Signs of Analyticity in Single-Field Inflation

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arXiv:1502.07304
with Baumann, Lee and Porto

Outline

The Nature of Inflation

Analyticity and EFT

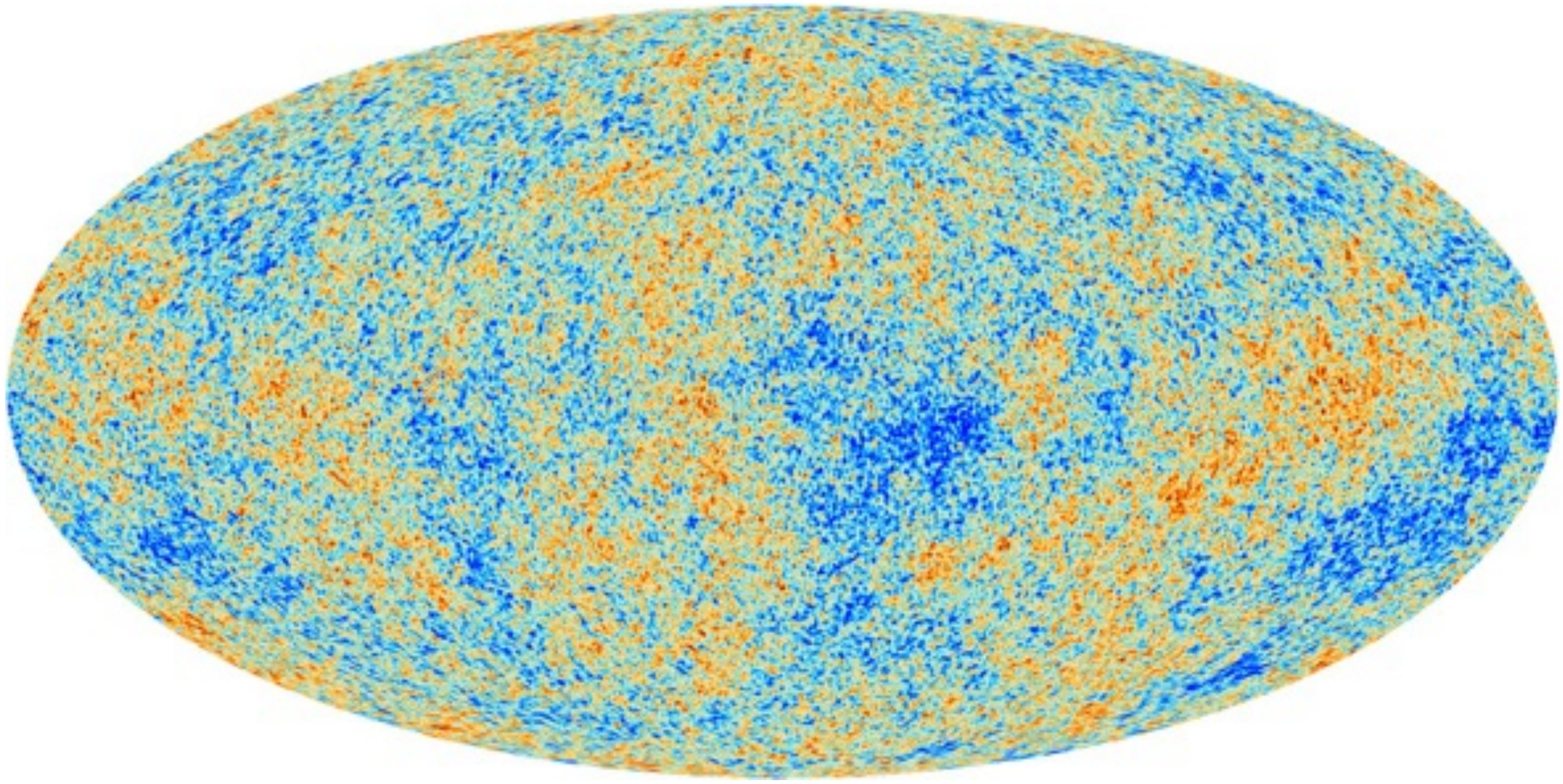
EFT of Inflation

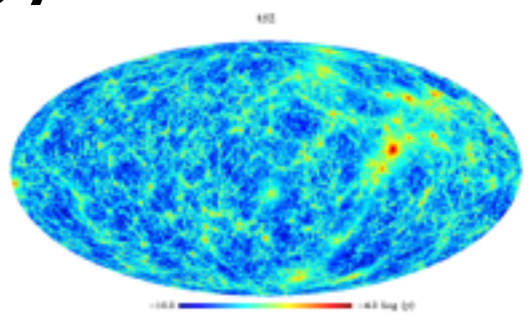
Sum Rules and Observations

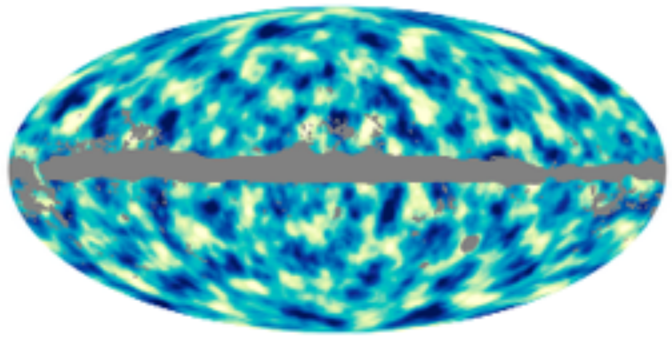
The Nature of Inflation

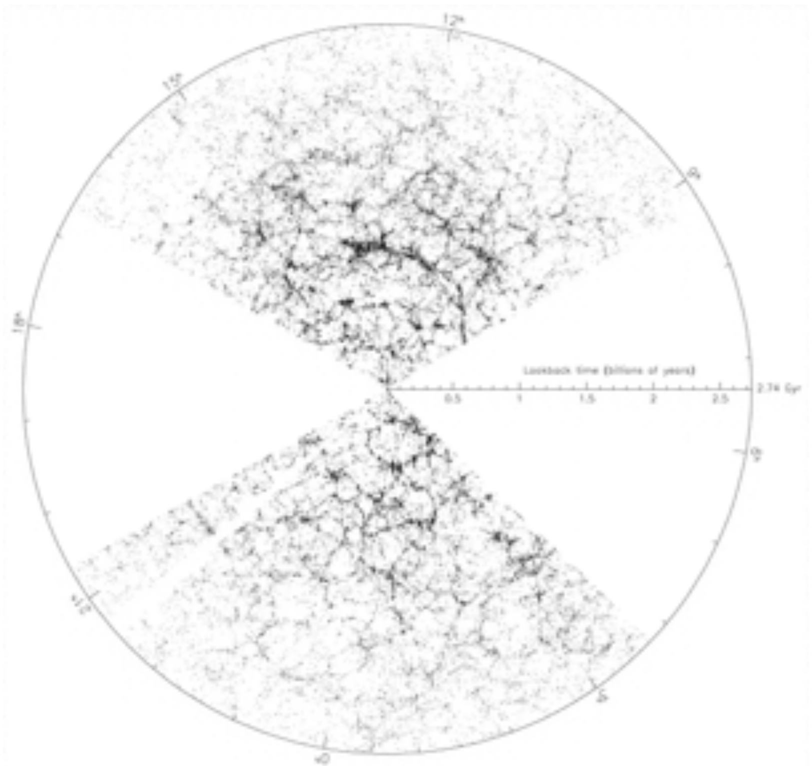
What is Inflation?

We are in the era of precision cosmology



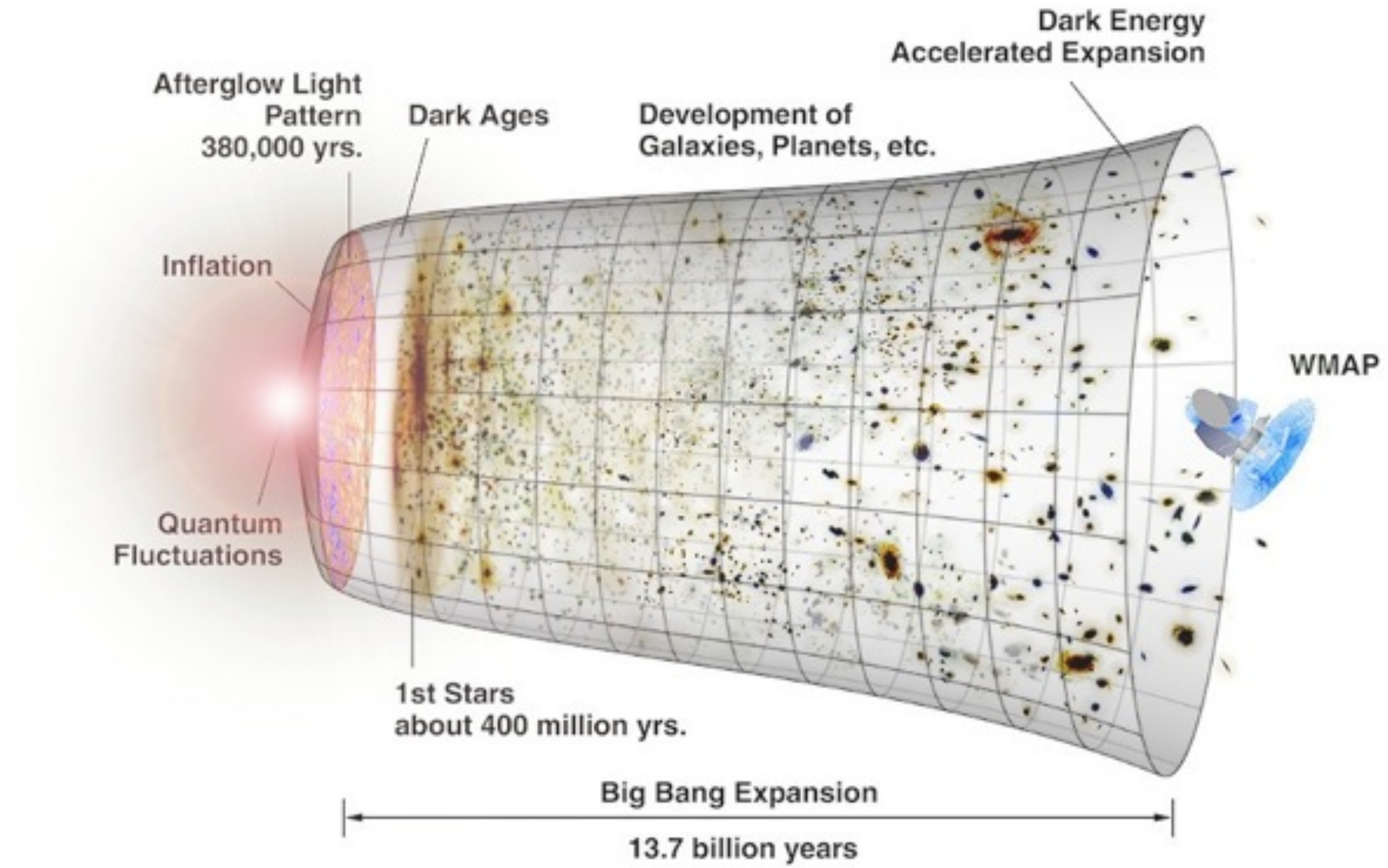






What is Inflation?

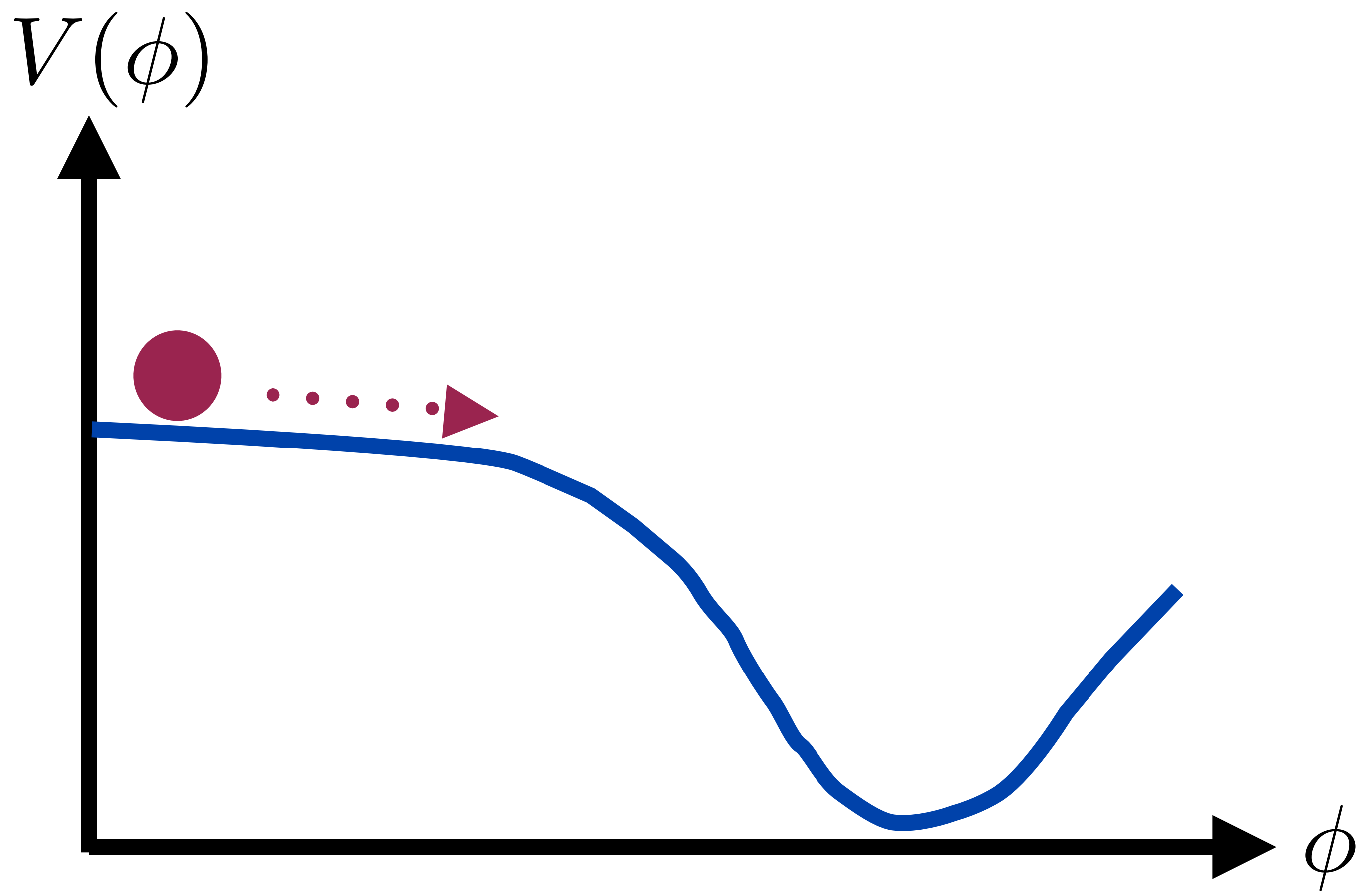
Data appears to have a single causal origin



Only compelling model is inflation

What is Inflation?

The conventional picture of inflation is slow-roll:



$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$$

All cosmological data is compatible with this picture

What is Inflation?

Inflation is a more general framework:

e.g. $\mathcal{L} = P(X, \phi) - V(\phi)$ where $X \equiv \partial_\mu \phi \partial^\mu \phi$

Armendariz-Picon et al.

This is very closely related to a superfluid with:

$X \to \mu$ chemical potential

$\delta\phi \to \pi$ superfluid phonon

$P(\mu) \to$ equation of state

Can inflation have a more exotic origin?

What is Inflation?

A definition:

1. A period of quasi-dS expansion

$$\frac{\dot{H}}{H^2} \ll 1$$

2. A physical clock

Needed to define the end of inflation Cheung et al.

In slow roll, the clock is defined by $\phi(t)$

What is Inflation?

Raises the question: what was the clock?

We have lots of ways to make clocks

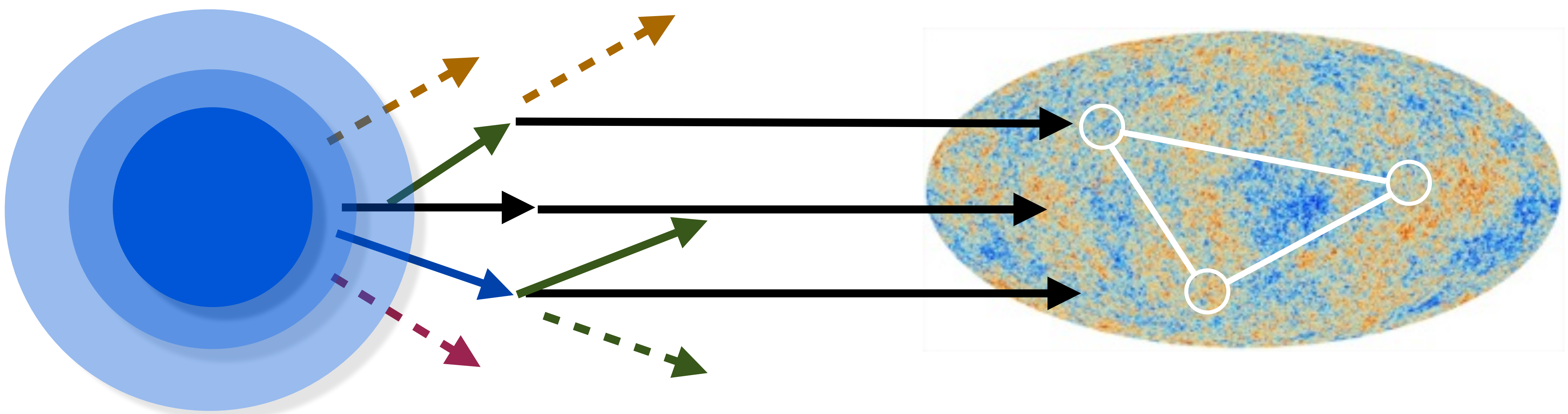
Slow-roll inflation is easiest to construct, because it is weakly coupled (like Higgs versus technicolor)

How can we tell from observations?

What is Inflation?

Current approach is to constrain EFT of clock

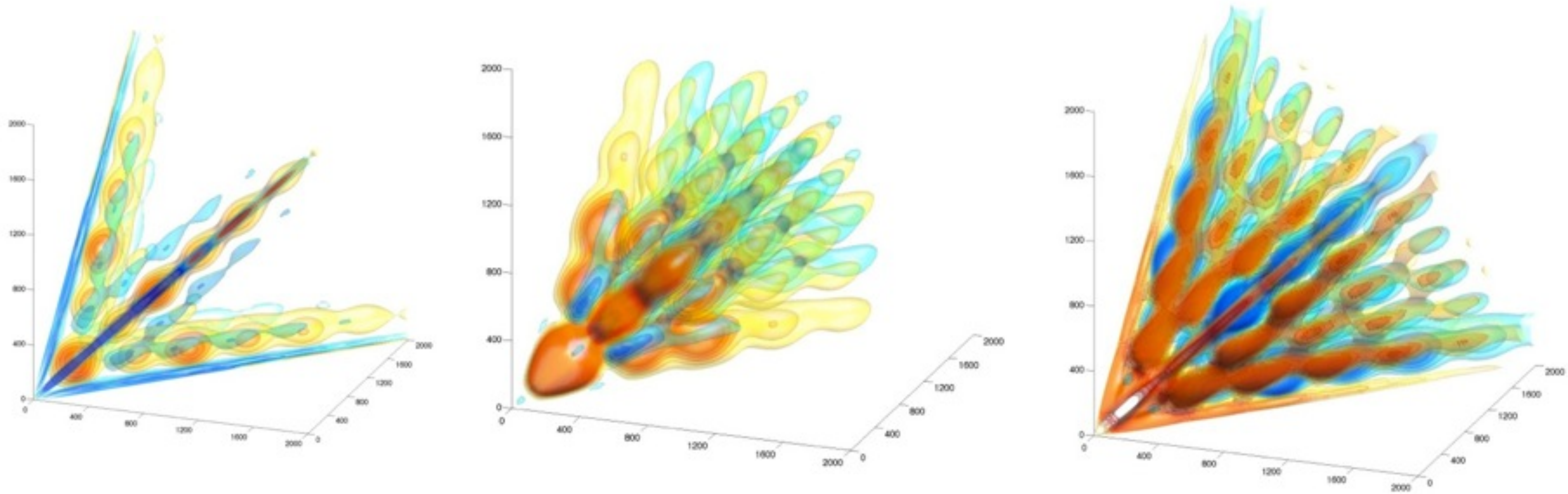
$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{\Lambda_1^2}\mathcal{O}_1 + \frac{1}{\Lambda_2^2}\mathcal{O}_2 + \ldots$$



Interactions produce non-gaussian fluctuations

What is Inflation?

Planck constrains many possible bispectra



Consistent with gaussanity at 10^{-3} level

Roughly implies that $\Lambda_i \gtrsim (5-10) \times H$

What is the Challenge?

EFT tests are great when you have lots of models

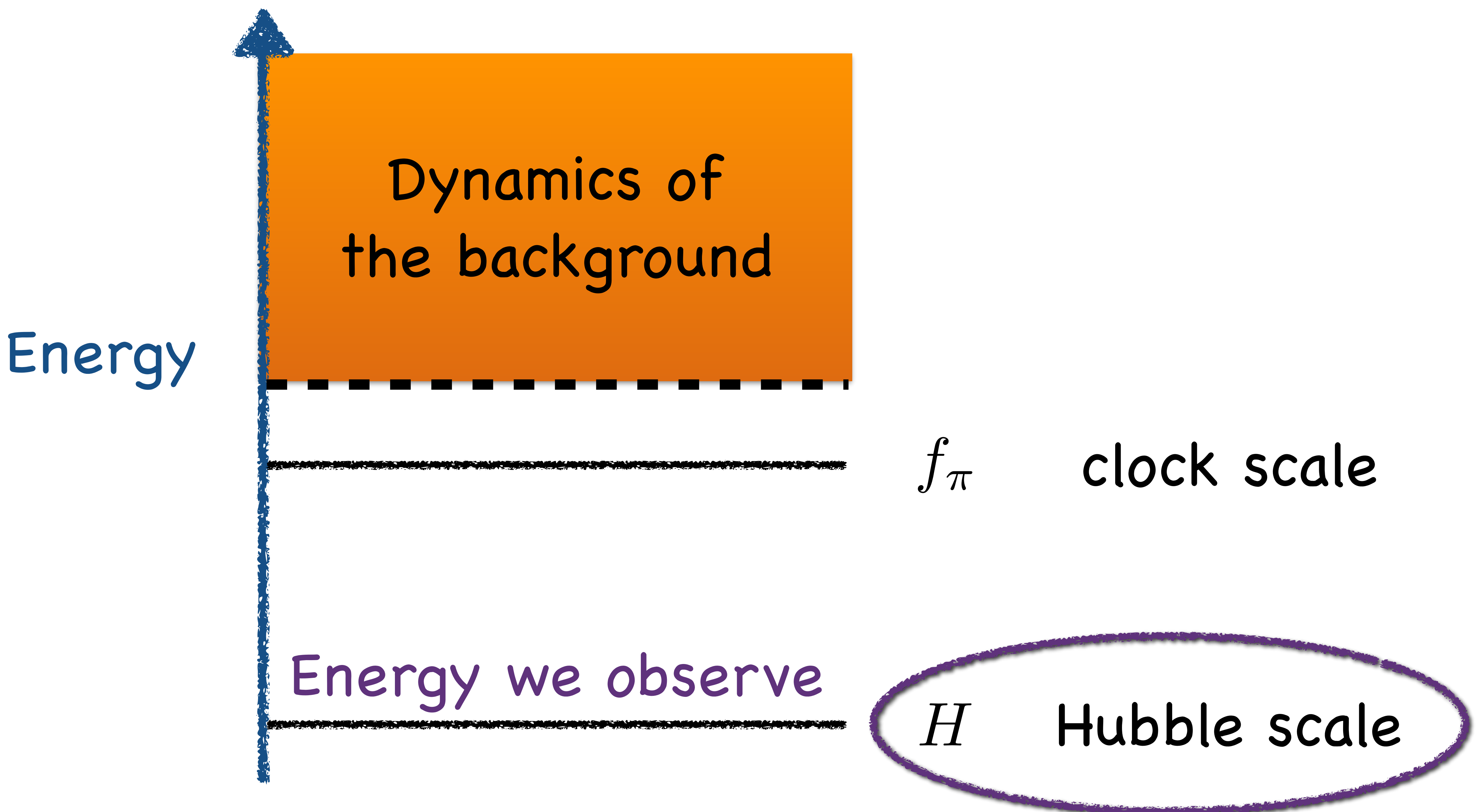
Inflation doesn't require a scalar field but there are no working examples of alternatives

We have only vague guesses for what a strongly coupled model might predict

Is there more we can learn from measurements?

What is the Challenge?

Scale of observations is separated from dynamics



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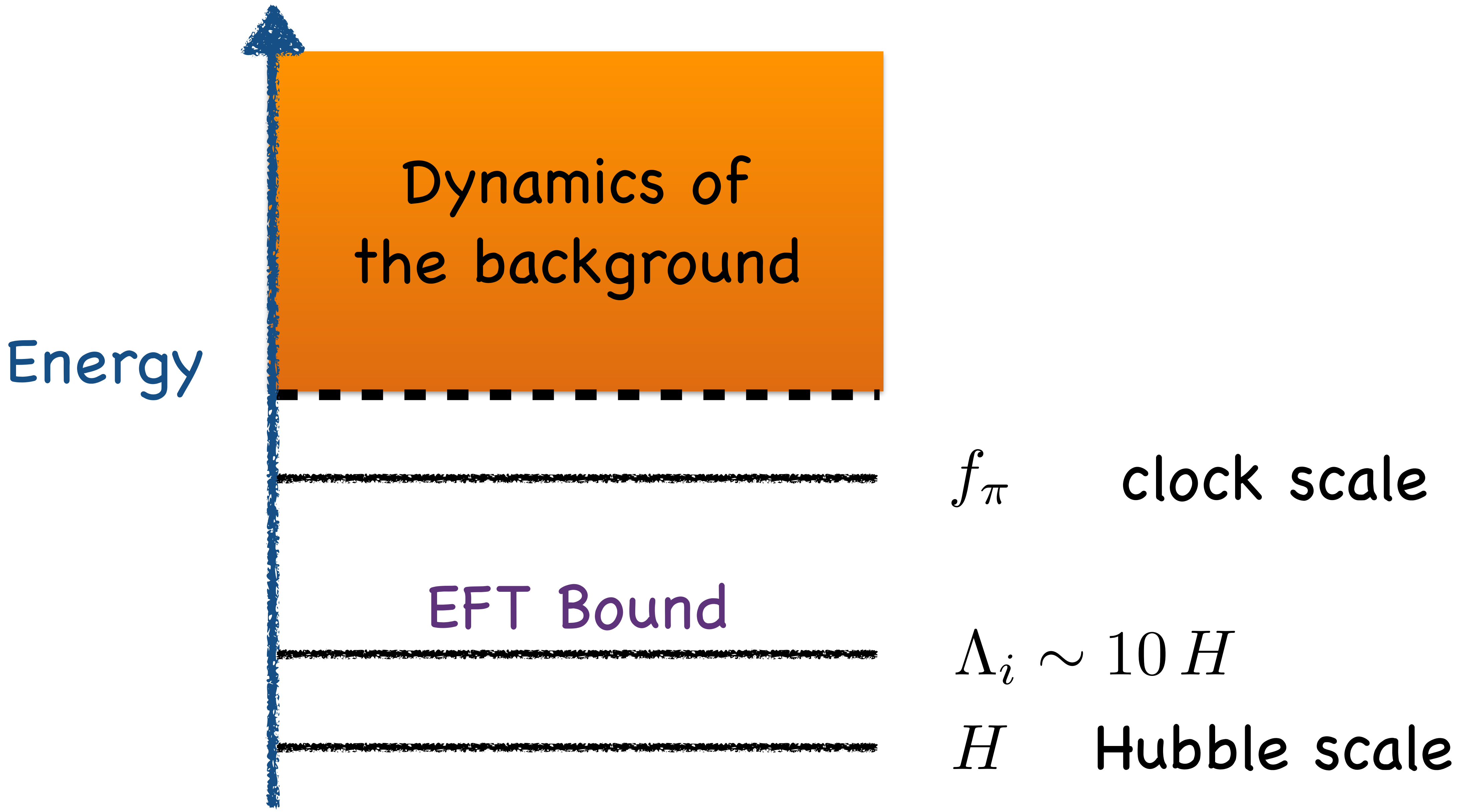

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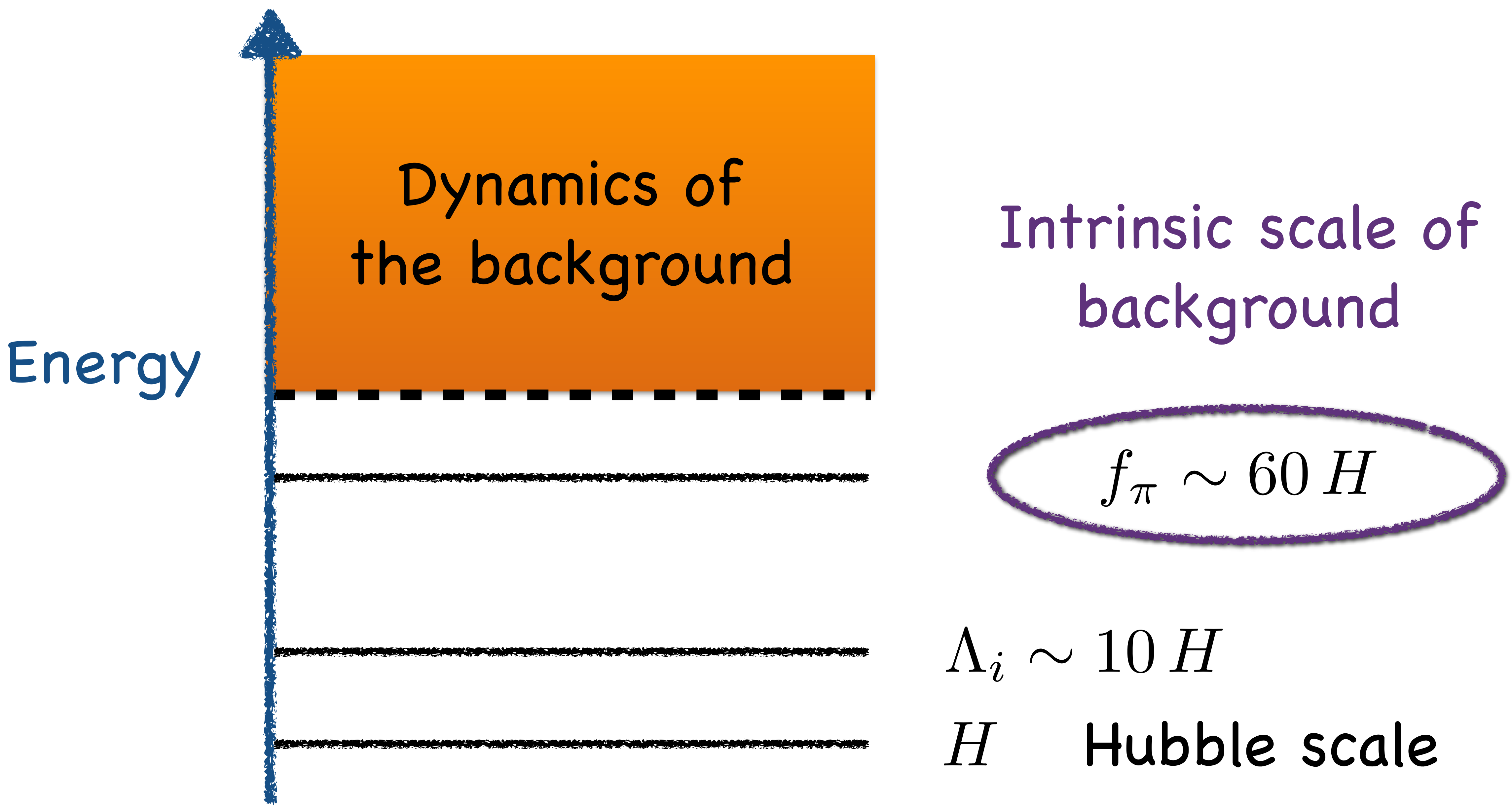

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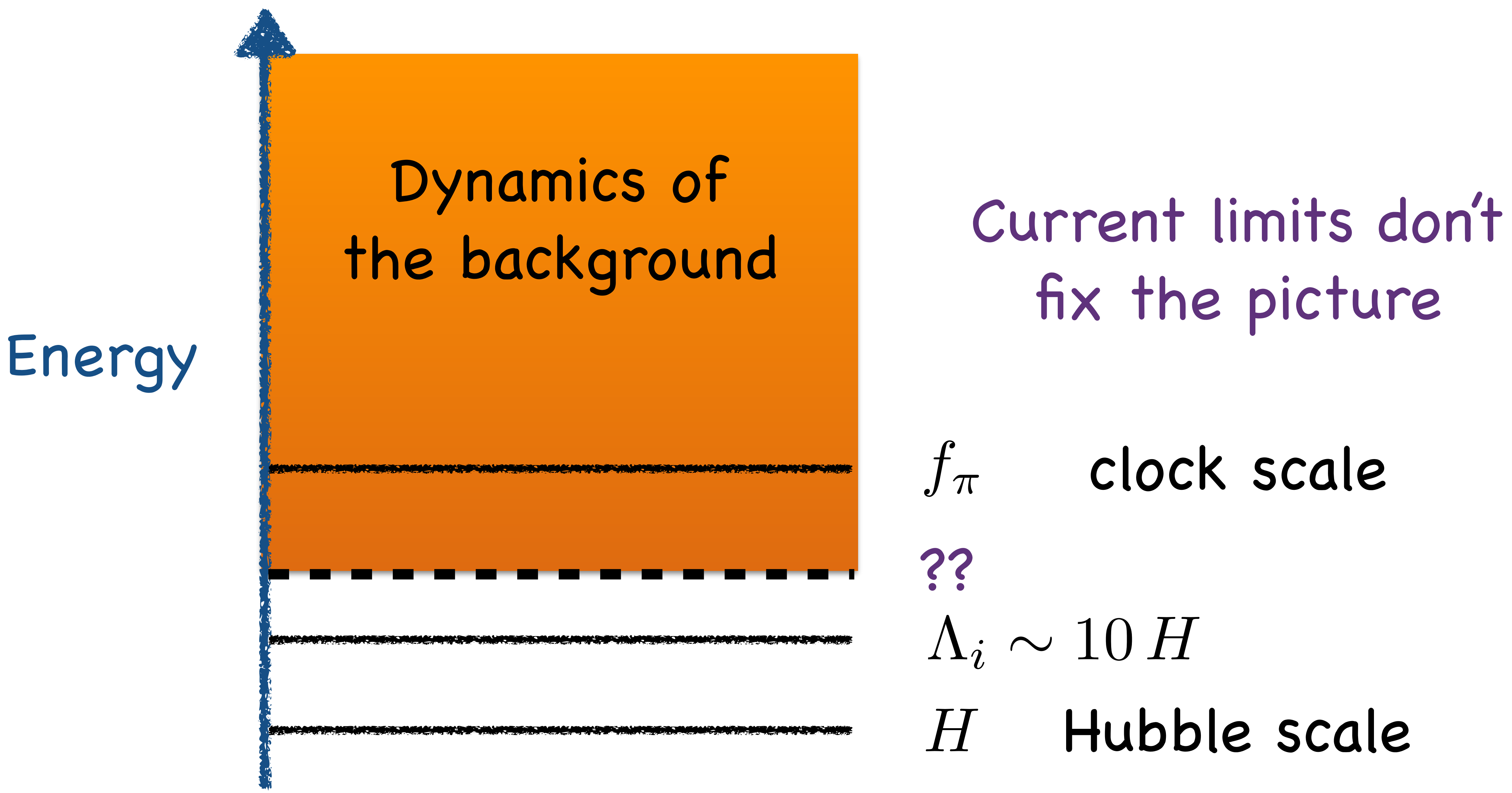

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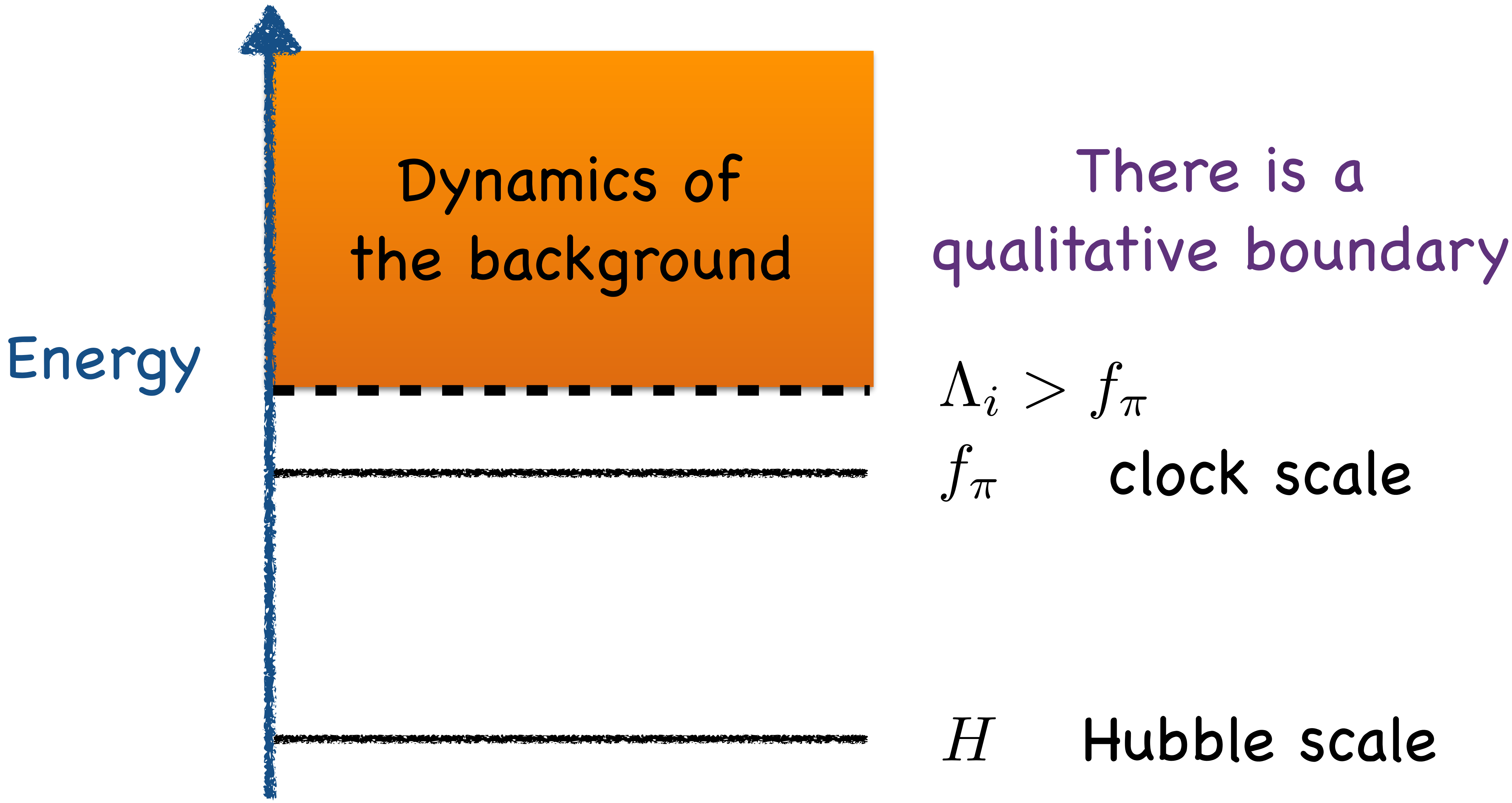

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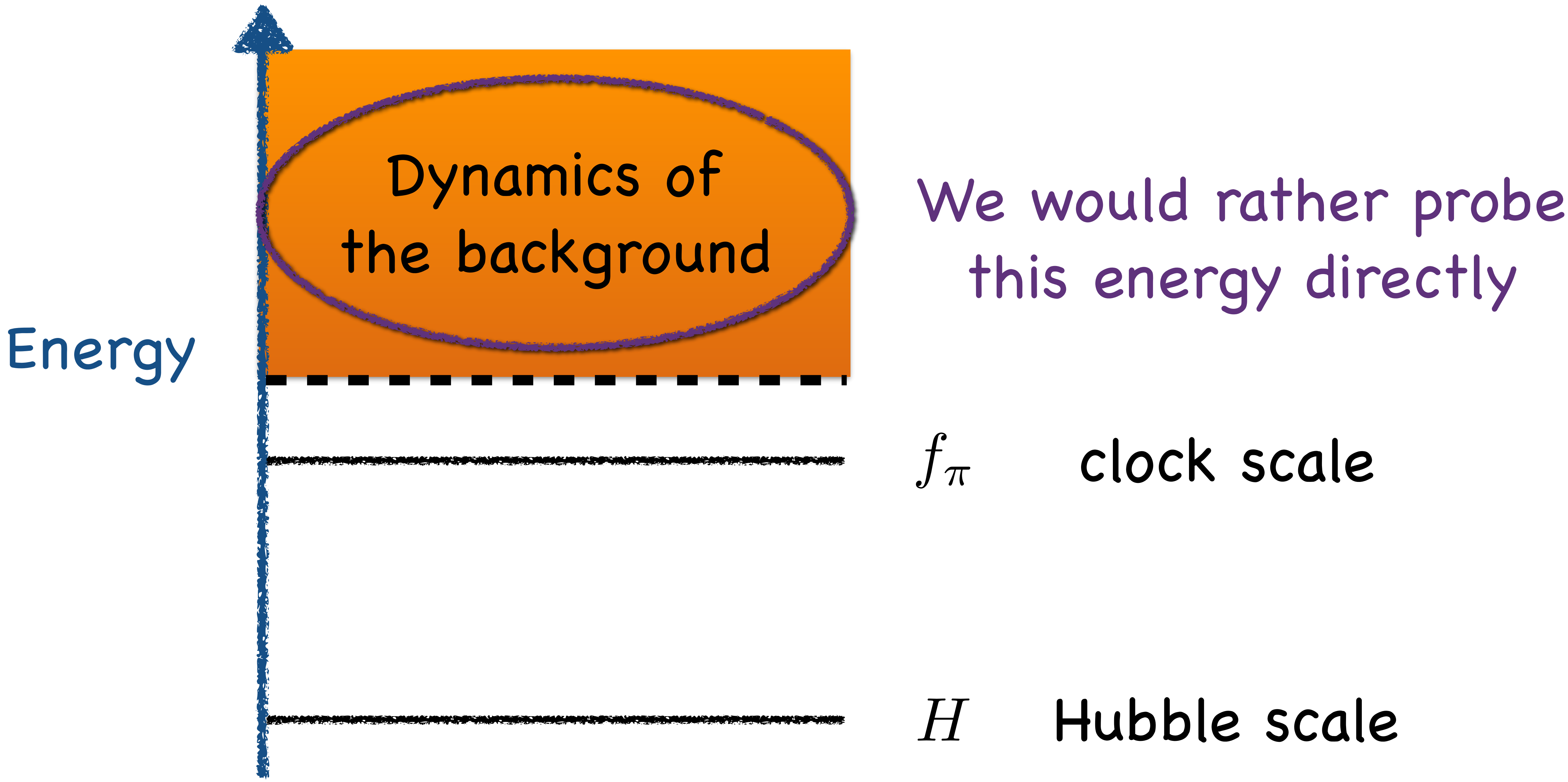


UV-IR connections

Problem common to any low(er) energy probe

Non-trivial relations between IR observables and UV physics have been found in a number of examples

E.g. Weinberg / QCD sum rules, Roy equations, etc.

Have been very valuable in pion physics where calculations and measurements are difficult

UV-IR connections

E.g. Application of Weinberg sum rules Das et al.

$$m_{\pi^+}^2 - m_{\pi^0}^2 = -\frac{3e^2}{16\pi^2 F_\pi^2} \int_0^\infty ds\, s \log s \left[\rho_V(s) - \rho_A(s)\right]$$

Uses asymptotic freedom (and analyticity in s)

Both sides are very difficult to calculate in QCD

We can measure the masses

Gives new (UV) meaning to the mass splitting

Analyticity and EFT

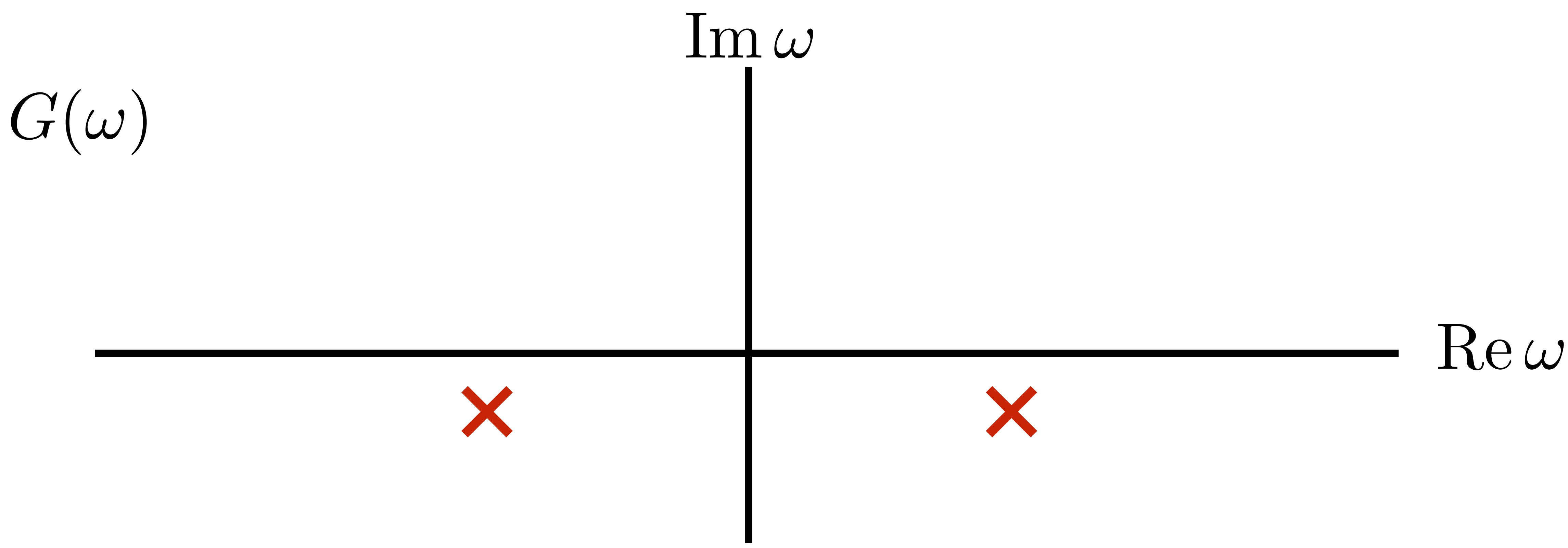
Causality and Analyticity

Causality is a basic property of physics

The response to a source is always delayed:

$$G_{\rm response}(t,t') = \theta(t-t')\langle[\mathcal{O}(t),\mathcal{O}(t')]\rangle$$

In frequency space, this implies analyticity in UHP



Causality and Analyticity

Analyticity connects physics at different scales

$$\mathrm{Re}\, G(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{\mathrm{Im}\, G(\omega)}{\omega' - \omega}$$

Each side is a different manifestation of the system

E.g. for light propagating in a medium

$$n - 1 = \frac{c}{\pi} P \int_{0}^{\infty} d\omega' \frac{\beta(\omega')}{\omega'^2 - \omega^2}$$

Refractive index / speed of propagation

Causality and Analyticity

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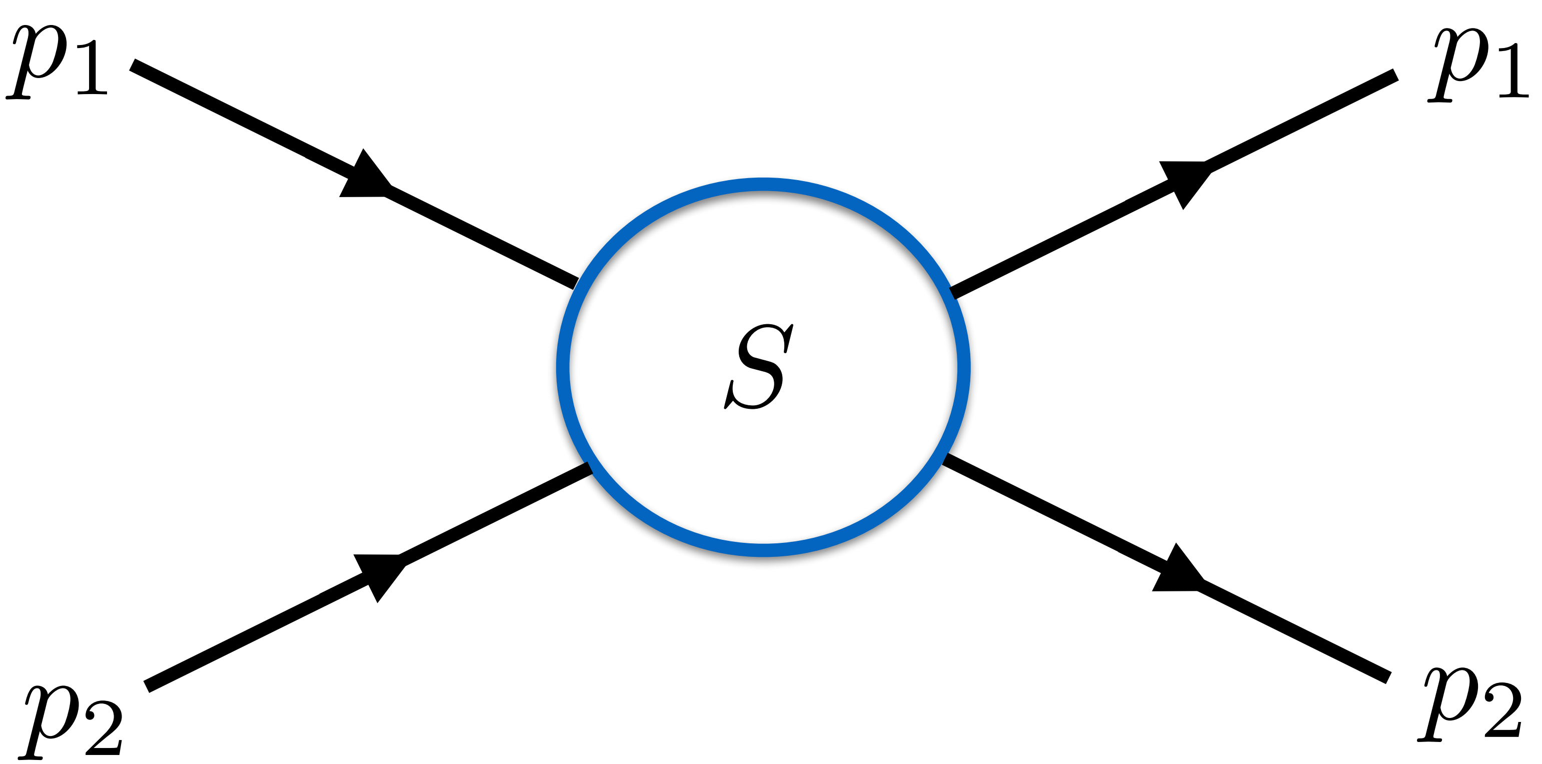
Extinction coefficient / attenuation

Forward Scattering

Similar logic applies to forward scattering amplitude

$$\mathcal{A}(s) \equiv \mathcal{M}(p_1, p_2 \to p_1, p_2)$$

Lorentz invariance: function only of $s = (p_1 + p_2)^2$

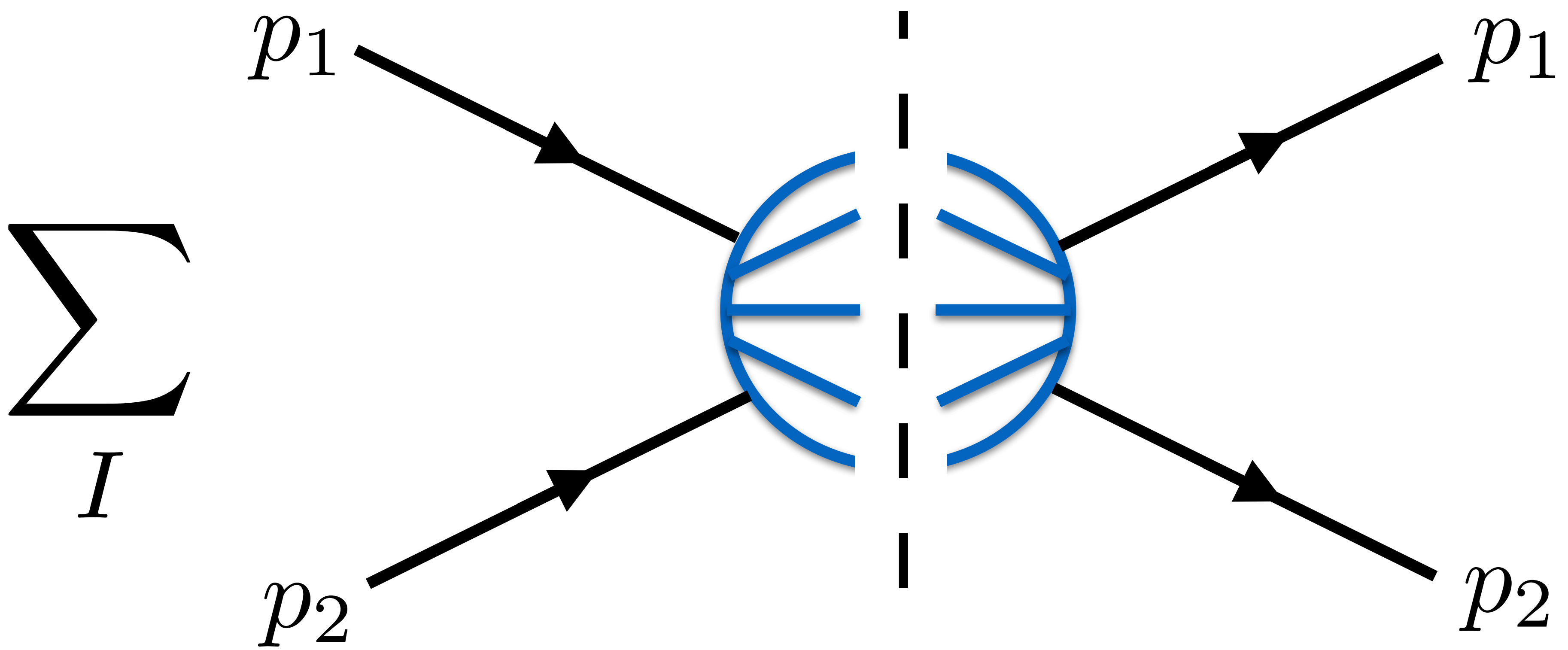


$$S = (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4)[1 + i\mathcal{M}]$$

Forward Scattering

Locations of poles and cuts given by optical theorem

$$2\mathrm{Im}[\mathcal{A}(s)] = \sum_I \int d\Pi_I |\mathcal{M}(p_1, p_2 \to I)|^2$$

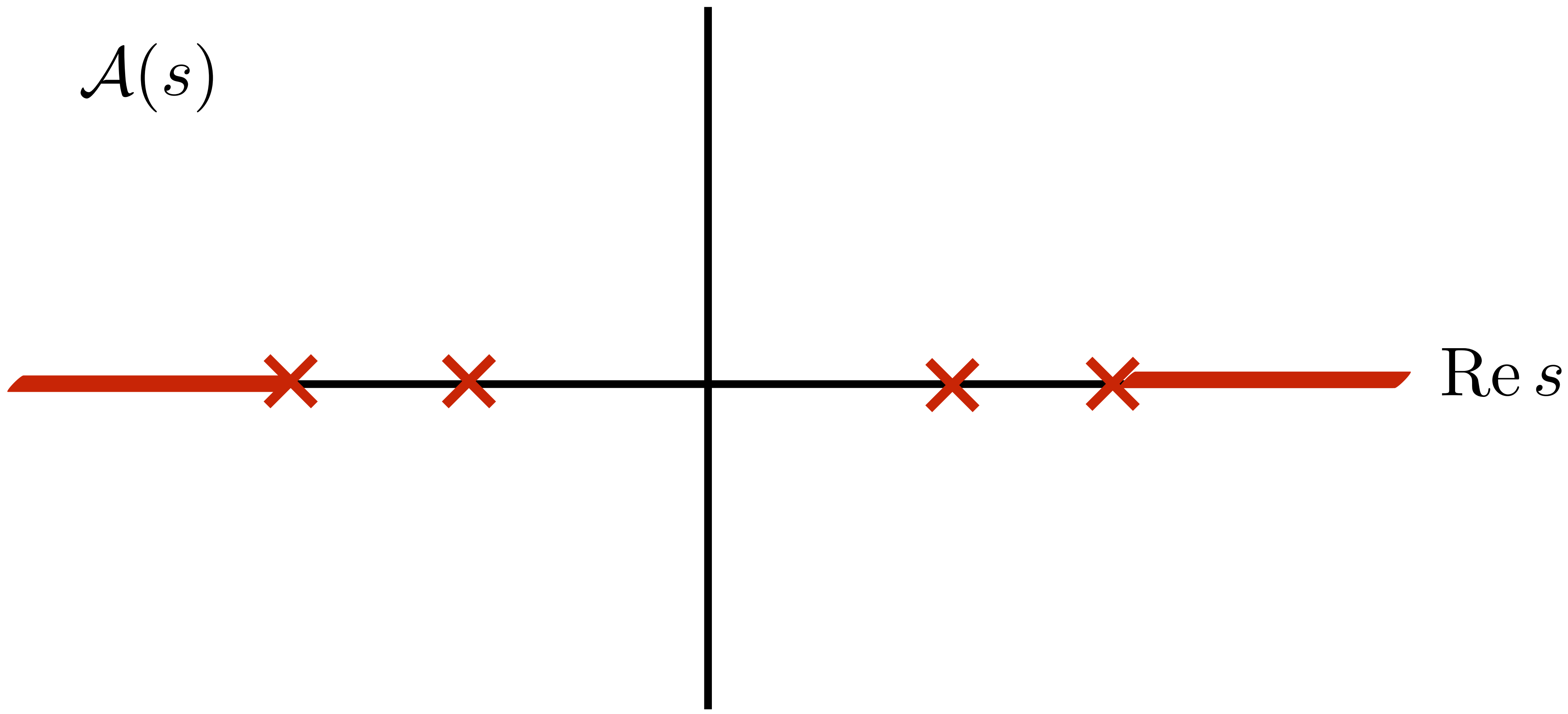


For an analytic function $\mathrm{Im}\mathcal{A}(s) = 0$ on the real line

Forward Scattering

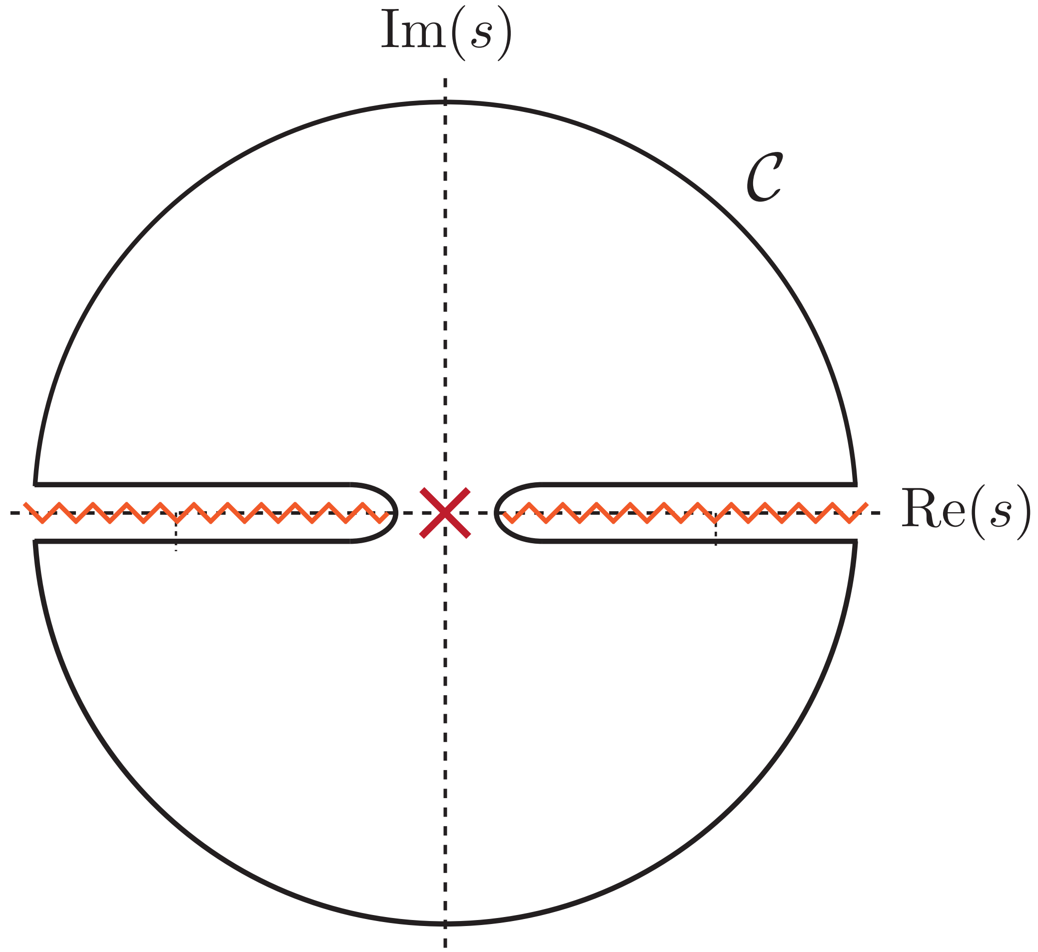
Poles and cuts on positive axis from new states

Same appear on negative axis from crossing $s \to -s$



Dispersion Relations

Analyticity allows us to derive "dispersion relations"



$$\frac{1}{2} A''(s)|_{s=0} = \frac{1}{\pi} \int_0^\infty ds' \, \frac{\mathrm{Im} A(s')}{s'^3}$$

Dispersion Relations

Analyticity allows us to derive "dispersion relations"

$$\frac{1}{2}A''(s)|_{s=0} = \frac{1}{\pi}\int_0^\infty ds' \frac{\mathrm{Im}A(s')}{s'^3}$$

Froissart bound $|\mathcal{A}(s)| \leq s\log^2 s$ lets us drop contour at infinity

Dispersion relation can be useful in two ways:

(1) Positivity : $\mathrm{Im}\mathcal{A}(s) \propto |\mathcal{M}|^2 > 0$

(2) As a "sum-rule" - Connects UV and IR behavior

Implications for EFT

Positivity is a non-trivial constraint on EFTs Adams et al.

Suppose we have some EFT

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{c}{\Lambda^4}(\partial_\mu\phi\partial^\mu\phi)^4 + \ldots$$

At low energies $\mathcal{A}(s) = 8\frac{c}{\Lambda^4}(s^2 + t^2 + u^2) + \ldots$

Dispersion relation + optical theorem: $c \geq 0$

Inequality can only be saturated by a free theory

Implications for EFT

Why didn't Lorentz invariance imply causality?

Around a non-trivial background $\phi = \alpha t + \delta\phi$

$$c_s^2 \simeq 1 - 4\frac{\alpha^2}{\Lambda^4}c + \mathcal{O}(\alpha^4)$$

For $c < 0$ we have superluminal propagation

That $c = 0$ is a free theory is more mysterious

In other situations, either constraint may be stronger

The EFT of Inflation

EFT of Inflation

"Clock" spontaneously breaks time translations

Operator gets time dependent vev – $\langle \phi \rangle \simeq \dot{\phi}_0 \times t$

In the absence of gravity, write an EFT for goldstone

Define field that transforms linearly: $U \equiv t + \pi$

Inflation requires (approx.) symmetry $U \to U + c$

EFT of Inflation

Now we write most general action

Creminelli et al.;
Cheung et al.

$$\mathcal{L} = \sum_n \frac{M_n^4}{n!}(\partial_\mu U \partial^\mu U + 1)^n + \text{ higher derivatives}$$

For "slow-roll inflation" : $M_1^4 = \frac{1}{2}\dot{\phi}_0^2 \qquad M_{n>1}^4 = 0$

Theory is free & fluctuations travel at speed of light

$$\mathcal{L} = \frac{1}{2}\dot{\phi}_0^2 \partial_\mu \pi \partial^\mu \pi$$

Natural to define "decay constant" : $f_\pi^4 = \dot{\phi}_0^2$

EFT of Inflation

Now we write most general action

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$$\mathcal{L} = \sum_n \frac{M_n^4}{n!}(\partial_\mu U \partial^\mu U + 1)^n + \text{ higher derivatives}$$

Small 'sound speed' : $M_1^4 \neq 0 \qquad M_2^4 = \frac{M_1^4}{2c_s^2}(1 - c_s^2)$

Speed of propagations introduces interactions

$$\mathcal{L} = \frac{M_1^4}{c_s^2}(\dot{\pi}^2 - c_s^2 \partial_i \pi^2) + \frac{M_1^4(1-c_s^2)}{c_s^2}[\dot{\pi}\partial_\mu \pi \partial^\mu \pi + \frac{1}{4}(\partial_\mu \pi)^4]$$

Natural to define "decay constant" : $f_\pi^4 \equiv M_1^4 c_s$

EFT of Inflation

What does this have to do with inflation?

Coupling to gravity “gauges” the time translations

Imposing that there is no tadpole for π fixes

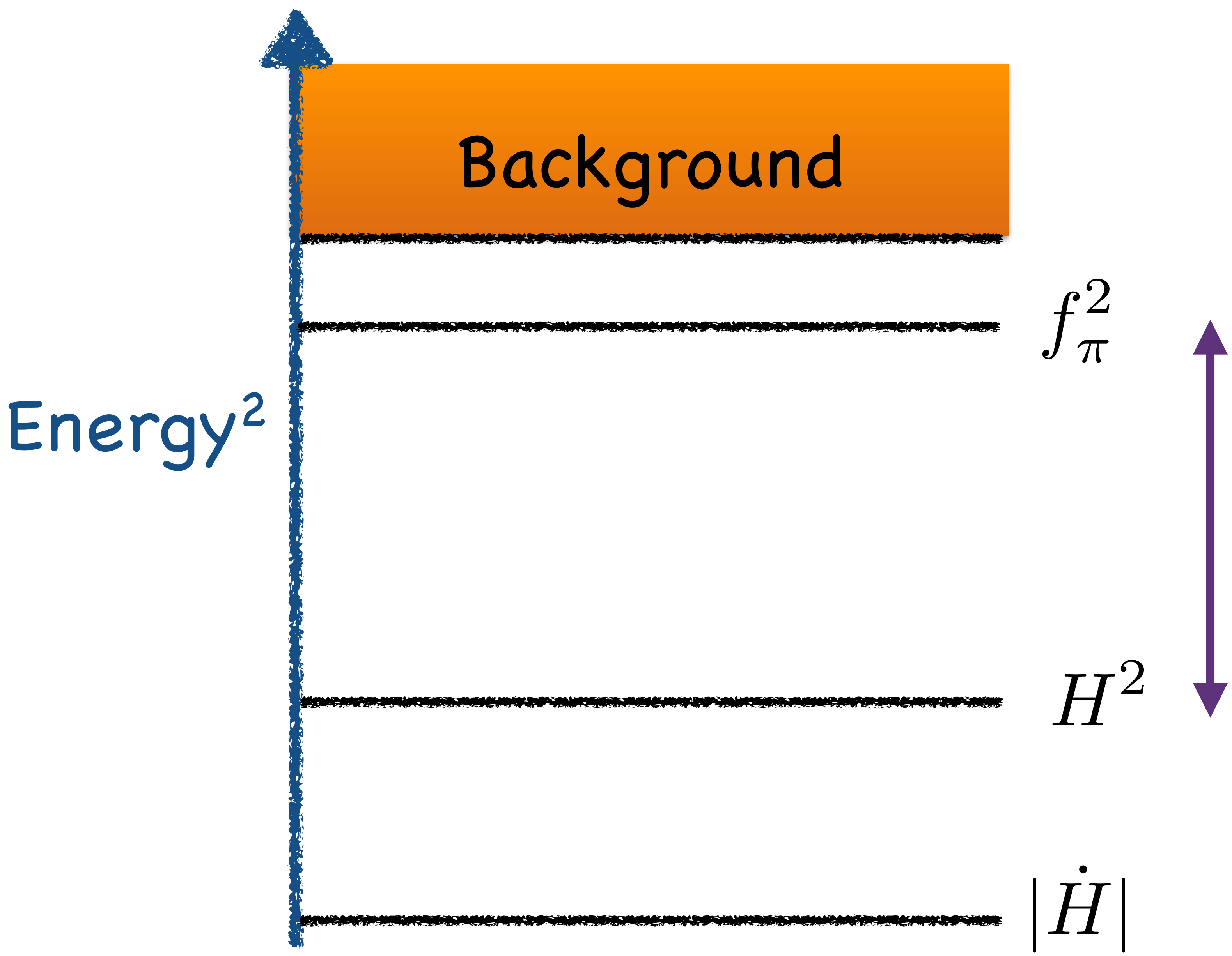
$$M_1^4 = M_{\rm pl}^2 \dot{H}$$

Goldstone boson equivalence from decoupling limit

$$M_{\rm pl}^2 \to \infty\,,\ \dot{H} \to 0 \qquad\qquad M_{\rm pl}^2 \dot{H} = M_1^4$$

EFT of Inflation

Goldstone boson equivalence

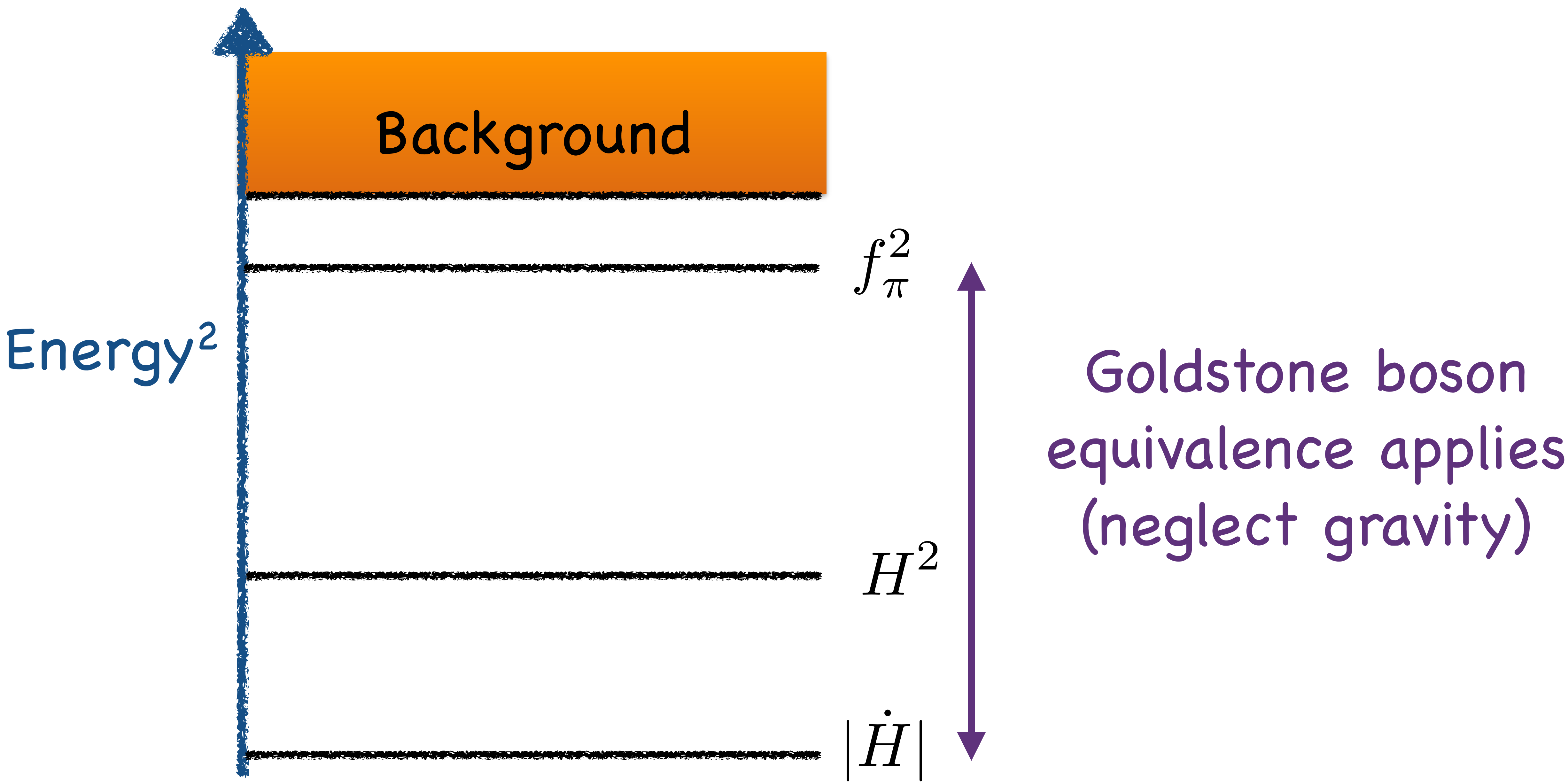


Hierarchy controls power spectrum:

$$\Delta_\zeta \sim 2 \times 10^{-4} = \frac{H^2}{2 f_\pi^2}$$

EFT of Inflation

Goldstone boson equivalence



EFT of Inflation

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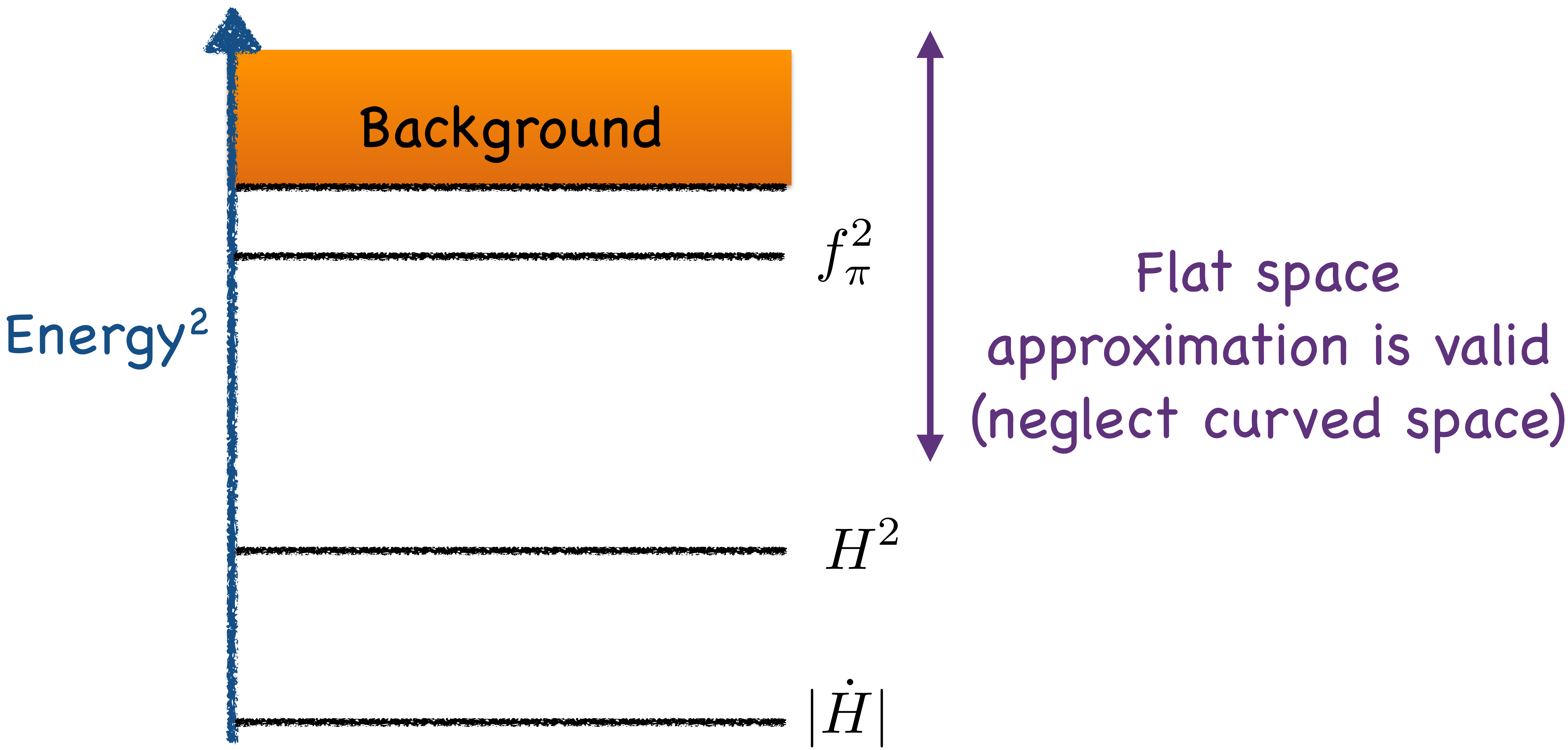
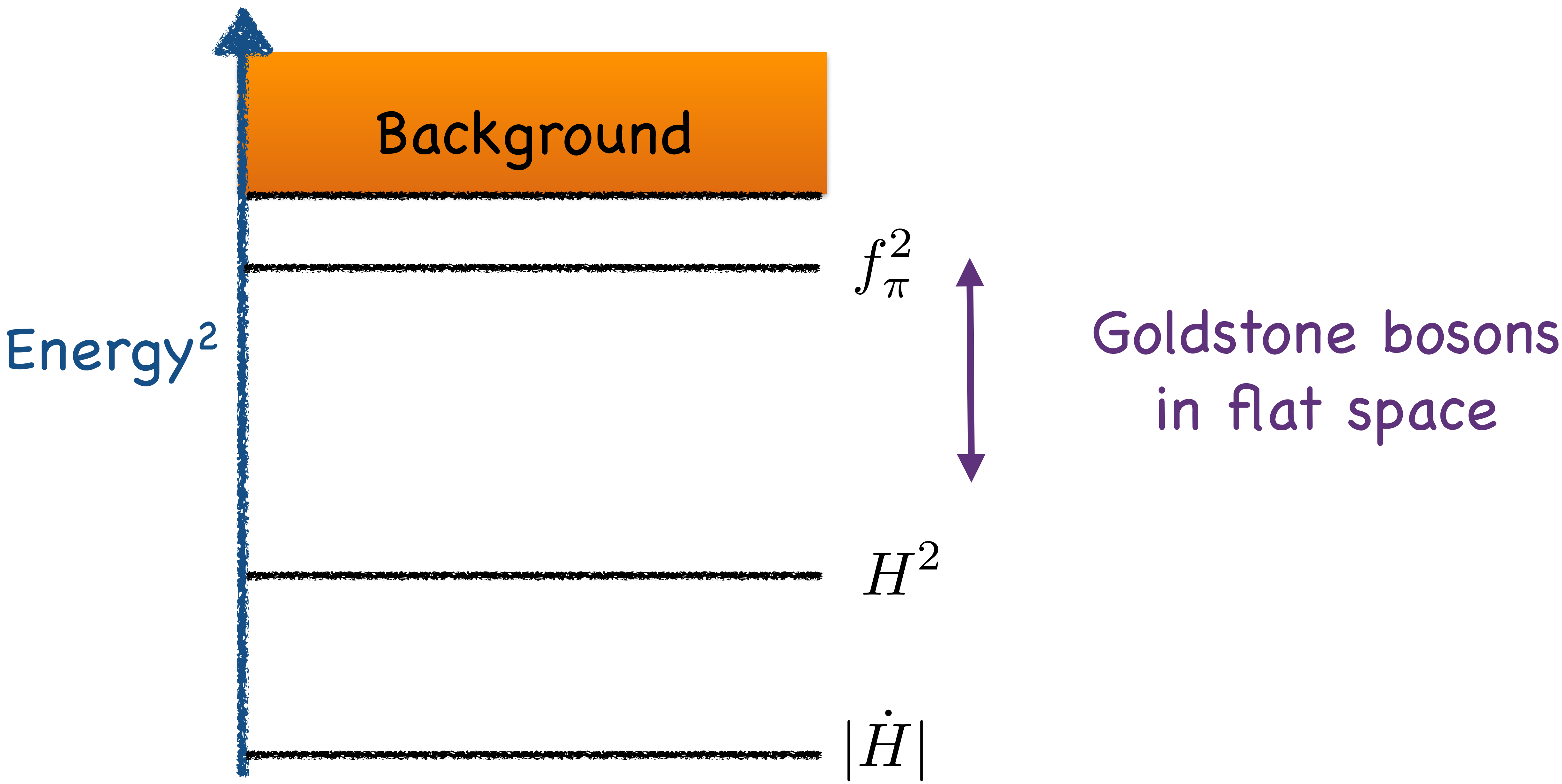


EFT of Inflation

Goldstone boson equivalence



Summary

There is a wide range of energies where we can use:

$$\tilde{\mathcal{L}} = -\frac{1}{2}(\tilde{\partial}\pi_c)^2 + \frac{1}{\Lambda^2}\left[\alpha_1\,\dot{\pi}_c^3 - \alpha_2\,\dot{\pi}_c(\tilde{\partial}\pi_c)^2\right] + \frac{1}{\Lambda^4}\left[\beta_1\,\dot{\pi}_c^4 - \beta_2\,\dot{\pi}_c^2(\tilde{\partial}\pi_c)^2 + \beta_3(\tilde{\partial}\pi_c)^4\right]$$

where $\tilde{x}^i = c_s x^i$ and $\Lambda = f_\pi \times c_s$

The α_i, β_i are determined by M_{2-4}^4

These parameters will be constrained by analyticity

Summary

This action determines cosmological observables

Adiabatic fluctuation : $\zeta \simeq -H\pi$

Interactions lead to non-gaussanity correlations

E.g. Absence of 3-point correlation in Planck

$$c_s > 0.02 \ (95\% \, \text{C.I.})$$

Planck just released constraint on quartic terms

Sum Rules and Positivity

Non-Relativistic Forward Scattering

The EFT is non-relativistic - revisit analyticity

Work in center of mass frame with $s = 4\omega^2$

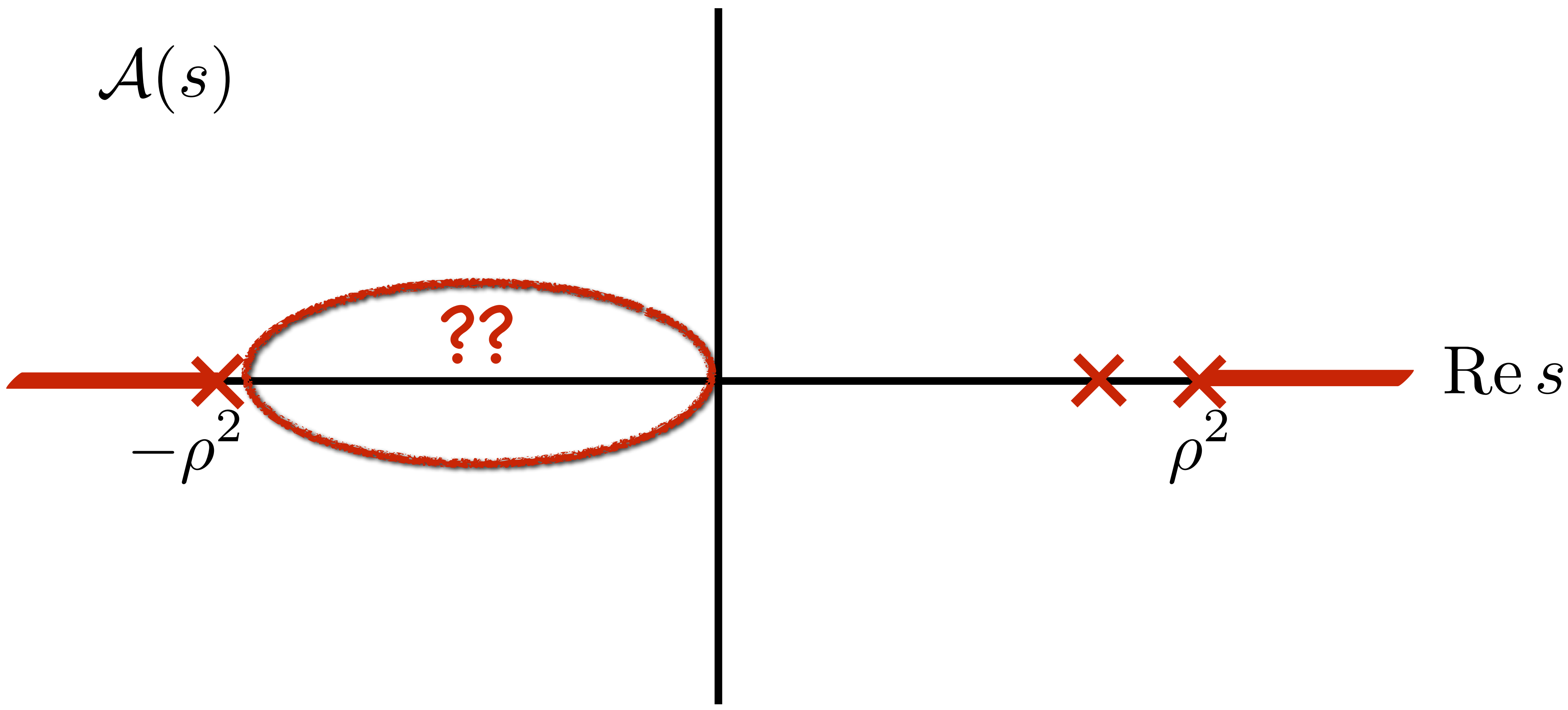
At high energies, $s \gg \rho^2$, becomes relativistic (previous results apply)

On positive axis, optical theorem applies

$$2\,\mathrm{Im}[\mathcal{A}(s)] = \sum_I \int d\Pi_I \, |\mathcal{M}(p_1, p_2 \to I)|^2 \geq 0$$

Non-Relativistic Forward Scattering

Negative axis not determined by $s \to -s$



Positivity is not guaranteed in general

Non-Relativistic Forward Scattering

Analyticity and Froissart bound allow us to write

$$\mathcal{A}''(s \to 0) = \frac{2}{\pi}\left(2\int_{\rho^2}^{\infty} + \int_{0}^{\rho^2}\right) ds\,\frac{\mathrm{Im}[\mathcal{A}(s)]}{s^3} + \int_{-\rho^2}^{0} ds\,\frac{\mathrm{Disc}[\mathcal{A}(s)]}{s^3}$$

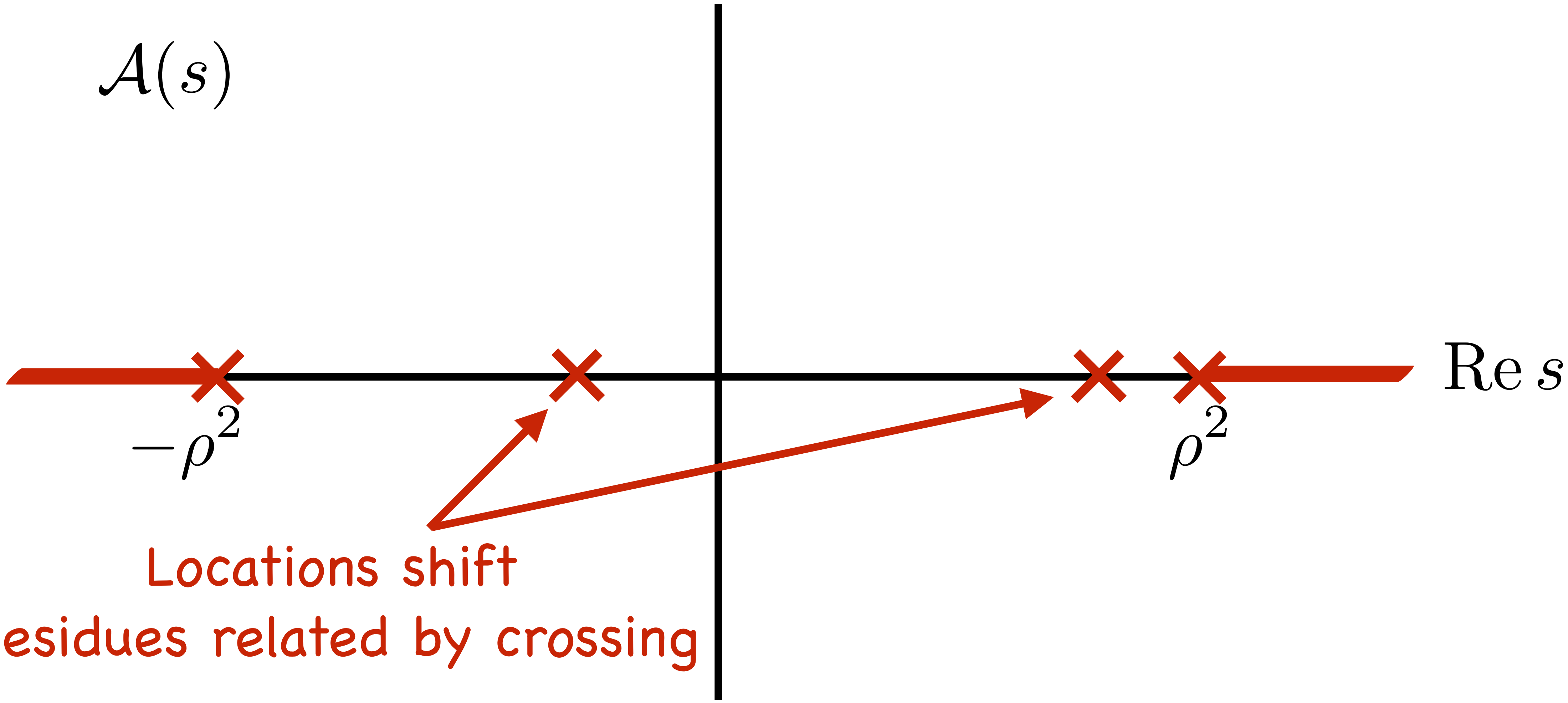
First term is manifestly positive by optical theorem

Normally the second term is positive by crossing

Even without positivity this is a useful sum rule

Non-Relativistic Forward Scattering

Negative axis not determined by $s \to -s$



For a given pole (cut), crossing symmetry acts non-trivially, but typically enforces positivity

Constraint from Positivity

Now let us assumption positivity of the residues
What does this tell us about the EFT of Inflation

Define $M_n^4 \equiv c_n \frac{f_\pi^4}{c_s^{2n-1}}$ (motivated by naturalness)

$$\mathcal{A}(s) = \left(c_4 + 1 - ((2c_3+1) - a(c_s))^2 - b(c_s)\right)\frac{s^2}{\Lambda^2}$$

where $b(c_s) \geq 0$

Positivity requires that $c_4 + 1 \geq 0$ for any c_3, c_s

Constraint from Positivity

Naturally large 4-point function Senatore & Zaldarriaga

$$c_4 \gg c_3^2 \gg 1$$

Stable under radiative corrections

These positivity bounds imply that $c_4 > 0$

Fixes the sign of the trispectrum amplitude

Constraint from Positivity

Also implies analogue of Suyama-Yamaguchi

$$c_4 \geq 4c_3^2 \gg 1$$

Consistent with size of radiative corrections

We cannot tune the trispectrum to vanish

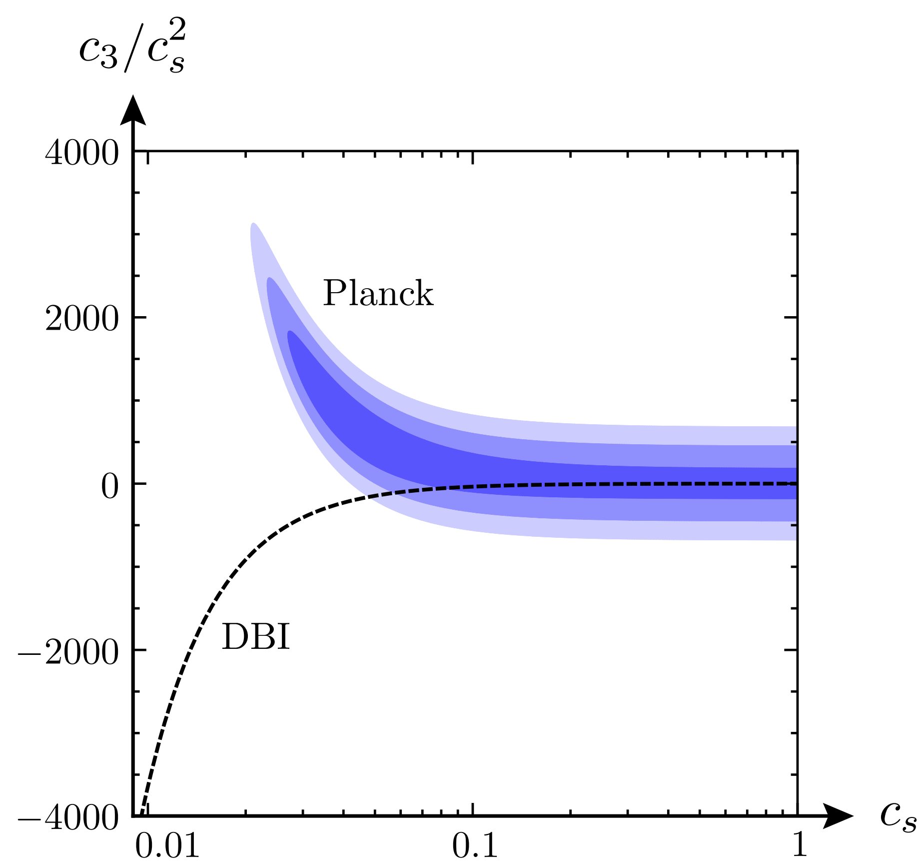
Difficult to measure $c_4 \sim c_3^2$ in practice
(given current constraints on the bispectrum)

Current Limits

Planck reports first limits on c_4 :

$$-8.3 \times 10^7 < \frac{c_4}{c_s^4} < 7.4 \times 10^7 \quad (95\% \, \text{C.I.})$$

Half of this parameter space violates positivity



Connection to superluminality

When $c_s = 1$, positivity requires that

$$c_4 > (2c_3 + 1)^2$$

Compare with speed around $\pi = \alpha t + \delta\pi$

Linear order: $c_s^2 = 1 - \alpha c_3$ $\qquad c_3 = 0$

Quadratic order: $c_s^2 = 1 - \alpha^2 c_4$ $\qquad c_4 \geq 0$

Superluminality gives a stronger constraint

Connection to superluminality

Ignoring angular dependence may weaken bound

Nicolis, Rattazzi & Trincherini

D-wave amplitude : $a_2 = \frac{1}{960\pi} \frac{1 - c_s^2}{c_s^4} \frac{s^2}{f_\pi^4}$

Natural conjecture is that theory is free, $c_{n>1} = 0$

Hope for proof via non-forward dispersion relation

Would imply only slow-roll inflation gives $c_s = 1$

Sum Rule in Action

Weakly coupled example: Tolly & Wyman; Baumann & DG; Achucarro et al.

$$-\frac{1}{2}(\partial\bar{\pi})^2 - \frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}m^2\sigma^2 - \rho\sigma\dot{\bar{\pi}} - \frac{\sigma(\partial\bar{\pi})^2}{2M}$$

At low energies leads to $c_s = \frac{m}{\rho} \ll 1$

Integrate out σ

Mixing term generates $\frac{\rho^2}{k^2 + m^2}\dot{\pi}^2$

Sum Rule in Action

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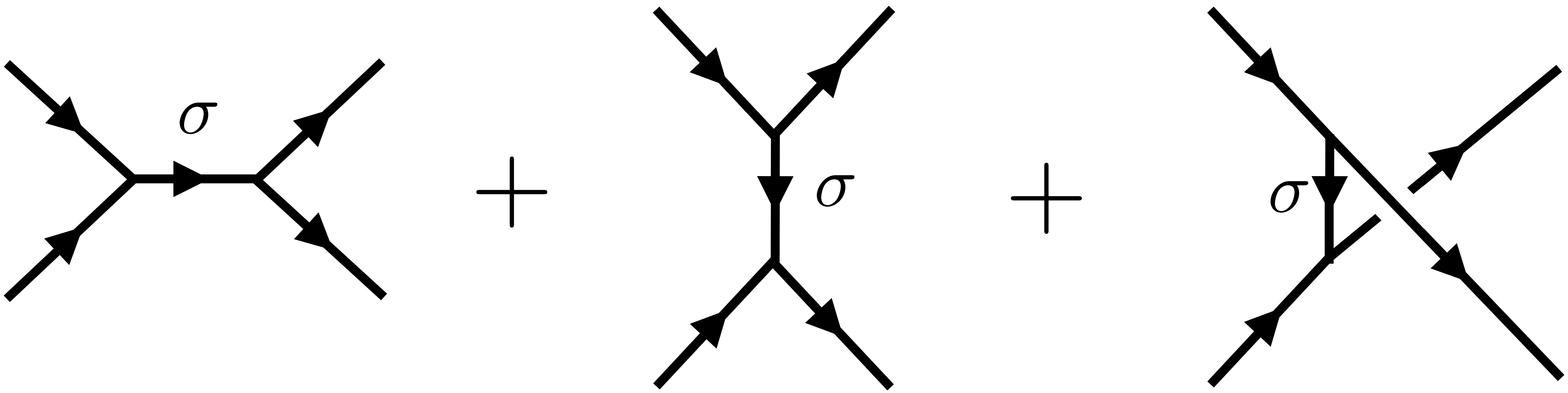
Non-linear realization of mixing gives interactions



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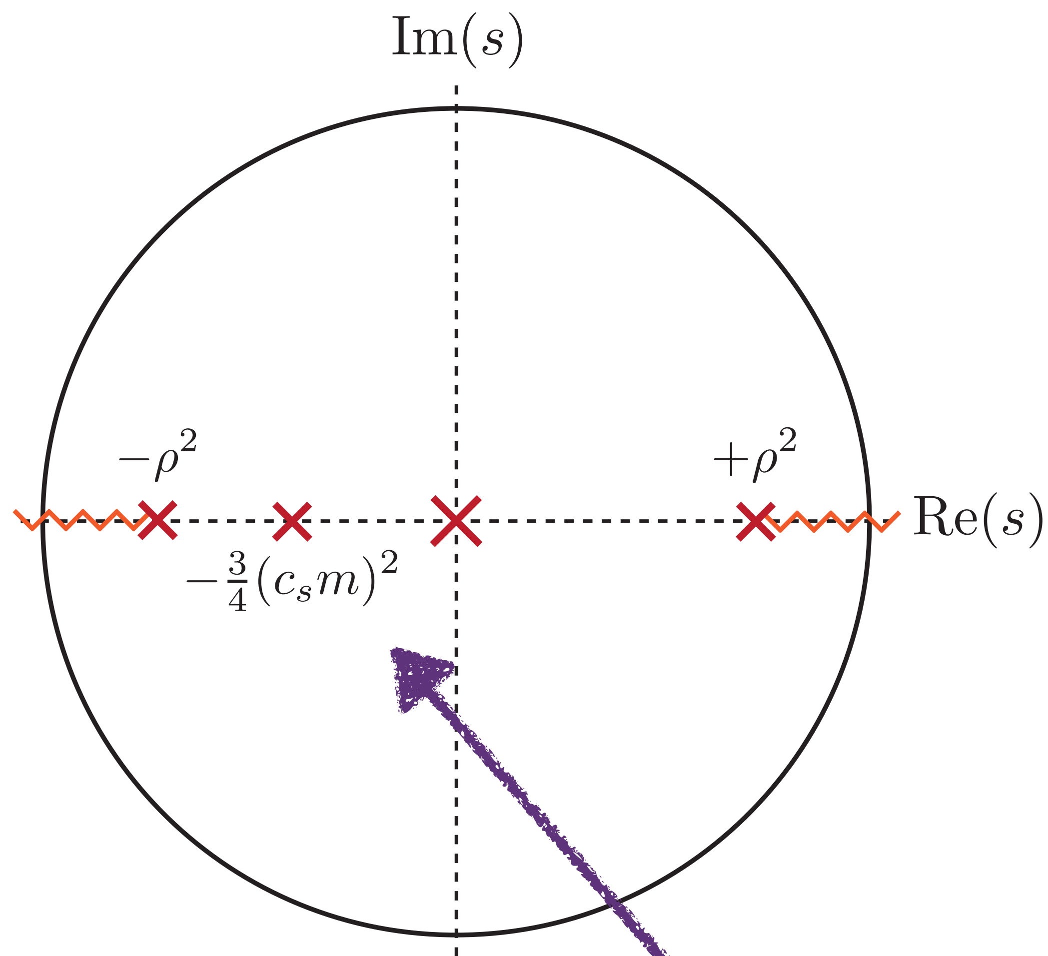
Forward amplitude for gapless mode

$$\mathcal{A} = \frac{i^2}{M^2}Z^4(\omega)\left\{(\omega^2+k^2)^2\left[\frac{1}{4\omega^2-m^2-\rho^2} - \frac{1}{4k^2+m^2}\right] - (\omega^2-k^2)^2\frac{1}{m^2}\right\}$$

See shift in s- and u- channel poles

Sum Rule in Action

Sum-rule dominated by u-channel pole



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Sum Rule in Action

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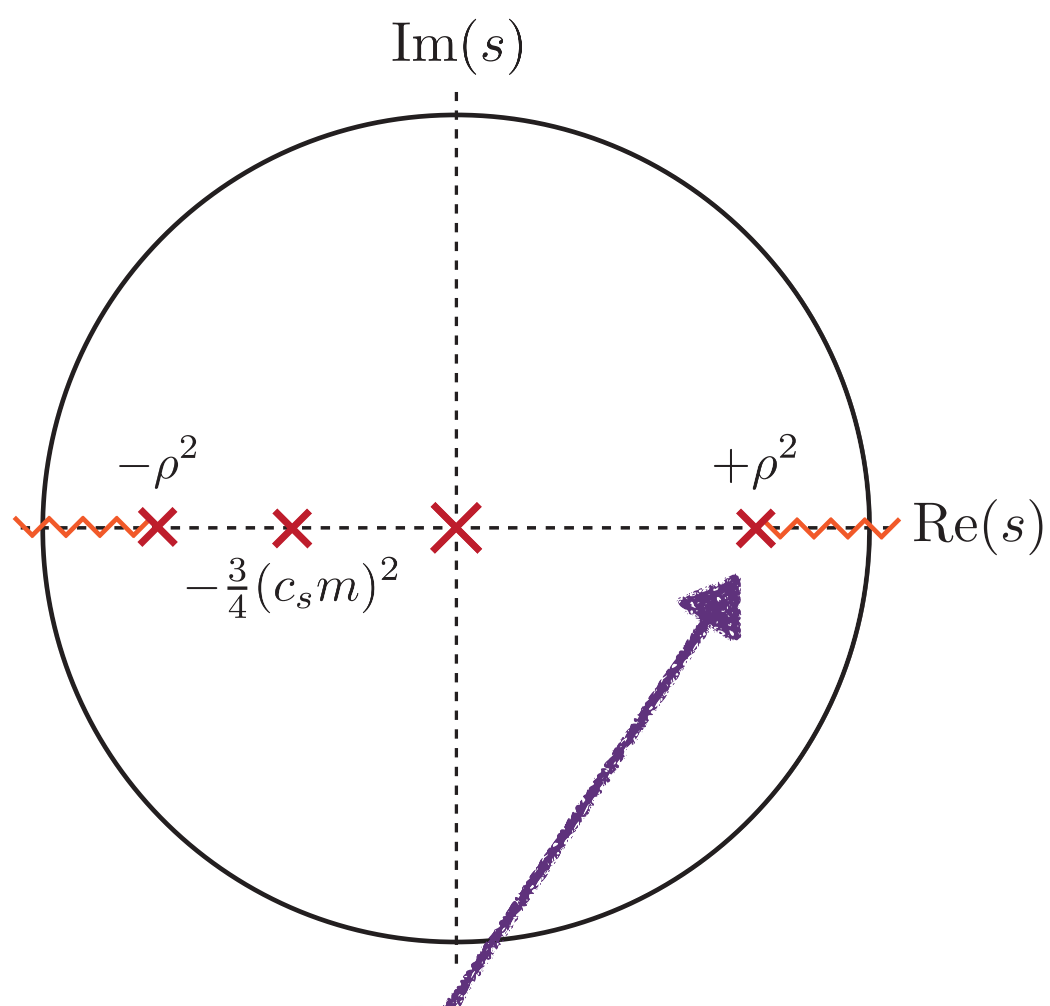

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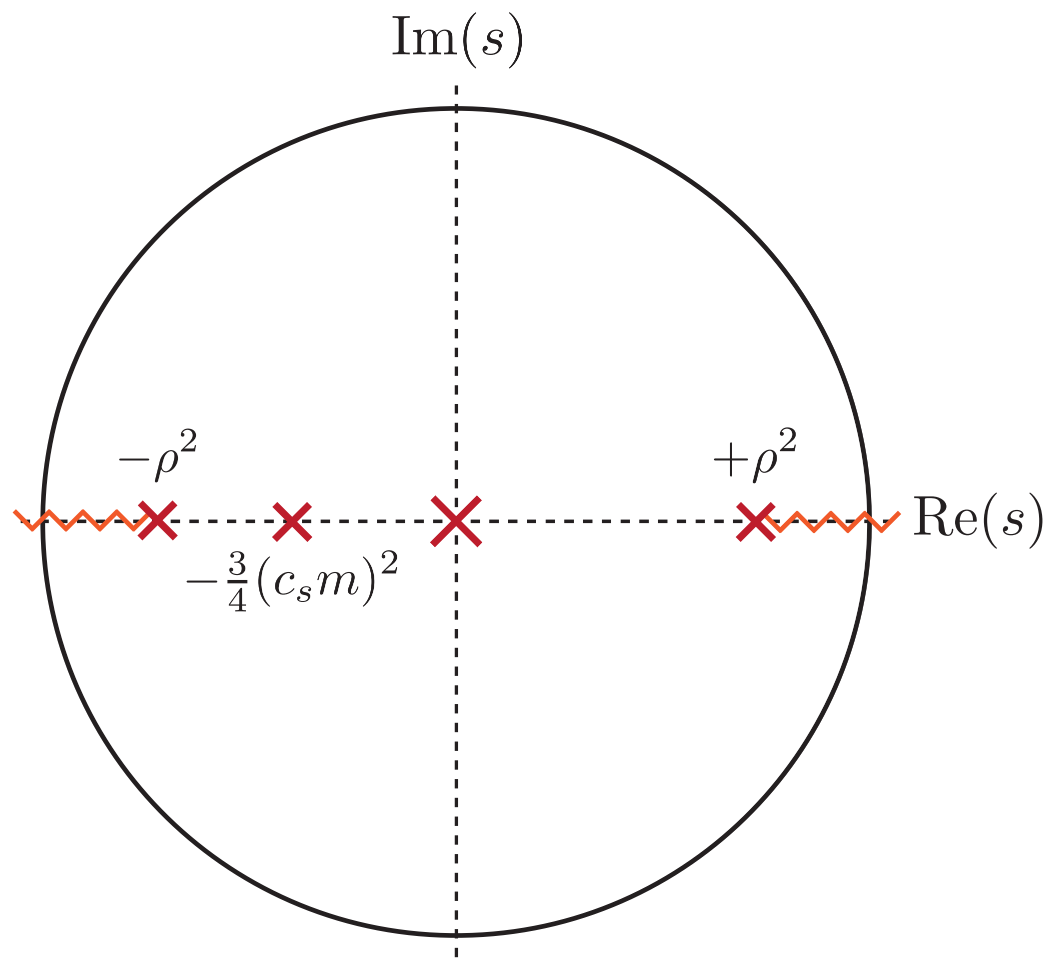
Sum Rule in Action

Sum-rule dominated by u-channel pole



Residues on negative axis are all positive

Positivity can be proven for all generalizations

Conclusions

Summary

Analyticity has non-trivial implications for EFTs

Studied the implications for Single-Field Inflation

Positivity restricts the sign of 4-point function

Relates 3- and 4-point amplitudes

Sum Rule connects UV with values of parameters

Future Directions

Hope to find sum rules for individual terms

One approach is to look at non-forward scattering

Spectral functions might also be useful

(easier to understand analytic structure)

Want to know if $c_s = 1$ requires slow-roll inflation