2404.16098 [Akers, JS]
2501.16437 [JS]

Information theory in the semiclassical limit

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History of black hole thermodynamics

- 1970-1972: In classical general relativity, the area of a black hole tends to increase [Christodoulou, Hawking]
- 1972: The combined black hole area + matter entropy of the universe tends to increase [Bekenstein]
- 1973: Black hole perturbations satisfy a “first law” [Bardeen, Carter, Hawking]
- 1975: Black holes radiate like blackbodies [Hawking]

Are black holes statistical systems?

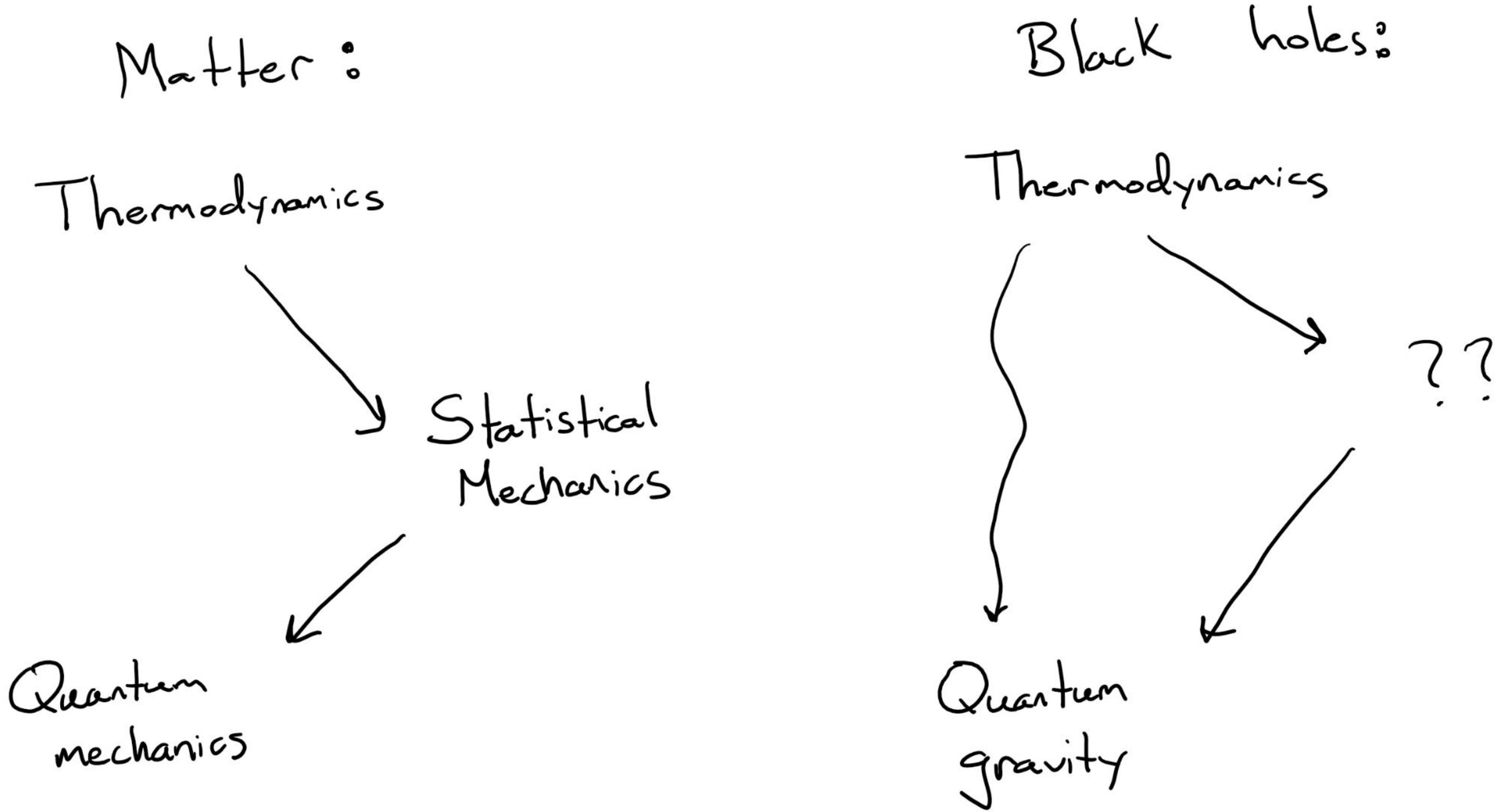
- Guess: In quantum gravity, black holes are complex systems with

$$\#\text{ microstates} = e^{A/4G_N}$$

- Lots of beautiful work verifying this in various settings:
    - BPS black holes in supergravity (D-branes, supersymmetric indices)
    - AdS_3/CFT_2 (Cardy formula for density of states)
    - Holographic gravitational path integral (diagonalization of nonperturbative overlaps)

- Strong evidence, but calculations strongly depend on setting.

Filling in a gap



Outline

- Boltzmann, Gibbs, and von Neumann
- Crossed product entropy
  [2021-2024; Witten, Chandrasekaran, Longo, Penington, Kolchmeyer, Jensen, JS, Speranza, Akers, Levine, Wildenhain, Kudler-Flam, Leutheusser, Satishchandran, Faulkner....]
- Relative state-counting for semiclassical black holes
  [Akers, JS 2024]
- Looking under the hood with classical stat mech
  [JS 2025]

Classical Ensembles

- Classically, a state is a point $(\vec{x}, \vec{p})$ in phase space.
- An ensemble is a probability distribution $P(\vec{x}, \vec{p})\, d\vec{x}\, d\vec{p}$; for example a thermal state of an ideal gas satisfying Maxwell-Boltzmann statistics.
- Classically, there is no preferred measure on phase space…

$$d\vec{x}\, d\vec{p}? \qquad 5 d\vec{x} d\vec{p}? \qquad \frac{1}{\pi\sqrt{e}} d\vec{x} d\vec{p}?$$

  so probability distributions are not canonically normalized, and are only defined up to a multiplicative factor.

- (In QM, $d\vec{x} d\vec{p}/(\hbar)^{\#}$ is preferred.)

Classical entropy

- The entropy of an ensemble is

$$S(P) \equiv -\int d\vec{x}\, d\vec{p}\, P(\vec{x},\vec{p}) \log P(\vec{x},\vec{p}),$$

and under

$$d\vec{x}\, d\vec{p} \mapsto C\, d\vec{x}\, d\vec{p} \qquad P(\vec{x},\vec{p}) \mapsto \tfrac{1}{C} P(\vec{x},\vec{p}),$$

have

$$S(P) \mapsto S(P) + \log C$$

Entropy difference

- Total entropy is not defined in classical stat mech, but entropy *difference* is.

- This is enough to explain thermodynamics: you can say one ensemble has **more entropy** than another, hence is statistically preferred, even if you don't know what the **total entropy** is.

Quantum statistical mechanics

- Quantumly, a pure state is a vector in Hilbert space.
- An ensemble, or mixed state, is a density operator.
- No dimensional ambiguity. The trace on Hilbert space provides a canonical normalization for states: $\mathrm{tr}(\rho) = 1$.

Quantum entropies

- Von Neumann entropy is unambiguous and absolutely defined:

$$S(\rho) \equiv -\mathrm{tr}(\rho \log \rho).$$

- For a microcanonical ensemble:

$$\rho \equiv \frac{\mathrm{Proj}_X}{\dim(X)}, \qquad S(\rho) = \log \dim X.$$

- The microcanonical quantum entropy is more or less what we think black hole entropy is getting at.

Going back to classical stat mech

- In a canonically quantized system, one has

$$\mathrm{tr} \sim \int \frac{dx\ dp}{\hbar}$$

- Taking the classical limit causes this expression to diverge — but this doesn't stop us from doing statistical mechanics, because we can drop the divergence and use regular old $dx\, dp$.

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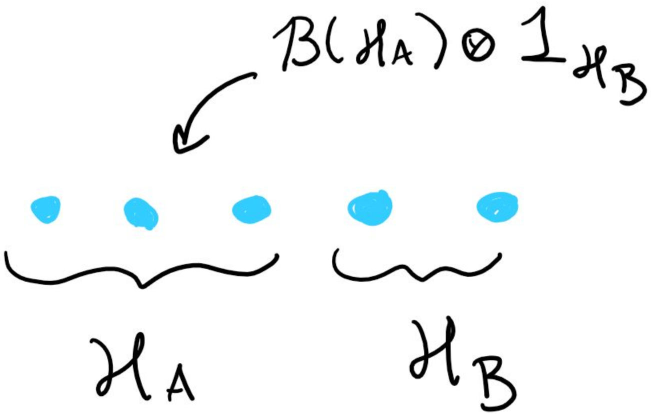
Algebras

- A general quantum system is described by a collection of operators, typically arranged into a **von Neumann algebra**.

e.g.



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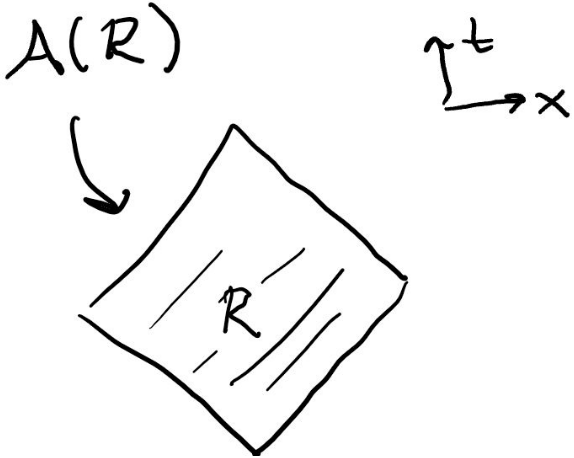


Density matrices

- Given a state $|\Psi\rangle$, a **density operator** for $|\Psi\rangle$ in the algebra $\mathcal{A}$ is a positive operator $\rho \in \mathcal{A}$ satisfying

$$\mathrm{tr}(\rho a) = \langle\Psi|a|\Psi\rangle, \qquad a \in \mathcal{A}.$$

- Some algebras don't have any density operators, since every positive operator has infinite trace.
    - (This is typical in QFT)

Renormalizing a trace?

- The non-existence of finite-trace operators in QFT seems like a problem for talking about entropy...

- But think about classical stat mech: the classical limit of $dx\,dp/\hbar$ is infinite, but the measures $C dx\,dp$ are perfectly good.

- Maybe we have something like

$$\mathrm{tr}_{\mathrm{QFT}} = \frac{1}{G_N}\mathrm{tr}_{QG},$$

  and we can still make sense of QG statistics in the QFT limit if we can get rid of the pesky divergent factor.

Traces

- Mathematicians have defined an abstract “trace” to be a map

$$\tau : \mathcal{A} \to \mathbb{C}$$

that is linear and cyclic, and has the additional “well behaved-ness” properties of being **faithful, normal, and semifinite**. We will call any of these a **renormalized trace**.

(see 2302.01958 for details)

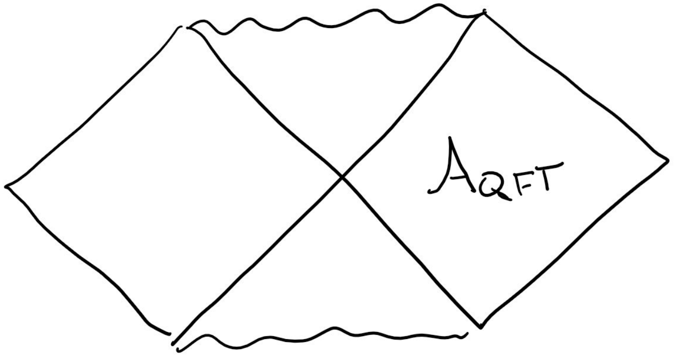
A beautiful observation

- Typically, quantum field algebras admit no renormalized traces. (“Type III”)

- In [Witten 2021; Chandrasekaran et al 2023], it was shown that for certain semiclassical algebras in perturbative quantum gravity, **a renormalized trace exists that is unique up to scaling**. (“Type II factor”) (see also [Jensen et al. 2023])

The explicit setting

- Start with the QFT algebra in the exterior of a Schwarzschild black hole:



- Add a Hilbert space $L^2(\Delta M)$ for G_N-suppressed mass fluctuations, and associated algebra $\mathcal{A}_{\text{mass}} \equiv \langle \Delta \hat{M} \rangle$.
- Impose a constraint:

$$\mathcal{A} \equiv \{a \in \mathcal{A}_{\text{QFT}} \otimes \mathcal{A}_{\text{mass}} \mid [H_{\text{QFT}} - \Delta \hat{M}, a] = 0\}.$$

The structure of this algebra

- This algebra describes a certain subsector of QG fluctuations around Schwarzschild.

- It has the structure of a **crossed product** [Takesaki 1973], and this structure allows one to show that the algebra has a unique family of renormalized traces.

- It even gives a formula! (Won't write it down; see [Jensen, Speranza, JS appendix C])

Upshot

- For states of the form $|\Phi_{\text{QFT}}\rangle \otimes |f_{\text{mass}}\rangle$, can find a density matrix in $\mathcal{A}$ and compute its entropy in a semiclassical approximation: [Chandrasekaran et al; Jensen et al; Kudler-Flam et al]

$$S = -S_{\text{QFT}}(\Phi||\Psi_{\text{HH}}) + \frac{\kappa}{2\pi}\langle\Delta\hat{M}\rangle_f - \int dx\, |f(x)|^2 \log|f(x)|^2 + \text{const}$$

- Regulating this formula gives

$$S = \frac{A}{4G_N} + S_{\text{matter}} + S_{\text{mass}} + \text{const}$$

Interpretation

- This structure lets you compute entropy differences in semiclassical gravity… but what do they mean? Do they have an interpretation as a count of states?
- In other words: **can we use this formalism to intuit that certain semiclassical black hole configurations have “more” or “fewer” states, even if the number of states is infinite?**
- Rest of this talk: Yes!

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Microcanonical density matrices

- We only really expect entropy to count states in a microcanonical ensemble.

- A microcanonical mass density matrix in our semiclassical algebra is

$$\rho_K \propto 1_{\mathrm{QFT}} \otimes \mathrm{Proj}_K \,,$$

  i.e., a (trace-normalized) projection onto a mass window K.

- We have

$$S(\rho_K) = -\tau\left(\frac{\mathrm{Proj}_K}{\tau(\mathrm{Proj}_K)} \log \frac{\mathrm{Proj}_K}{\tau(\mathrm{Proj}_K)}\right) = \log \tau(\mathrm{Proj}_K)$$

Entropy differences and trace ratios

- In our semiclassical algebra, only entropy differences are well defined, so we should only consider

$$S(\rho_{K_1}) - S(\rho_{K_2}) = \log \frac{\tau(\mathrm{Proj}_{K_1})}{\tau(\mathrm{Proj}_{K_2})}.$$

- Does this have a meaning?

- Yes!

Trace equality

- General theorem about type II factors [Murray-von Neumann]:

$$\tau(P) = \tau(Q) \quad \Leftrightarrow \quad P = VQV^\dagger, Q = V^\dagger PV, \text{ for some } V \in \mathcal{A}$$

- So we have $S(\rho_{K_1}) = S(\rho_{K_2})$ if and only if the microcanonical supports are isomorphic via an isomorphism contained the algebra.

- **"Contained in the algebra"** is important. We are in infinite dimensions, so any two infinite-dimensional Hilbert spaces are isomorphic by *some* isomorphism; the key here is that **the entropies of microcanonical black holes are equal iff there exists a gauge-invariant operator that maps between their supports.**

General statement

- Generally, we have

$$\mathcal{H}_{K_2} \otimes \mathbb{C}^n \hookrightarrow_{\mathcal{A}} \mathcal{H}_{K_1} \quad \Leftrightarrow \quad n \geq e^{\Delta S_{K_1,K_2}}$$

- Key phrase: **the "smaller" microcanonical space can be made to contain the "bigger" one iff you attach an auxiliary space with dimension at least as big as exp(entropy difference).**

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What’s going on?

- Our fundamental statement was: the entropy difference of two states is zero iff their supports are related by an isometry in the algebra.

- Where does this restriction “in the algebra” come from?

- Idea: All of the properties of the semiclassical theory should be avatars of the true, underlying quantum theory in the semiclassical limit.

- Idea: A unitary operator acting on the underlying quantum gravity Hilbert space should become an isometry “in the algebra” in the semiclassical limit. More general isometries should not arise via consistent limits of the underlying theory.

A helpful analogy

- Let’s return to the correspondence between quantum and classical stat mech.

- In classical stat mech, a “microcanonical state” is a uniform probability distribution on a portion of phase space:

Symplectomorphisms

- The entropy difference of two "microcanonical states" is related to the ratio of the phase space volumes:

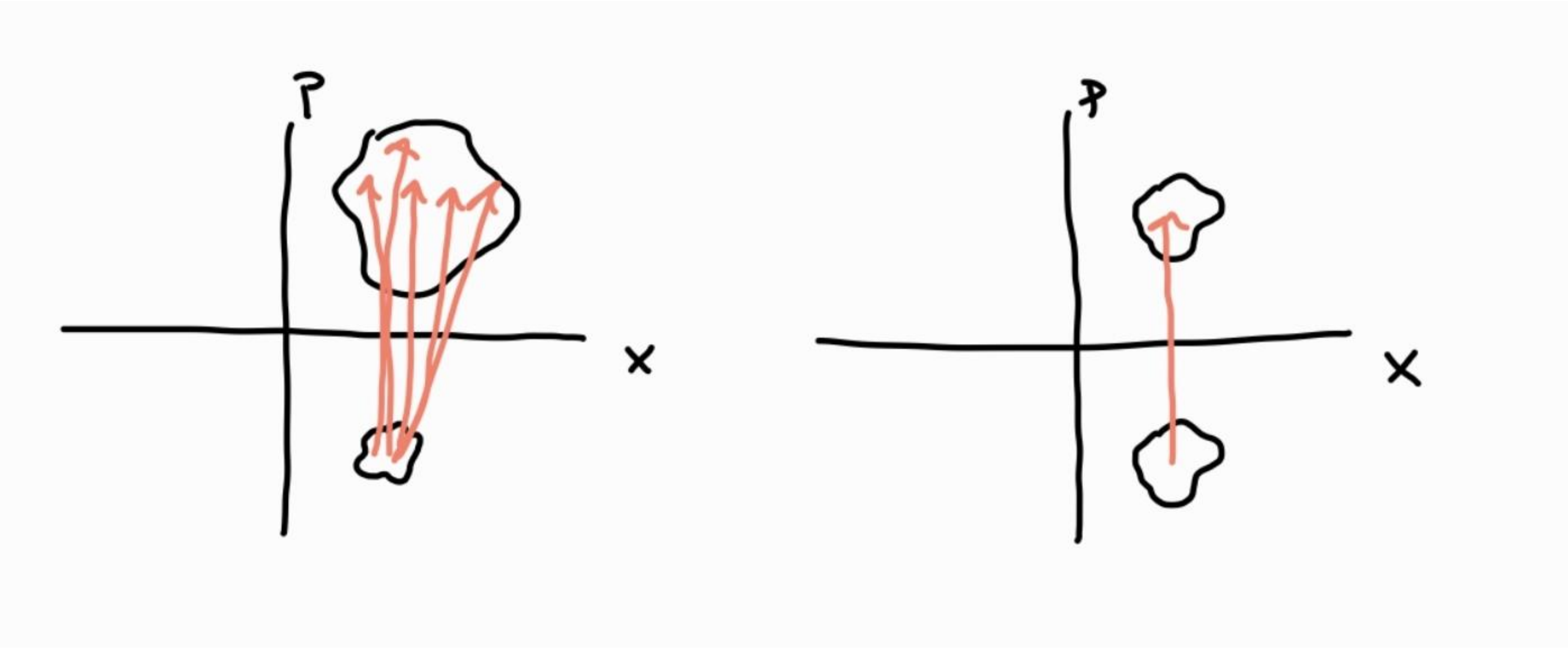
$$\Delta S = \log \frac{\mathrm{Vol}_1}{\mathrm{Vol}_2}$$

- The individual volumes are ill defined, but the ratio is independent of the choice

$$\lambda \, dx \, dp$$

- Any two phase space regions are related by a *diffeomorphism* — but they cannot be related by a *symplectomorphism* unless their volumes are the same

Figures



Revisiting the analogy

- This is exactly like the setting in semiclassical QG: in order for two microcanonical states to have the same entropy, they must be related by a map that preserves the fundamental structure of the theory — in this case, a symplectomorphism.

- In this setting, can we see how the symplectomorphism restriction arises from the underlying quantum theory?

- Yes! [JS 2025]

Sketch of 2501.16437

- Quantum mechanics can be written explicitly as a deformation of classical mechanics using the technology of *Wigner functions*.

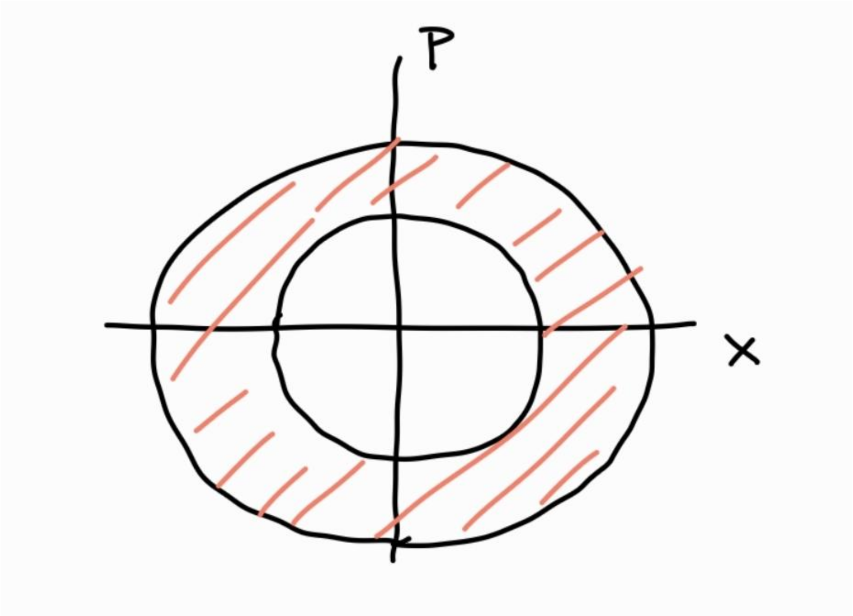
- In this language, density matrices are written as “quasiprobability distributions” on phase space that become true probability distributions in the classical limit.

- A microcanonical state for the simple harmonic oscillator is of the form

$$\rho^{(E_-,E_+)} \sim \sum_{n=E_-/\hbar\omega-1/2}^{E_+/\hbar\omega-1/2} |n\rangle\langle n|$$

- In the classical limit, the Wigner function converges to a uniform probability distribution on phase space.

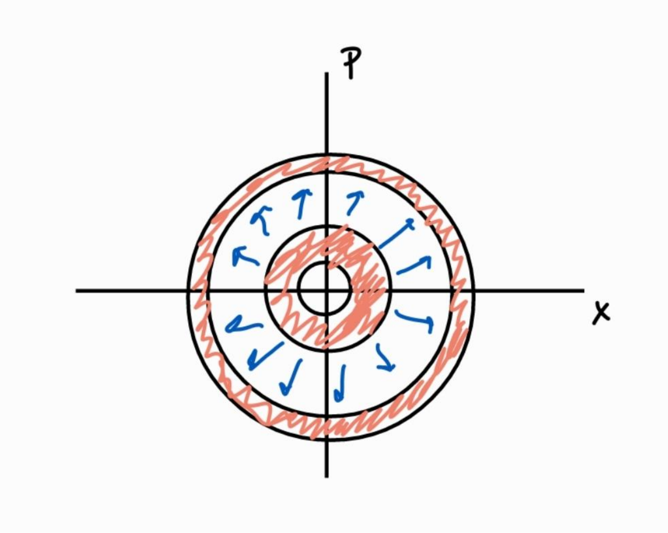
Figure



Origin of symplectomorphisms? [1/2]

- If two annuli have the same phase space volume, then they are related by a symplectomorphism:

$$(x,p) \mapsto \left(x\sqrt{1+\frac{\Delta E}{H(x,p)}}, p\sqrt{1+\frac{\Delta E}{H(x,p)}}\right)$$



Origin of symplectomorphisms? [2/2]

- In this example, one can explicitly show that this symplectomorphism is the classical limit of the quantum isometry

$$V|n\rangle = \left|n + \frac{\Delta E}{\hbar\omega}\right\rangle$$

- For details, see [JS 2025]. Basic idea is to compute how this acts on the position and momentum operators in the classical limit.

- **Upshot: Symplectomorphisms arise as classical limits of isometries in the underlying theory; I think a similar thing is happening in the information theory of the semiclassical limit of QG.**

Summary

- Recent developments have created a “bottom-up” technology for computing entropy differences of black hole fluctuations in the semiclassical limit.

- Tools from operator theory let us interpret these entropy differences as counting the “relative index of infinity” by telling us how much bigger one (infinite-rank) microcanonical state is than another.

- Everything about this construction should arise by taking a consistent limit of a fundamental theory of quantum gravity, but we can’t control many of these explicitly.

- In a simple analogy, everything works out in a natural, intuitive way that provides a guide for understanding how semiclassical physics arise from the underlying theory of quantum gravity.