New ‘t Hooft Anomalies in the Hamiltonian Picture

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Contents

Introduction to Hamiltonian Anomalies

Extremely briefly

Anomalies

- The breaking of a classical symmetry through quantum effects

- Often shown as a phase ambiguity in the path integral

- Can be recast in terms of the operators of the theory:

$$\partial_\mu \hat{J}^\mu \neq 0$$

- Can rewrite this in terms of the charge:

$$\left[\hat{H}, \hat{Q}\right] \neq 0 \quad \text{with } \hat{Q} = \int d^3x \, \hat{J}^0$$

Anomalous algebras

- In fact, we could extend this to the symmetry operators:

$$\widehat{U}_\alpha = e^{i\alpha\,\hat{Q}}$$

- Then the operator algebra definition of symmetry breaking is

$$\left[\widehat{H}, \widehat{U}_\alpha\right] \neq 0$$

- This form can be generalized to all symmetries, hence new anomalies of discrete and higher-form symmetries

‘t Hooft anomalies

$$\left[\hat{H}, \hat{U}\right] = 0$$

Introduce background for
U symmetry

$$\left[\hat{H}, \hat{U}\right] \neq 0$$

Mixed 't Hooft anomalies

$$[\hat{H}, \hat{U}] = 0$$
$$[\hat{H}, \hat{T}] = 0$$

Introduce background
for just T symmetry

$$[\hat{H}, \hat{U}] = 0$$
$$[\hat{H}, \hat{T}] = 0$$

But modify $[\hat{U}, \hat{T}]$

Introduce backgrounds for both
T and U symmetries

$$[\hat{H}, \hat{U}] \neq 0$$

or

$$[\hat{H}, \hat{T}] \neq 0$$

Symmetry breaking via an algebra

Diagonalize $\widehat{H}$ and $\widehat{T}$

$$|E_2, t_2\rangle \longrightarrow |E_2, t'_2\rangle$$

Apply $\widehat{U}$

$$|E_1, t_1\rangle \longrightarrow |E_1, t'_1\rangle$$

Apply $\widehat{U}$

$$|E_0, t_0\rangle \longrightarrow |E_0, t'_0\rangle$$

Apply $\widehat{U}$

Anomaly implies $t_0 \neq t'_0$

This talk

- Work out the algebras explicitly for new mixed ‘t Hooft anomalies in Yang-Mills and QCD-like theories
- Specifically the center-chiral anomaly in QCD(adj) and the center-parity anomaly at $\theta = \pi$
- Show exactly the implication of these algebra on symmetry breaking patterns

Center stability

	Center unbroken	Center broken
Parity (chiral) symmetry unbroken	X	
Parity (chiral) symmetry broken		

Caveats and Motivations

Caveats:

- We only work on a 3-torus
- Algebra only forces the symmetry breaking in the presence of a non-trivial boundary conditions

Motivations:

- Works for any torus size, so we can take the torus really big making the first two caveats unimportant

"Old Bottles"

The mixed 0-form/1-form anomaly in Hilbert space: pouring the new wine into old bottles

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Abstract:

We study four-dimensional gauge theories with arbitrary simple gauge group with 1-form global center symmetry and 0-form parity or discrete chiral symmetry. We canonically quantize on $\mathbb{T}^3$, in a fixed background field gauging the 1-form symmetry. We show that the mixed 0-form/1-form 't Hooft anomaly results in a central extension of the global-symmetry operator algebra. We determine this algebra in each case and show that the anomaly implies degeneracies in the spectrum of the Hamiltonian at any finite-size torus. We discuss the consistency of these constraints with both older and recent semiclassical calculations in $SU(N)$ theories, with or without adjoint fermions, as well as with their conjectured infrared phases.

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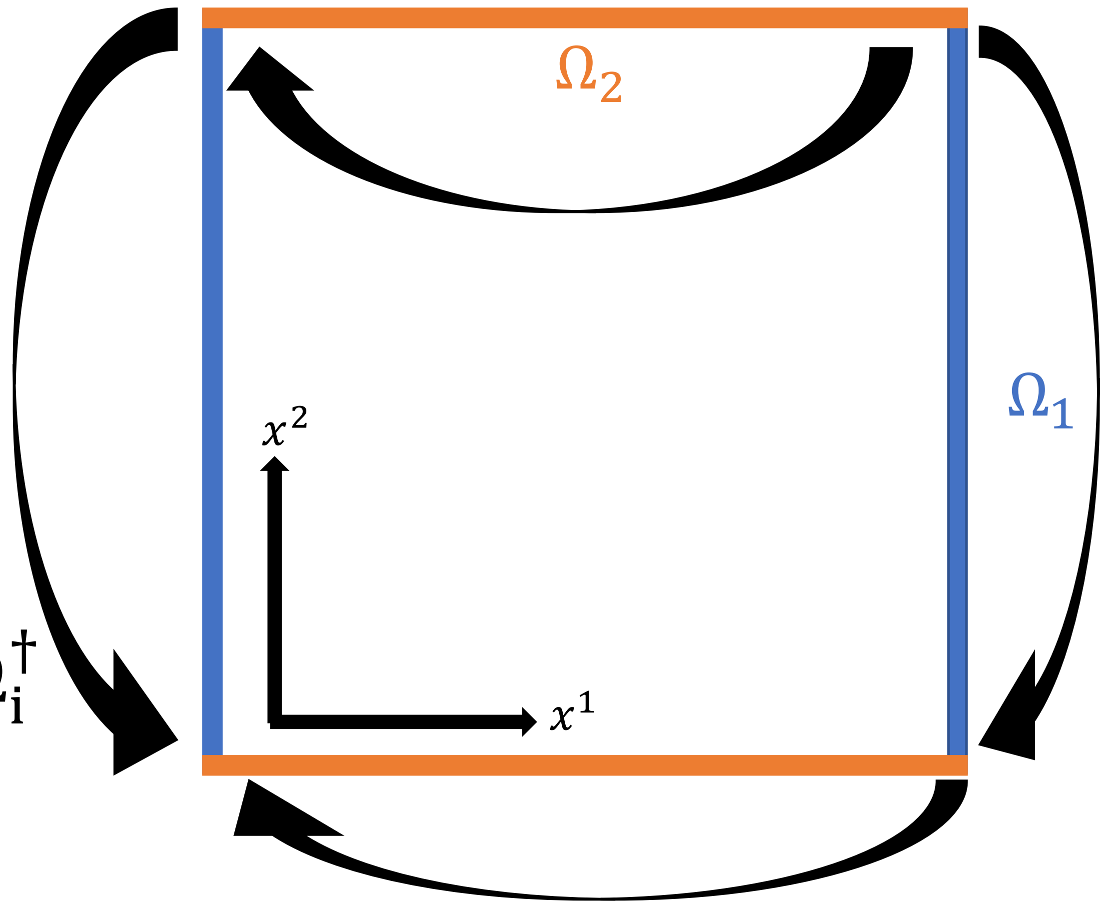
Quantizing Yang-Mills

A brief overview

Boundary conditions



- Boundary conditions:

$$A(x^i = L_i) = \Omega_i(A(x^i = 0) - id)\Omega_i^\dagger$$

- Usual cocycle condition:

$$\Omega_i(x^j = L_j)\Omega_j(x^i = 0) = \Omega_j(x^i = L_i)\Omega_i(x^j = 0)$$

- With only adjoint fields we can add:

$$\Omega_i(x^j = L_j)\Omega_j(x^i = 0) = \Omega_j(x^i = L_i)\Omega_i(x^j = 0)e^{i2\pi\, n_{ij}/N}$$

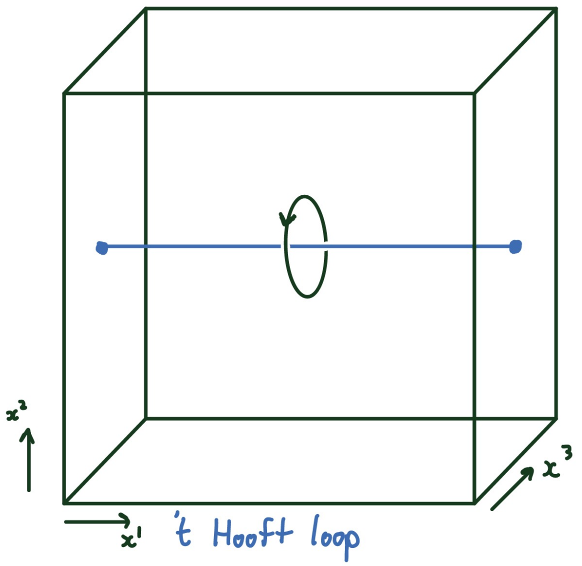
Center backgrounds

- Related to background center field via:

$$\oint C^{(2)} = \frac{2\pi n_{ij}}{N} + 2\pi\mathbb{Z}$$

- This is often described using:

$$m_i = \frac{1}{2}\varepsilon_{ijk} n_{jk}$$



- This is often called the "magnetic flux" since introducing an 't Hooft loop in the i direction increases m_i by 1

The large Hilbert space

- Fix boundary conditions

- Create a Hilbert space out of all possible gauge field configurations $|A_i^a(x)\rangle$ that respect boundary conditions and have $A_0 = 0$

- Define the field operators $\hat{A}_j^b(x)$ via
$$\hat{A}_j^b(y)|A_i^a(x)\rangle = |A_i^a(x)\rangle A_j^b(y)$$

- This is known as the large Hilbert space because it contains physical Hilbert space as a subspace

Gauge transformations & physical Hilbert space

- Now define gauge transformation operators on the large Hilbert space by

$$\hat{U}|A(x)\rangle = |U(A - id)U^{-1}\rangle$$

- To remain in the large Hilbert space, we only consider gauge transformations that do not change the boundary conditions

- Define the physical Hilbert space as the subspace of the large Hilbert space that is invariant under all gauge transformations

$$\hat{U}|\psi\rangle = |\psi\rangle$$

$\hat{E}$, $\hat{B}$, and $\hat{H}$

- To define the Hamiltonian we need the colour electric and magnetic field operators:

$$\hat{E}_i^a(x) = -i \frac{\delta}{\delta \hat{A}_i^a(x)}$$

$$\hat{B}_i^a(x) = \frac{1}{2}\varepsilon_{ijk}\left(\partial_j \hat{A}_k^a(x) - \partial_k \hat{A}_j^a(x) - f^{abc}\hat{A}_j^b(x)\hat{A}_k^c(x)\right)$$

- Then the Hamiltonian is

$$\hat{H} = g^2 tr\hat{E}_i\hat{E}_i + \frac{1}{g^2}\, tr\hat{B}_i\hat{B}_i$$

Theta angle

- The theta angle may be incorporated into the Hamiltonian via
$$\hat{H}_\theta = g^2 tr\left(\hat{E}_i - \frac{\theta}{8\pi^2}\hat{B}_i\right)\left(\hat{E}_i - \frac{\theta}{8\pi^2}\hat{B}_i\right) + \frac{1}{g^2}\, tr\hat{B}_i\hat{B}_i$$
- Alternatively, use $\hat{H}_0$ and redefine Hilbert space to have
$$\hat{U}_\nu\left|\psi\right\rangle = e^{-i\nu\theta}\left|\psi\right\rangle$$
- Map between these using the operator
$$\hat{V}_\theta = \exp\left(i\theta \int K^0(\hat{A})\right)$$
$$K^0(A) = \frac{1}{8\pi^2} tr\left(A \wedge F - \frac{i}{3} A \wedge A \wedge A\right)$$

$\hat{V}_{2\pi}$

- $\hat{V}_\alpha$ has the following properties:

$$\hat{U}_\nu \hat{V}_\alpha |\psi\rangle_\theta = e^{-i(\theta-\alpha)\nu} \hat{V}_\alpha |\psi\rangle_\theta$$

$$\hat{V}_\alpha \hat{H}_\theta \hat{V}_\alpha^\dagger = \hat{H}_{\theta+\alpha}$$

- $\hat{V}_\alpha$ takes states out of the physical Hilbert space unless $\alpha \in 2\pi\mathbb{Z}$
- Hence, $\hat{V}_{2\pi}$ is a good operator of the theory and corresponds to 2π rotations of the theta angle
- This will be a very useful operator for us

Center symmetry

- Consider maps $\mathrm{C}\left[\vec{k}\right]: T^3 \to G$ with $\vec{k} \in \mathbb{Z}^3$ which could be applied as a gauge transformation except they change the boundary conditions by
$$\Omega_i \to e^{i2\pi k_i/N}\Omega_i$$
- These are not gauge transformations of our theory, but they preserve our physical Hilbert space, so we may use them to define operators $\hat{C}\left[\vec{k}\right]$
- These operators form the 1-form center symmetry group $Z(G)^3 = \mathbb{Z}_N^3$
- Consider a set of generators of the symmetry:
$$\hat{T}_1 = \hat{C}[(1,0,0)] \;\; \hat{T}_2 = \hat{C}[(0,1,0)] \;\; \hat{T}_3 = \hat{C}[(0,0,1)]$$

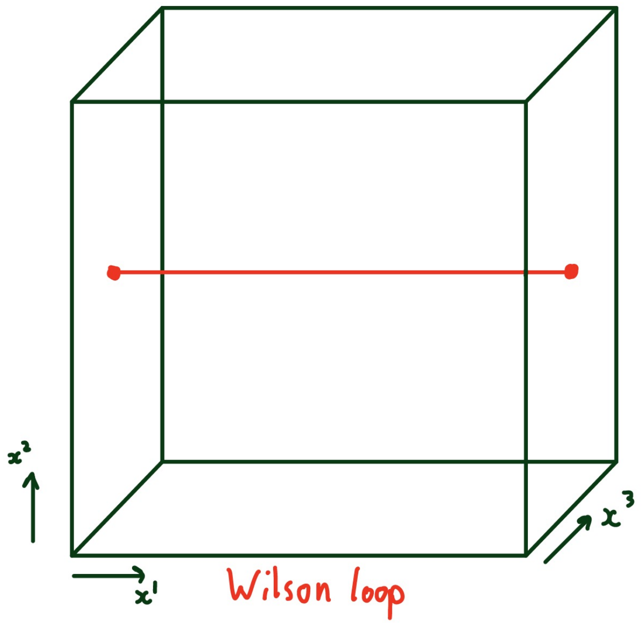
Eigenvalues of the $\hat{T}_l$ operators

- Each $\hat{T}_l$ generates a $Z(G) = \mathbb{Z}_N$ subgroup so we can denote eigenstates by a vector of integers $\vec{e}$ mod N

$$\hat{T}_l|\vec{e}\rangle = e^{i2\pi e_l/N}|\vec{e}\rangle$$

- $\vec{e}$ is known as the "electric flux" because a Wilson loop increases it by 1

- Since $\hat{T}_l$ are symmetry operators, $\vec{e}$ is a good quantum number



The algebra

- One can work out explicitly that

$$\hat{T}_l \hat{V}_{2\pi} = e^{i2\pi m_l/N}\, \hat{V}_{2\pi} \hat{T}_l$$

- This commutation relation will give us the anomalous symmetry algebras (since $\hat{V}_{2\pi}$ shows up as part of other symmetries)

- Notice that the phase is the same as one gets in the path integral when doing a 2π shift of θ in a center background field

The center-parity anomaly at $\theta = \pi$

The parity symmetry operator

- The parity operator $\hat{P}_0$ at $\theta = 0$ is defined via

$$\hat{P}_0 |A(x^1, x^2, x^3)\rangle = |-\Gamma_P A(L_1 - x^1, L_2 - x^2, L_3 - x^3)\Gamma_\mathrm{P}\rangle$$

- $\hat{P}_0$ commutes with $\hat{B}$ but anticommutes with $\hat{E}$:

$$\hat{P}_0 \widehat{H}_\theta \hat{P}_0 = \widehat{H}_{-\theta}$$

- Hence not a symmetry unless $\theta = 0$ or $\theta = \pi$

Center-parity algebra

$\boldsymbol{\theta = 0}$

- $\hat{P}_0$ is a symmetry operator on its own
- The following algebra holds:

$$\hat{P}_0 \hat{T}_l \hat{P}_0 = \hat{T}_l^\dagger$$

- Hence $\hat{P}_0 |\vec{e}\rangle = |-\vec{e}\rangle$

$\boldsymbol{\theta = \pi}$

- $\hat{P}_\pi = \hat{V}_{2\pi} \hat{P}_0$ is the symmetry operator to account for shift of θ
- The algebra is

$$\hat{P}_\pi \hat{T}_l \hat{P}_\pi = e^{i 2\pi m_l / N} \hat{T}_l^\dagger$$

- Hence $\hat{P}_\pi |\vec{e}\rangle = |\vec{m} - \vec{e}\rangle$

Implications

- Consider $\vec{m} = (0, 0, 1)$ and look at just the e_3 eigenstates:
- At $\theta = 0$, $\hat{P}_0|e_3\rangle = |-e_3\rangle$ means $e_3 = 0$ is a valid eigenvalue for a unique ground state
- At $\theta = \pi$, $\hat{P}_\pi|e_3\rangle = |1 - e_3\rangle$ means:
  - At odd N, $e_3 = \frac{N-1}{2}$ is the only candidate for a unique ground state
  - At even N, there are no e_3 states invariant under $\hat{P}_\pi$

$SU(3)$	$SU(4)$
$|0\rangle \leftrightarrow |1\rangle$	$|0\rangle \leftrightarrow |1\rangle$
$|2\rangle$	$|2\rangle \leftrightarrow |3\rangle$

Implications

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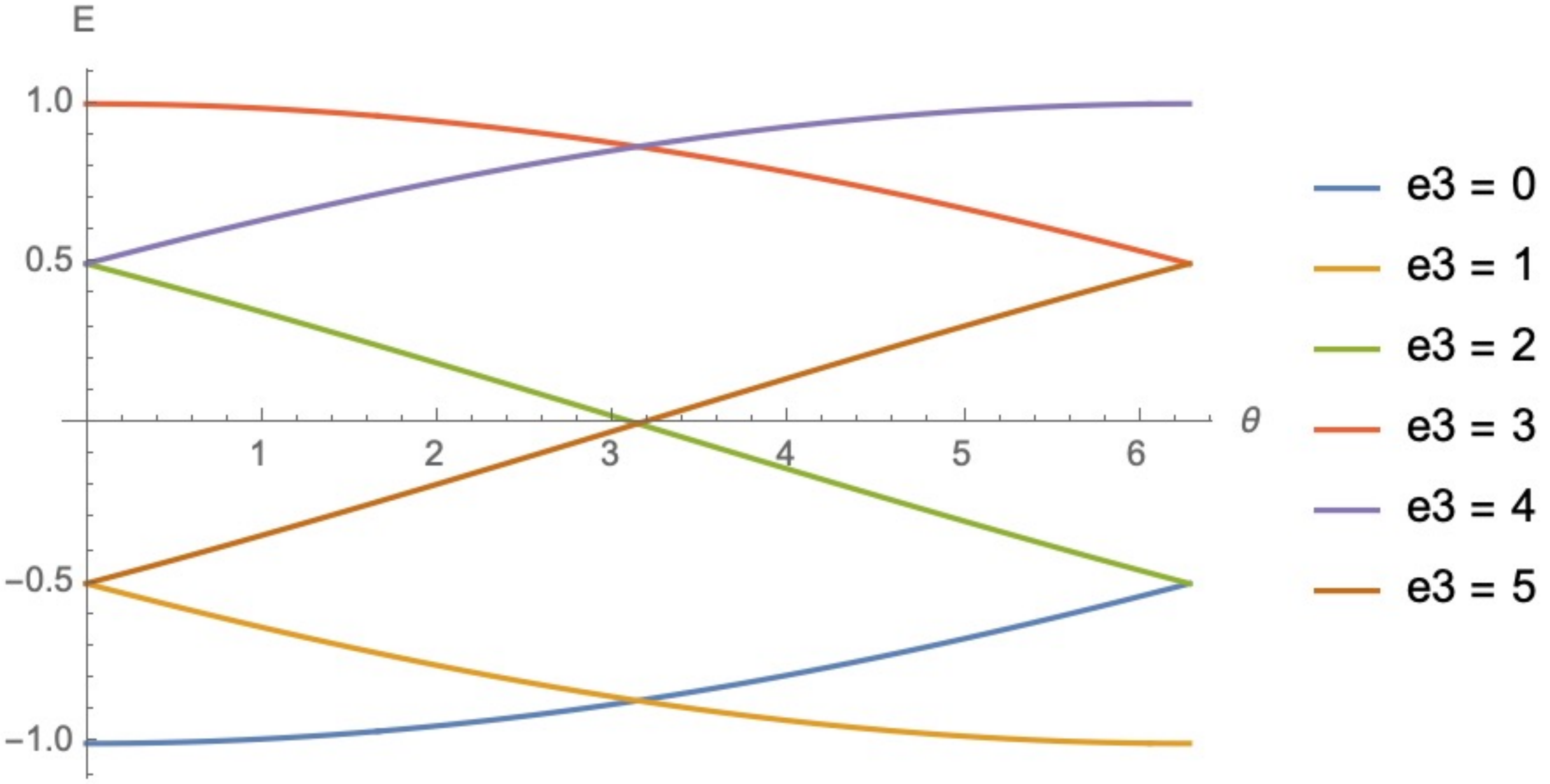
  - At odd N, $e_3 = \frac{N-1}{2}$ is the only candidate for a unique ground state
  - At even N, there are no e_3 states invariant under $\hat{P}_\pi$
- Hence, at even N we must have at least 2 ground states
- This corresponds to spontaneous breaking of parity and it only shows up in the theories with the anomaly

Explicit example ($SU(6)$)

- On a very small torus we can find the lowest energies explicitly



The center-chiral anomaly in QCD(adj)

Chiral symmetry operator

- Consider adding n_f flavours of adjoint Weyl fermions, λ_I
- There is a classical $U(1)$ symmetry given by $\lambda_I \to e^{i\alpha}\lambda_I$
- Broken by usual anomaly:

$$\partial_\mu \hat{J}^\mu = 2n_f N \partial_\mu \widehat{K}^\mu$$

- We can define the operator:

$$\hat{J}_5^\mu = \hat{J}^\mu - 2n_f N \widehat{K}^\mu$$

- Operator is conserved, but not gauge invariant
- However, the operator

$$\hat{\mathrm{X}} = e^{i\frac{2\pi}{2n_f N}\int \hat{J}_5^0}$$

Is gauge invariant and generates the unbroken subgroup $\mathbb{Z}_{2n_f N}$

Center-parity anomaly

- Notice that

$$\hat{X} = e^{i\frac{2\pi}{2n_f N}\int \hat{J}^0 - i2\pi \int K^0} = e^{i\frac{2\pi}{2n_f N}\int \hat{J}^0} \hat{V}_{2\pi}^{-1}$$

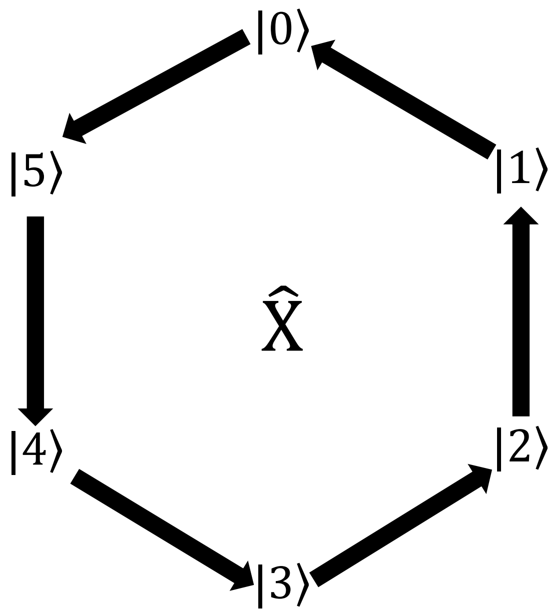
- Hence, since $\hat{J}^0$ only depends on fermion operators, and fermion operators are unaffected by $\hat{T}_l$:

$$\hat{T}_l \hat{X} = e^{-i2\pi m_l/N} \hat{X}\hat{T}_l$$

- Hence

$$\hat{X}|\vec{e}\rangle = |\vec{e} - \vec{m}\rangle$$

Implications



- Again consider $\vec{m} = (0, 0, 1)$
- This gives

$$\hat{X}|e_3\rangle = |e_3 - 1\rangle$$

- No e_3 eigenstates can be invariant under this and it forces a minimum N-fold degeneracy
- Hence, there is are at least N vacua related by chiral symmetry, so we have spontaneous chiral symmetry (partial) breaking

Results for all gauge groups

Group, G	Center, Z(G)	Parity breaking?	Chiral symmetry	Minimal degeneracy in chiral theory
$SU(N)$	$\mathbb{Z}_N$	Only for N even	$\mathbb{Z}_{2n_f N}$	N
$Sp(2k+1)$	$\mathbb{Z}_2$	Yes	$\mathbb{Z}_{2n_f(2k+2)}$	2
$Spin(4N)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	Yes	$\mathbb{Z}_{2n_f(4N-2)}$	2
$Spin(4N+2)$	$\mathbb{Z}_4$	Yes	$\mathbb{Z}_{8n_f N}$	4
E_6	$\mathbb{Z}_3$	No	$\mathbb{Z}_{24n_f}$	3
E_7	$\mathbb{Z}_2$	Yes	$\mathbb{Z}_{36n_f}$	2

Summary

- New anomalies can be understood in Hamiltonian formalism
- We showed that anomalies imply exact degeneracies at any finite volume (unexpected without anomalies)
- Can more concrete calculations of the IR spectra reveal something about how the anomalies force cancellation of semiclassical effects (dYM)?
- Could more degeneracy be implied by anomalies of non-invertible symmetries?

Thank you!