Chiral Lattice Theories from Staggered Fermions

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Outline:

- Why chiral lattice fermions, why is it difficult ...
- Fermion doubling, no-go theorems, staggered and reduced staggered fermions.
- Symmetric mass generation for vector-like theories.
- Chiral theories: Yukawa terms: site parity and Kitaev structure.
- Discrete anomalies and continuum limit.
- Relation to Kähler-Dirac fermions.
- Pati-Salam model.
- Future ...

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Motivation and Problems

Why ?

Would like a non-perturbative definition of a chiral gauge theory - pedagogical and practical reasons – eg dynamical symmetry breaking

Only non-perturbative regulator we have is lattice ...

Putting chiral fermions on lattice is hard (impossible ?)

- Mirror models – start with vector-like theory and try to gap right handed states (mirrors) ...
- Many lattice formulations tried eg. naive, DWF, overlap, ... no success. Typically bilinear fermion condensates form

In this talk:

New lattice mirror model. Reduced staggered fermions. Gapped by Yukawas with structure borrowed from CMT
Automatically satisfy discrete anomaly cancellation conditions

Back to basics: fermion doubling

Naive discretization of Dirac action:

$$S = \frac{1}{2}\sum_x \sum_\mu \overline{\psi}(x)\gamma_\mu \left[\psi(x+\mu) - \psi(x-\mu)\right] + m\sum_x \overline{\psi}(x)\psi(x)$$

Momentum space:

$$D_\mu = i\sin\left(ap_\mu\right), \quad p_\mu = \left(0, \ldots, \tfrac{\pi}{a}\right)$$

Two zeroes per dimension: 16 fermions in 4d !

Nielsen-Ninomiya theorem:

Doubling topologically guaranteed for reasonable choice of massless lattice action (local, translation invariant, chirally symmetric)
Doublers come in L/R pairs. Continuum limit is vector-like.

Staggered fermions: a step in the right direction ?

Workarounds - reduce numbers of doublers (but lattice theories still vector-like)

- Add chiral symmetry breaking mass term for doublers (Wilson)
- Formulate 4d theory as boundary of 5d world (DWF)
- Overlap fermions - non-local Dirac op. and $\hat{\gamma}_5$
- Staggering - spin diagonalize naive fermions

$$\psi_\alpha(x) = \Omega_{\alpha\beta}(x)\chi_\beta(x) \quad \text{with } \Omega(x) = \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$$

Staggered action:

$$S = \frac{1}{2}\sum_x \sum_\mu \eta_\mu(x)\overline{\chi}(x)\left[\chi(x+\mu) - \chi(x-\mu)\right] + m\sum_x \overline{\chi}(x)\chi(x)$$

with $\eta_\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$

$\chi(x)$ - 4 uncoupled components - discard 3 copies.

Describes 4 Dirac fermions in continuum limit

Reduced staggered fermions (RSF)

Can we do any better ?

If $m = 0$:

$$S = \sum_x \sum_\mu \eta_\mu(x) \left[\overline{\chi}_+(x) D_\mu \chi_-(x)\right] + \left[\overline{\chi}_-(x) D_\mu \chi_+(x)\right]$$

where $\quad [2D_\mu f(x) = f(x+\mu) - f(x-\mu)]$

$$\psi_\pm(x) = P_\pm \psi(x) = \tfrac{1}{2}\left(1 \pm \epsilon(x)\right)\psi(x)$$

$$\epsilon(x) = (-1)^{\sum_{i=1}^4 x_i} \text{ lattice site parity}$$

Relabel $\overline{\chi}_+ \to \chi_+$ and write:

$$S = \frac{1}{2} \sum_{x,\mu} \eta_\mu(x) \chi(x) D_\mu \chi(x)$$

U(1) symmetry: $\chi(x) \to e^{i\alpha\epsilon(x)} \chi(x)$

Describes two massless Dirac or four Majorana fermions in continuum

Generating mass - four fermion interactions for RSF

$$S = \frac{1}{2}\sum_{x,\mu}\eta_\mu(x)\chi^a(x)D_\mu\chi^a(x) + \frac{1}{2}G\sum_x \sigma^+_{ab}(x)\chi^a(x)\chi^b(x) + \frac{1}{2}\left(\sigma^+_{ab}\right)^2(x)$$

Symmetries:

$SO(4)$ invariant. $\sigma^+ \in SO(3)$ subgroup.

Shift symmetries $\chi^a(x) \to \xi_\mu(x)\chi^a(x+\mu)$ with $\xi_\mu(x) = (-1)^{\sum_{i=\mu+1}^4}$

Z_8 symmetry $(\omega, \omega^2, \ldots, \omega^8)$ with $\omega = e^{i\epsilon(x)\frac{\pi}{4}}$

$$\psi^a(x) \to \omega\psi^a(x)$$

$$\sigma^+_{ab}(x) \to \omega^{-2}\sigma^+_{ab}(x) = -i\epsilon(x)\sigma^+_{ab}$$

Note:

Symmetries protect against all fermion bilinear terms

Where does Z_8 come from ?

Why not Z_4 ?

- Z_4 on fermions means $Z_2 \in (1, -1)$ on σ
- Hence Z_8 implies $Z_4 \in (1, -1, i, -i)$ on σ
- Like $g \to ig$ in Yukawa or $g^2 \to -g^2$ in four fermi term

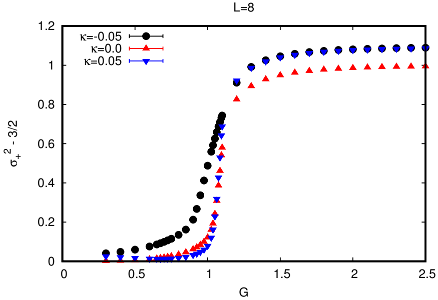
Original $SO(4) = SU_+(2) \times SU_-(2)$ symmetry

Used $\sigma^+ = \frac{1}{2}(\sigma^{ab} + \frac{1}{2}\epsilon^{abcd}\sigma^{cd})$. But could have chosen σ^-
Swapping $g^2 \to -g^2$ preserves partition function !

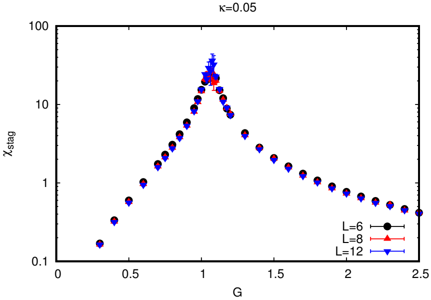
True symmetry is Z_8 !

Symmetric mass generation

- $\lambda = 0$. Massless fermions
- $\lambda \to \infty$. Four fermion condensate. Interpret as mass term $\chi^a \Pi^a$ with $\Pi^a = \epsilon_{abcd}\chi^b\chi^c\chi^d$. Gapped symmetric phase.



(a) Four fermion condensate



(b) Susceptibility

Phase transition $\lambda \sim 1$. Need to add kinetic term for σ^+ coeff. κ

Gapping subsets of RSF

Using four fermion interactions full RSF can generate masses for both χ_- and χ_+ leading to 2 massive Dirac fermions without breaking symmetries
Can we apply interactions only to (say) χ_+ ?

Consider

$$S_{\text{Yuk}} = \sum_x (GP_+ + gP_-)\sigma_A(x)\chi^a(x)\Gamma_A^{ab}\chi^b(x) + \frac{1}{2}\sigma_A^2$$

Assume Γ_A Dirac gamma matrices for $2N$ fermions transforming under some global rotation symmetry G

Question:

Assume $G >> g$. Can we gap χ_+ fermions and leave χ_- massless ?
What do we get ? Are there constraints ?

Effective action for σ_A

Let

$$M = \eta_\mu(x) D_\mu \delta^{ab} + (gP_- + GP_+)\,\sigma_A \Gamma_A^{ab}$$

Consider eigenvalue equation:

$$M(\sigma_A)\phi_n = \mu_n \phi_n$$

Invariance under Z_8 with $\omega = e^{i\epsilon(x)\frac{\pi}{4}}$

$$M(\omega^2\sigma)\omega^{-1}\phi_n = (\omega^2\lambda_n)\omega^{-1}\phi_n$$

eigenvalues shift by phase ω^2 under $\sigma \to \omega^2\sigma$

Thus $\mathrm{Pf}(\sigma_A) = e^{-S_{\rm eff}}$ naively invariant

Need to take care with zero modes with $D_\mu \chi^0_\pm = 0$

Arising when $g, G \to 0$ on torus with pbc

Discrete Z_8 anomaly

Consider configuration $\sigma_A(x) = \hat{\sigma}_A$ with $\hat{\sigma}_A^2 = 1$ and $g = G \to 0$
Changing phase of $\hat{\sigma}_A$ corresponds to $g \to g' = e^{i\epsilon(x)\frac{\pi}{2}} g$.

Bring $\Lambda = \hat{\sigma}_A \Gamma_A$ to 2×2 block diagonal form J using $SO(2N)$ rotation.

$$J = \mathrm{diag}\left(\mu_1 i\tau_2 \oplus \mu_2 i\tau_2 \ldots \oplus \mu_N i\tau_2\right) \quad \text{with } \mu_i = 1 \text{ for } \chi^0_\pm$$

$$\mathrm{Pf}(g'J) = \prod_{\epsilon(x)=\pm 1} e^{iN\epsilon(x)\frac{\pi}{2}}\ \mathrm{Pf}(gJ)$$

Pfaffian invariant for vector-like theory

But suppose gap out χ_+ with large G:

Pfaffian is not invariant under $g \to g'$ if N not multiple of 4.
Global anomaly in Z_8 symmetry!
Cancel anomaly using multiples of 8 fermions

Turning this around

Will not be able to generate mass for χ_+ and decouple from I.R unless we have a multiple of eight RSF.
Minimal model fermions must be in 8 dim real irrep. of global symmetry G
simplest solution – $G = \mathrm{Spin}(7)$

Comments:

- Will show that this implies continuum theory has 16 Weyl fermions in agreement with continuum arguments on discrete anomalies (Dai Freed ...)
- $ig \equiv g$. Sign of $g^2 \left(\chi^T \Lambda \chi\right)^2$ irrelevant. No energetic reason to condense the bilinear and break symmetry.
- Argument formal. Assumed χ_+ gapped above cut-off to decouple....
- Likely need kinetic term for scalars in four dimensions.

Continuum fields

Assemble staggered fields into matrix fermion (neglect global $\mathrm{Spin}(7)$ indices)

$$\Psi = \sum_{\{n_\mu=0,1\}} \chi(x + n_\mu \hat{\mu})\Omega(n_\mu)$$

with Ψ residing on lattice with twice the lattice spacing. Has block structure (chiral basis):

$$\Psi = \left(\begin{array}{cc} E & O' \\ O & E' \end{array} \right) \quad \text{(E, E') even parity sites etc.}$$

Columns of Ψ give continuum spinors

In continuum limit recover $\mathrm{Spin}(4)_{\mathrm{Lorentz}} \times \mathrm{Spin}(4)_{\mathrm{Flavor}}$ symmetry. Act by left and right multiplication of Ψ. Majorana nature of Ψ:

$$O' = \sigma_2 O^* \sigma_2 \text{ and } E' = -\sigma_2 E^* \sigma_2.$$

Generalized charge conjugation

Continuum Symmetries

Symmetries:

$$[SU(2) \times SU(2)]_{\mathrm{Lorentz}} \otimes [SU(2) \times SU(2)]_{\mathrm{Flavor}} \otimes Z_8 \otimes \mathrm{Spin}(7)$$

2×2 block	Lorentz rep.	Flavor rep.
E	$(\mathbf{2}, \mathbf{1})$	$(\mathbf{2}, \mathbf{1})$
O	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{2}, \mathbf{1})$
E'	$(\mathbf{1}, \mathbf{2})$	$(\mathbf{1}, \mathbf{2})$
O'	$(\mathbf{2}, \mathbf{1})$	$(\mathbf{1}, \mathbf{2})$

If we succeed in gapping the blocks (E, E') left with:

(O, O') containing a pair of Majorana fermions.
Expect in I.R $g \to 0$ - massless
Equivalent to pair left handed Weyl fermions transforming in $(8, 2, 1)_L$ representation of $\mathrm{Spin}(7) \otimes [SU(2) \otimes SU(2)]_{\mathrm{Flavor}}$

Continuum as Kähler-Dirac

This continuum fermion Ψ can be mapped into a set of p-forms $\Omega = (\omega_0, \omega_1, \ldots, \omega_D)$ via

$$\Psi = \sum_p \omega_{n_1 \ldots n_p(x)} \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$$

Satisfy the Kähler-Dirac equation:

$$(d - d^\dagger)\Omega = 0$$

Analog of $\epsilon(x)$ is Γ acting on forms: $\Gamma : \omega_p \rightarrow (-1)^p \omega_p$

Reduced Kähler-Dirac fermion $\equiv$ real forms. Implies $\Psi^\dagger = \gamma_2 \Psi^T \gamma_2$

$$\chi_\pm \rightarrow \Psi_\pm = \frac{1}{2}\left(\Psi \pm \gamma_5 \Psi \gamma_5\right)$$

$$S = \int d^4x \, \mathrm{Tr}\left[\Psi^{\dagger a} \gamma_\mu \partial_\mu \Psi^a\right] + \Lambda^{ab} \mathrm{Tr}\left[\left(GP_+ + gP_-\right) \Psi^{\dagger a} \Psi^b\right]$$

Connection to Pati-Salam

- Gap out Ψ_+. $4 \to 2$ massless Majoranas per Kähler-Dirac field. Equivalent to 2 Weyl.
- All flavor symmetries explicit: $SU(2) \times SU(2) \times \mathrm{Spin}(7)$.
- Gauge flavor

Spontaneously break $\mathrm{Spin}(7) \to \mathrm{Spin}(6) \equiv SU(4)$

eg. $\sigma_A^2 = 1$

$$\mathbf{8} \to \mathbf{4} + \overline{\mathbf{4}}$$

Pati Salam GUT:

fermions $(\mathbf{8}, \mathbf{2}, \mathbf{1}) \to (\mathbf{4}, \mathbf{2}, \mathbf{1})_{\mathbf{L}} \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2})_{\mathbf{R}}$

unbroken gauge symmetry: $SU(4) \times SU(2) \times SU(2)$

Summary

- RSF with carefully chosen Yukawa interactions may allow for lattice theory whose continuum limit is chiral.
- Requires symmetric mass generation (SMG) - non-perturbative.
- A necessary condition for SMG is cancellation of discrete anomalies.
- Shown how these arise in context of model. Cancellation requires 8 RSF or 16 Weyl in continuum. Agrees with known results.
- $\mathrm{Spin}(7)$ symmetry can be gauged - chiral lattice gauge theory !
- Continuum limit given by Pati-Salam after breaking $\mathrm{Spin}(7) \to \mathrm{Spin}(6)$. How to gauge internal $SU(2) \times SU(2)$?
- SMG remains to be checked by simulation (sign problems ?)

Thanks!