal{U}_a\overline{\mathcal{U}}_a\right] - N\right)^2$

U(1) plaquette determinant $\sim G \sum_{a \neq b} \left(\det \mathcal{P}_{ab} - 1\right)$

Scalar potential **softly** breaks $\mathcal{Q}$ supersymmetry

susy-violating operators vanish as $\mu^2 \to 0$

Plaquette determinant can be made $\mathcal{Q}$-invariant $\longrightarrow$ improved action

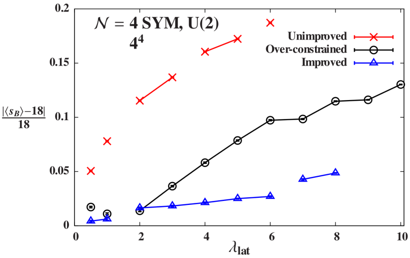
Backup: More on supersymmetric constraints

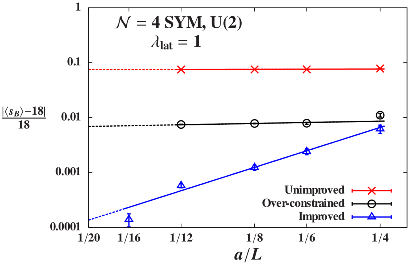
Improved action from arXiv:1505.03135
imposes $\mathcal{Q}$-invariant plaquette determinant constraint

Basic idea: Modify the equations of motion $\longrightarrow$ moduli space

$$d(n) = \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) \longrightarrow \overline{\mathcal{D}}_a^{(-)}\mathcal{U}_a(n) + G\sum_{a\neq b}\left[\det \mathcal{P}_{ab}(n) - 1\right]$$

Produces much smaller violations of $\mathcal{Q}$ Ward identity $\langle s_B\rangle = 9N^2/2$



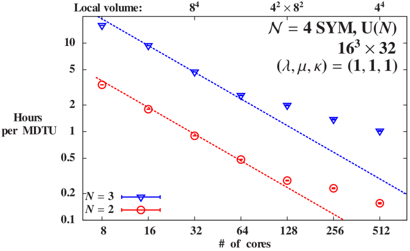


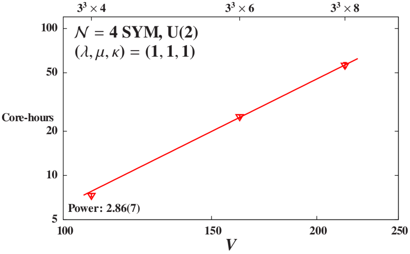
Backup: Code performance—weak and strong scaling

Results from arXiv:1410.6971 using the unimproved action

Left: Strong scaling for U(2) and U(3) $16^3 \times 32$ RHMC

Right: Weak scaling for $\mathcal{O}(n^3)$ pfaffian calculation (fixed local volume)
$n \equiv 16 N^2 L^3 N_T$ is number of fermion degrees of freedom





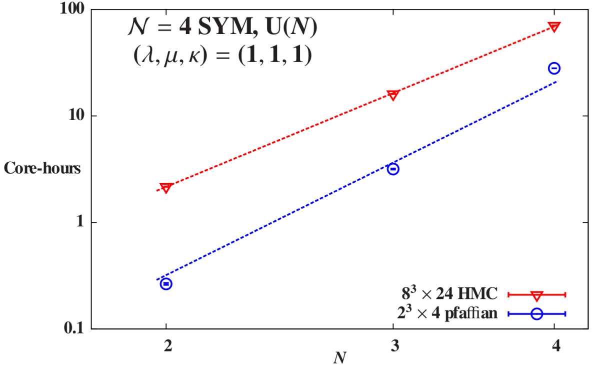
Both plots on log–log axes with power-law fits

Backup: Numerical costs for 2, 3 and 4 colors

Red: Find RHMC cost scaling $\sim N^5$ (recall adjoint fermion d.o.f. $\propto N^2$)

Blue: Pfaffian cost scaling consistent with expected N^6

Additional factor of $\sim 2\times$ from improved action, but same scaling

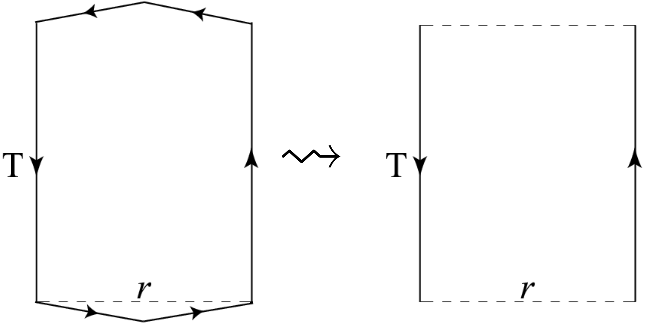


Backup: $\mathcal{N} = 4$ static potential from Wilson loops

Extract static potential $V(r)$ from $r \times T$ Wilson loops

$$W(r, T) \propto e^{-V(r)\,T} \qquad\qquad V(r) = A - C/r + \sigma r$$

Coulomb gauge trick from lattice QCD reduces A_4^* lattice complications



Backup: Real space RG for susy lattices

Exact lattice symmetries ($\mathcal{Q}$, S^5, ghost number, gauge invariance) + power counting lead to remarkable result: only a single marginal coupling needs to be tuned for lattice theory to flow to continuum $\mathcal{N} = 4$ theory as $L \to \infty,\ g = \text{fixed}$. (arXiv:1408.7067)

However

This analysis implicitly assumes existence of RG that preserves $\mathcal{Q}$

One simple blocking exists:

$$\mathcal{U}'_a(x') \overset{a'=2a}{=} \xi\, \mathcal{U}_a(x)\mathcal{U}_a(x+a)$$

$$\psi'_a = \xi \left(\psi_a(x)\mathcal{U}_a(x+a) + \mathcal{U}_a(x)\psi_a(x+a)\right)$$

.....

ξ is free parameter obtained by matching vevs of observables computed on initial and blocked lattices.

RG also yields a tool for extracting beta functions and anomalous dimensions from Monte Carlo data

Backup: Scaling dimensions from Monte Carlo RG

Write system as (infinite) sum of operators $\mathcal{O}_i$ with couplings c_i

Couplings c_i flow under RG blocking transformation R_b

n-times-blocked system is $H^{(n)} = R_b H^{(n-1)} = \sum_i c_i^{(n)} \mathcal{O}_i^{(n)}$

Consider linear expansion around fixed point $H^\star$ with couplings $c_i^\star$

$$c_i^{(n)} - c_i^\star = \sum_j \left.\frac{\partial c_i^{(n)}}{\partial c_j^{(n-1)}}\right|_{H^\star} \left(c_j^{(n-1)} - c_j^\star\right) \equiv \sum_j T_{ij}^\star \left(c_j^{(n-1)} - c_j^\star\right)$$

$T_{ij}^\star$ is the stability matrix

Obtained from measured correlators of $\mathcal{O}_i$

Eigenvalues of $T_{ij}^\star \longrightarrow$ scaling dimensions of corresponding operators

Backup: The sign problem

In lattice gauge theory we compute operator expectation values

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B[\mathcal{U},\overline{\mathcal{U}}]}\ \text{pf}\, \mathcal{D}[\mathcal{U},\overline{\mathcal{U}}]$$

pf $\mathcal{D} = |\text{pf}\, \mathcal{D}| e^{i\alpha}$ can be complex for lattice $\mathcal{N} = 4$ SYM
$\longrightarrow$ Complicates interpretation of $\left[e^{-S_B}\ \text{pf}\, \mathcal{D}\right]$ as Boltzmann weight

Instead absorb $e^{i\alpha}$ into phase-quenched (pq) observables $\mathcal{O} e^{i\alpha}$
and reweight using $Z = \int e^{i\alpha}\, e^{-S_B}\, |\text{pf}\, \mathcal{D}| = \left\langle e^{i\alpha} \right\rangle_{pq}$

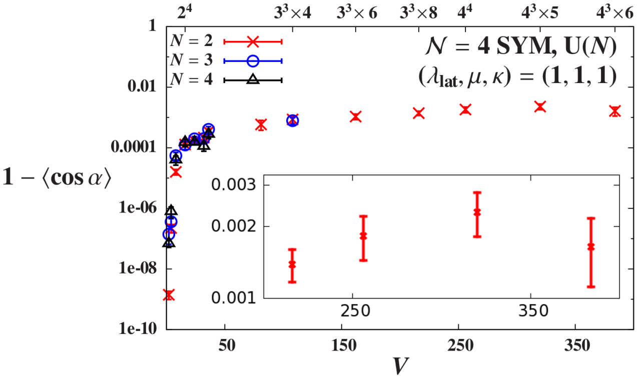
$$\langle \mathcal{O} \rangle_{pq} = \frac{1}{\mathcal{Z}_{pq}} \int [d\mathcal{U}][d\overline{\mathcal{U}}]\, \mathcal{O}\, e^{-S_B}\, |\text{pf}\, \mathcal{D}| \qquad\qquad \langle \mathcal{O} \rangle = \frac{\left\langle \mathcal{O} e^{i\alpha} \right\rangle_{pq}}{\left\langle e^{i\alpha} \right\rangle_{pq}}$$

Sign problem: This breaks down if $\left\langle e^{i\alpha} \right\rangle_{pq}$ is consistent with zero

Backup: Pfaffian phase volume dependence

No indication of a sign problem at $\lambda_{\text{lat}} = 1$ with anti-periodic BCs

- Results from arXiv:1411.0166 using the unimproved action
- Fluctuations in pfaffian phase don't grow with the lattice volume
- Insensitive to number of colors $N = 2, 3, 4$



Backup: Lattice quiver theory

Construction: Sugino, Matsuura via orbifolding

Simple derivation:
– Take a lattice with just 2 timeslices in z-direction and free bc.
– Choose gauge groups $U(N_c)$ and $U(N_f)$ on the 2 timeslices.
– To retain gauge invariance fields on links between 2 slices *must* transform as bifundamental fields under $U(N_c) \times U(N_f)$
– Relabel fields as follows

N_c-lattice x	bifundamental fields $(x, \overline{x})$, $(\overline{x}, x)$	N_f-lattice $\overline{x}$
$\mathcal{U}_\mu(x)$	$\mathcal{U}_3 \to \phi(x, \overline{x})$	$\hat{\mathcal{U}}_\mu(\overline{x})$
$\eta(x)$	$\psi_3 \to \lambda(x, \overline{x})$	$\hat{\eta}(\overline{x})$
$\psi_\mu(x)$	$\chi_{3\mu} \to \lambda_\mu(\overline{x} + \mu, x)$	$\hat{\psi}_\mu(\overline{x})$
$\chi_{\mu\nu}(x)$	$\theta_{3\mu\nu} \to \lambda_{\mu\nu}(x, \overline{x} + \mu + \nu)$	$\hat{\chi}_{\mu\nu}(\overline{x})$