Quasinormal Corrections to Near-Extremal Black Hole Thermodynamics

Dan Kapec

Harvard University

Outline

- Review: old puzzles about cold black holes
- Low temperature quantum corrections to Kerr thermodynamics
- Questions about the calculation
- Rotating BTZ: $T^{3/2}$ from the full determinant, lessons for Kerr
- Derivation of the DHS formula from the BTZ spectral measure

First half based on [2310.00848] with Sheta, Strominger, Toldo

Second half based on [2409.14928] with Albert Law, Chiara Toldo

See also: interesting work [2409.16248] by [Kolanowski, Marolf, Rakic, Rangamani, Turiaci]

Introduction and Motivation

There is a very simple list of black holes in 4d general relativity: (M, J, Q)

Today we think of this simplicity as due to complexity. Black holes are simple like statistical ensembles are simple, not simple like hydrogen atoms

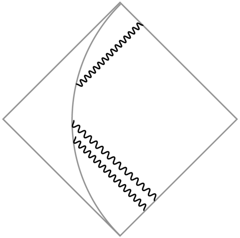
This analogy between the behavior of black holes and the laws of thermodynamics turned out to be extremely powerful.

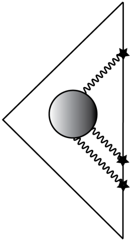
The laws of thermodynamics are not fundamental laws: they have a statistical, microscopic origin. They emerge from other, more fundamental laws of nature whose dynamics are complex and ergodic.

There is a lot of evidence that the geometric black hole entropy also has a microscopic, quantum mechanical origin. **Geometry from quantum mechanics.**

Most of the evidence comes from the AdS/CFT correspondence.

The idea that led to the AdS/CFT correspondence was to study the response of the black hole to low energy probes in two ways.





Replace the probes with sources at the boundary of AdS, do the calculation in the throat geometry, and then match back onto the far region.

But if the BH is really a quantum system, and the dynamics is occurring right outside the black hole, then you can do the calculation another way by studying the response of the quantum system to these probes.

In the best understood examples we can do the calculation on both sides and they match. In the other cases we use this idea to learn about the QM.

Identifying the quantum mechanical system provides a statistical interpretation of the black hole entropy [Strominger, Vafa]

$$e^{S_{\text{BH}}} = \text{number of states}$$

Since the black hole was extremal, what was counted were the ground states in the quantum mechanics. The extremal black hole has zero temperature but finite entropy (at tree level).

That is slightly nonstandard thermodynamic behavior. Most systems do not have a huge ground state degeneracy unless there is a symmetry.

In the examples that were understood, there was extra supersymmetry coming from the superstring and that protected the degeneracy.

It is impossible to make the 4d Kerr black hole supersymmetric. Yet the leading order calculation predicts huge ground state degeneracy at $T = 0$. **Do quantum corrections lift the ground states?**

[Preskill, Schwarz, Shapere, Trivedi, Wilczek '91] noted that the statistical description should break down when the specific heat becomes order one.

$$S(T,J) = S_0 + 8\pi^2 J^{3/2} T + O(T^2) \, , \qquad C = T\frac{\partial S}{\partial T} \sim 8\pi^2 J^{3/2} T$$

The specific heat controls the size of thermodynamic fluctuations in non-equilibrium processes.

So at temperatures $T \sim J^{-3/2}$ the emission of a single Hawking quantum can lead to relatively large fluctuations in temperature.

Above this temperature we trust the thermodynamic description of the black hole. Hawking evaporation can be treated as a small effect.

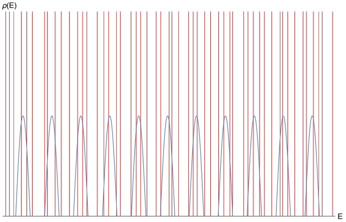
Below this temperature we need a better approximation to the black hole partition function to know what happens. That is what we compute.

If the spectrum of the black hole had an energy gap $E_{gap} \sim J^{-3/2}$ then we wouldn't expect to be able to apply thermodynamics below that temperature anyway. You need a dense band of states to coarse grain.

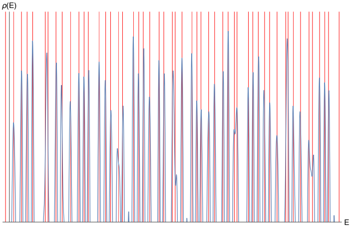
Another possibility is that the e^{S_0} ground states get lifted and fill out a dense energy band above the vacuum. In that case the thermodynamic description would extend to lower temperatures.

The black hole is large, curvatures are small, but the leading-order semiclassical analysis receives important quantum corrections either way.

We know that BH thermodynamics involves course graining. We don't know the exact density of states, we only have a smooth approximation to it. Including more corrections will change the density of states.



Coarse approximation



With subleading corrections

The AdS/CFT dictionary relates the partition function $Z_{QM}(\beta) = \text{Tr}e^{-\beta H}$ in quantum mechanics to the gravitational path integral in the AdS throat

$$Z_{QM}(\beta) = Z_{Grav}(\beta)$$

$Z_{Grav}(\beta)$ means we integrate over metrics and matter fields in AdS, subject to some boundary conditions fixed by the ensemble.

So we are trying to do the functional integral

$$Z_{Grav}(\beta) = \int [Dg] e^{-S[g]} \,, \qquad \text{with} \qquad g \to \bar{g}(\beta) \quad \text{at the boundary}$$

and the only way we know how to treat this integral is via saddle point.

Saddle point means solution to the Einstein equation subject to the boundary condition, and the solution is NHEK (near-horizon extreme Kerr)

$$ds^2 = J(1+\cos^2\theta)(-\sinh^2\eta dt^2 + d\eta^2 + d\theta^2) + \frac{4J\sin^2\theta}{1+\cos^2\theta}(d\phi + [\cosh\eta - 1]dt)^2$$

At zero temperature this computation reproduces the extremal Kerr entropy

$$Z_{Grav} \sim e^{2\pi J} \sim e^{S_0}$$

The first correction comes from integrating over the quantum fluctuations about the saddle (divergent in local QFT).

In a UV complete theory of quantum gravity this would be finite. But we can't solve string theory in NHEK so we have to use EFT.

There is some universal information in this 1-loop correction. Part of the correction comes from short distance UV physics, but part of it comes from the low-energy fluctuations of massless fields.

The massless fields generate universal "log corrections" to the entropy

$$S_{BH} = S_0 + \# \log S_0 + \frac{\#}{S_0} + \cdots$$

These corrections were computed by Sen and collaborators in a number of papers and matched to microscopic calculations in string theory [Sen '14].

There is a subtlety in Sen's calculation, which turns out to be irrelevant for SUSY black holes but crucial for non-SUSY black holes.

The 1-loop approximation amounts to calculating a functional determinant.

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}} \qquad \int \prod_i e^{-\lambda_i x_i^2} dx_i = \sqrt{\prod_i \frac{\pi}{\lambda_i}}$$

Write $g = \bar{g}_{NHEK} + h$ and expand the action to quadratic order

$$Z \sim e^{2\pi J} \int [Dh] e^{-\int h(x) \mathcal{D} h(x)}$$

$\mathcal{D}$ is a 2^{nd}-order linear differential operator, an infinite dimensional matrix.

If we decompose h into orthogonal eigenvectors $h = \sum c_i h_i$ of $\mathcal{D}$ then $\int h \mathcal{D} h = \sum \lambda_i c_i^2$ and the path-integral computes a determinant

$$\int [Dh] e^{-\int h(x) \mathcal{D} h(x)} \sim \frac{1}{[\text{det}\mathcal{D}]^{1/2}}$$

For NHEK the differential operator is simply

$$h_{\alpha\beta} D^{\alpha\beta,\mu\nu}_{\text{NHEK}} h_{\mu\nu} = -\frac{1}{16\pi} h_{\alpha\beta} \left(\frac{1}{4} \bar{g}^{\alpha\mu} \bar{g}^{\beta\nu} \bar{\Box} - \frac{1}{8} \bar{g}^{\alpha\beta} \bar{g}^{\mu\nu} \bar{\Box} + \frac{1}{2} \bar{R}^{\alpha\mu\beta\nu} \right) h_{\mu\nu}$$

So the 1-loop correction is a regularized determinant. Sen extracted some universal information from it and compared to quantum mechanics.

But there can be a subtlety if $\mathcal{D}$ has zero modes

$$\int_{-\infty}^{\infty} e^{-0x^2} dx = \infty$$

And it turns out that $\mathcal{D}$ actually has infinitely many zero modes

$$h^{(n)} \sim (1 + \cos^2\theta) e^{in\tau} \frac{(\sinh\eta)^{n-2}}{(1 + \cosh\eta)^n} \left(d\eta^2 + 2i \sinh\eta d\eta d\tau - \sinh^2\eta d\tau^2 \right)$$

Perfectly explicit and normalizable. So the path integral is IR divergent.

This signals a subtlety in the extremal limit for black holes. The problem doesn't arise for higher-dimensional black branes and their $AdS_{d>2}$ throats.

Treated quantum mechanically, this mode dramatically alters the low temperature thermo. This was all put together for Reissner-Nordstrom in [Iliesiu, Murthy, Turiaci '22], see also [Banerjee, Saha '23]. We adapted it to Kerr, see also [Rakic, Rangamani, Turiaci '23]. More general cases: [Maulik, Pando Zayas, Ray, Zhang '24]

The idea is to take the scaling limit to isolate the NHEK geometry, but keep the subleading term

$$\bar{g} = g_{NHEK} + T\delta g$$

The temperature is treated as a small parameter. The subleading correction couples NHEK to the far region but is very complicated.

$$\begin{aligned}
\frac{\delta g_{\mu\nu}dx^\mu dx^\nu}{4\pi J^{3/2}T} &= (1+\cos^2\theta)(2+\cosh\eta)\tanh^2\frac{\eta}{2}(d\eta^2 - \sinh^2\eta d\tau^2) + \\
&\frac{2\sin^2\theta}{1+\cos^2\theta}(\cosh\eta - 1)\left((\sin^2\theta\sinh^2\eta - 3) - 4\frac{\cos^2\theta}{1+\cos^2\theta}\cosh\eta(\cosh\eta-1)\right)d\tau^2 \\
&+ \frac{2i\sin^2\theta}{1+\cos^2\theta}\left((\sin^2\theta\sinh^2\eta - 3) - 8\frac{\cos^2\theta}{1+\cos^2\theta}\cosh\eta(\cosh\eta-1)\right)d\tau d\phi \\
&+ 8\cosh\eta\frac{\sin^2\theta\cos^2\theta}{(1+\cos^2\theta)^2}d\phi^2 + \sin^2\theta\cosh\eta(d\eta^2 + \sinh^2\eta d\tau^2) + 2\cosh\eta\, d\theta^2 \ .
\end{aligned}$$

The metric perturbation induces a perturbation in the kinetic operator

$$\mathcal{D} \to \mathcal{D} + \delta\mathcal{D}$$

Using perturbation theory to compute the eigenvalues

$$\delta\lambda_n = \int d^4x\sqrt{\bar{g}}h_n\delta\mathcal{D}h_n$$

one finds

$$\delta\lambda_n = \frac{3nT}{64\sqrt{J}}$$

This is because

$$h^{(n)} = \mathcal{L}_{\xi^{(n)}} g_{\textsf{NHEK}} \qquad h^{(n)} \neq \mathcal{L}_{\zeta} g_{\textsf{not-NHEK}}$$

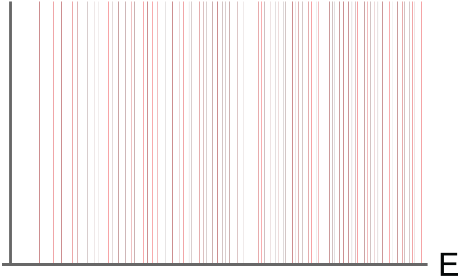
So the finite temperature lifts the eigenvalues and regulates the partition function. Computing the determinant gives

$$\delta \log Z \sim \log \prod_{n=2}^{\infty} \frac{1}{nT} \sim \frac{3}{2}\log T$$

So $Z[T]$ is becoming small at low temperatures, not exponentially large: the ground state degeneracy has been lifted.

$$Z[T] \sim T^{3/2} e^{S_0} \qquad \text{as} \qquad T \to 0$$

Instead the states fill out a dense energy band above the vacuum



We expect the eigenvalue spacing in this region of the spectrum to be roughly $e^{-S_0} \sim e^{-1/G_N}$ which is non-perturbatively small. Thermodynamics still applies.

Recap

For many questions, the leading approximation to the black hole density of states, as computed using the Euclidean black hole saddle, is sufficient.

$$Z_{AF}(\beta,\mu,\Omega) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\substack{\text{Asymptotically flat metrics with} \\ (\beta,\mu,\Omega)\text{ boundary conditions at } i^0}} \quad \sim \quad \frac{1}{\sqrt{\det -\nabla^2}} \exp\left[-I_{\text{on-shell}}\right]$$

Because the exponential term is so large, the tree level calculation typically dominates the thermodynamics.

We view the determinant as a small correction which in most circumstances does not change the qualitative behavior of the thermodynamic system.

Recent observation: the gas of gravitons at low temperatures in a black hole background becomes important even when curvatures are small.

[Preskill, Schwarz, Shapere, Trivedi, Wilczek '91] identified this problem, but did not resolve it because they could not calculate the one loop correction to $Z_{AF}(\beta)$.

In the meantime we learned that for low temperatures and certain black brane observables, we can replace $Z_{AF}(\beta)$ with a throat path integral

$$Z_{\text{throat}}(\beta, \mu, \Omega) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\substack{\text{Asymptotically AdS}_{d+1}\text{ metrics with} \\ (\beta,\mu,\Omega)\text{ boundary conditions at }\partial\text{AdS}}}$$

That is basically the AdS/CFT duality, but it is subtle for AdS_2. Sen found

$$Z_{\text{throat}}(\beta = \infty, Q, J) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\substack{\text{Asymptotically AdS}_{2}\text{ metrics with} \\ (\beta,Q,J)\text{ boundary conditions at }\partial\text{AdS}}} \quad \sim \infty \times e^{S_0 + c\log S_0}$$

We interpret the infinity as an infrared divergence due to an unsupressed Goldstone mode. We regulate it by turning on an irrelevant deformation.

Questions and concerns

So the quantity that we actually compute is a regularized partition function in the deformed "not-NHEK" throat.

$$Z_{\text{reg}}(\beta, Q, J) \quad = \quad \underbrace{\int [Dg]\, e^{-I_{EH}-I_{GH}-I_{ct}}}_{\text{Asymptotically ``not-NHEK'' metrics}} \quad \sim \quad T^{3/2} e^{S_0 + c \log S_0}$$

But this calculation involves several subtle assumptions.

The whole effect comes from an integral over diffeomorphisms with non-compact support, which depends delicately on boundary conditions.

There can be ambiguities in connecting the far region to the throat.

Modes which are (non)normalizable in the throat might not complete to (non)normalizable modes in the full asymptotically flat geometry.

Example: the source and response terms for the gauge field flip.

Climbing out of the throat

Could the large diffeomorphisms in AdS_2/NHEK extend to diffeomorphisms with compact support in the full Kerr geometry?

Or can we show that they complete to physical non-zero modes in Kerr which simply reduce to diffeomorphisms in the throat region?

Given these apparent subtleties, one would like to reproduce the $T^{3/2}$ behavior using the full asymptotically flat geometry and verify that the contribution of the zero modes to the throat calculation is physical.

Seems hard: we cannot even perform the full not-NHEK path integral, only the piece responsible for the $T^{3/2}$ behavior.

The calculation in near-extremal Kerr is more complicated.

The DHS formula

How are we going to get a discrete factor like $\prod \frac{1}{nT}$ without the throat?

Clue: even when the spectrum of $-\nabla^2$ on eigenfunctions is continuous, the spectrum of **resonances** is discrete. *Resonances replace eigenfunctions*.

There is an interesting formula due to [Denef, Hartnoll, Sachdev '09] which expresses the **Euclidean** determinant in terms of the **Lorentzian** quasinormal modes

$$\frac{1}{\sqrt{\det(-\nabla^2)}} = \prod_{k,l\in\mathbb{Z}} \prod_{z_l} \left(\omega_{|k|,l} + i z_l\right)^{-1/2}$$

Here the z_l are the quasinormal modes of the field whose determinant we are calculating. The $\omega_{k,l}$ are the Matsubara frequencies

$$\omega_{k,l} = \frac{2\pi k}{\beta} - i\Omega l$$

These frequencies are required for periodicity on the Euclidean section.

The derivations of this formula are indirect. They assume certain analytic properties of the determinant.

Argument: The left hand side is a determinant. Let's assume that the spectrum is discrete and generalize the problem to allow for a mass.

Then we can formally write

$$\frac{1}{\det(-\nabla^2 + m^2(\Delta))} = \prod_j \frac{1}{\lambda_j(\Delta)}$$

where the λ_j are discrete eigenvalues of the Euclidean kinetic operator. The eigenvalues depend on Δ.

When can the left hand side have a pole? When there is an L^2 eigenfunction with a zero eigenvalue.

But if a Euclidean mode has eigenvalue $\lambda = 0$, then its Lorentzian continuation solves the Lorentzian equation of motion (and is a QNM).

So there is a pole whenever we have a quasinormal mode with a nonsingular Euclidean continuation.

That means that the real and imaginary parts of the QNM frequency must match the Matsubara frequency.

This does not happen for generic Δ, so you have to tune the mass to make a pole. Matching the poles as a function of Δ gives the DHS formula.

Strong assumptions: analyticity, discrete spectrum, $\cdots$

Every time it has been used to recompute a determinant it always agreed, modulo some important subtleties for **spinning fields** crucial for the $T^{3/2}$.

Later I will explicitly derive this formula for the BTZ black hole and we will see what the right hand side is actually computing.

So the DHS formula has a chance of producing a discrete product, as we encounter in the throat.

But it does not obviously simplify the problem since we cannot analytically compute the full quasinormal spectrum for black holes in $D > 3$.

However, there is a particular branch of quasinormal modes whose frequencies can be computed analytically and which are closely related to the existence of the throat in the near-extremal Kerr geometry.

These "lightly-damped" modes have real parts that accumulate at the superradiant bound and small imaginary parts spaced evenly in units of the temperature

$$\omega = m\Omega_H - 2\pi i T_H(n + 1/2)$$

The imaginary parts are small precisely because waves with $\omega = m\Omega_H$ penetrate and spend a long time in the throat region.

This is **the** spectral signature of the conformal symmetry of NHEK.

Cartoon: Manipulate the DHS formula to separate out the contribution of the throat region, discarding the rest of the terms that don't really have anything to do with extremality

$$Z(\beta,\Omega) = \left[\prod_{\text{throat piece}}\right]\left[\prod_{\text{All other QNM}}\right] \sim T^{3/2}\left[\prod_{\text{All other QNM}}\right] ???$$

The second incalculable term will correspond to a nonuniversal contribution which does not have a singular limit as $T \to 0$ since it is not really sensitive to the geometry near the throat.

I will discuss this calculation for near-extremal BTZ. In this case, we know:

- the quasinormal mode spectrum exactly [Datta, David '11]
- the 1-loop Euclidean determinant [Maloney, Witten '07; Giombi, Maloney, Yin '08]
- the relevant limit of the corresponding CFT_2 character which reproduces the $T^{3/2}$ behavior [Ghosh, Maxfield, Turiaci '19]

The DHS formula has already been applied in this case [Datta, David '11; Castro, Keeler, Szepietowski '17] although the low temperature limit was not investigated.

We want to take the low temperature limit of the determinant in quasinormal variables to see what aspects of the spectrum are responsible for the scaling. Are they also there in Kerr?

It turns out to be crucial that certain QNMs for spinning fields do not continue to regular Euclidean solutions with low Matsubara frequencies [Datta, David '11; Castro, Keeler, Szepietowski '17; Grewal, Law, Parmentier '22].

Their exclusion from the DHS product formula plays the same role as the exact $SL(2,\mathbb{R})$ symmetry in the extremal throat. Should be true in Kerr.

Other parts of the calculation do not have obvious analogs in Kerr.

The zero temperature Euclidean throat for rotating BTZ is $(r_0^2 = \ell J/2)$

$$ds^2_{\text{throat}} = \ell^2 \sinh^2\left(\frac{\eta}{2}\right) d\bar{t}^2 + \frac{\ell^2}{4} d\eta^2 + i\ell^2 r_0 (1 - \cosh\eta) d\bar{t} d\phi + \ell^2 r_0^2 d\phi^2$$

It has zero modes

$$h^{(n)} \sim e^{in\bar{t}} \frac{(\sinh\eta)^{|n|-2}}{(1+\cosh\eta)^{|n|}} \left(d\eta^2 + 2i\frac{n}{|n|}\sinh\eta\, d\eta\, d\bar{t} - \sinh^2\eta\, d\bar{t}^2\right)$$

The corrected geometry is

$$\begin{aligned}\delta g_{\mu\nu} dx^\mu dx^\nu = &\frac{\ell^4\pi}{2r_0}\sinh^4\left(\frac{\eta}{2}\right) d\bar{t}^2 + \frac{\ell^4\pi}{8r_0}(2+\cosh\eta)\tanh^2\left(\frac{\eta}{2}\right) d\eta^2 \\ &+ \ell^4\pi r_0 \cosh\eta d\phi^2 - i\frac{\ell^4\pi}{2}(\cosh\eta - 3)\sinh^2\left(\frac{\eta}{2}\right) d\bar{t} d\phi\end{aligned}$$

At first order in perturbation theory

$$\delta\lambda^{(n)} = nT/32r_0 \qquad\qquad \delta\log Z_{\text{throat}} \sim \frac{3}{2}\log T$$

The full BTZ determinant

[Maloney, Witten '07; Giombi, Maloney, Yin '08] calculated the graviton 1-loop determinant in the full BTZ geometry.

The calculation makes use of the fact that the Euclidean BTZ geometry is the modular transform of the thermal AdS_3 geometry: $\tau \to -1/\tau$.

The thermal AdS determinant is simply the identity character in CFT_2

$$Z^{\text{graviton}}_{\text{TAdS}_3}(\tau, \bar{\tau}) = \chi_1(\tau)\chi_1(\bar{\tau}) \,, \qquad \chi_1(\tau) = \frac{(1-q)q^{\frac{1-c}{24}}}{\eta(\tau)}$$

This was argued indirectly in [Maloney, Witten '07], verified in [Giombi, Maloney, Yin '08] using heat kernel techniques and the method of images (thermal AdS and BTZ are $\mathbb{Z}$ quotients of $\mathbb{H}_3$).

[Ghosh, Maxfield, Turiaci '19] took the low-T limit of the modular transform of this character and got a $T^{3/2}$.

In terms of the left and right temperatures of BTZ

$$\frac{2}{T} = \frac{1}{T_L} + \frac{1}{T_R} \,, \qquad \Omega = \frac{T_R - T_L}{T_R + T_L}$$

the limit is

$$T_L \to 0 \,, \qquad T_R \to \infty$$

The scaling

$$Z_{BTZ} \sim 2\pi \left(\frac{2\pi}{\beta_L}\right)^{3/2}$$

comes from χ, the left moving character is responsible for the $T^{3/2}$. Not interested in the exponentials.

If you are only interested in the $T^{3/2}$, there is a quick manipulation that uses a different representation. Ignoring the tree-level piece

$$Z_{BTZ} \sim \prod_{n=2}^{\infty} \frac{1}{1-q^n} \prod_{n=2}^{\infty} \frac{1}{1-\bar{q}^n} \qquad q = e^{-(2\pi)^2 T_L} \ , \quad \bar{q} = e^{-(2\pi)^2 T_R}$$

That looks like the determinant we calculated in the throat. To see that expand $q \sim 1-(2\pi)^2 T_L$

$$Z_{BTZ} \quad \sim \quad \prod_{n=2}^{\infty} \frac{1}{nT_L} \prod_{n=2}^{\infty} \frac{1}{1-\bar{q}^n} \quad \sim \quad T^{3/2} \times \prod_{other} [\cdots]$$

We want to understand how to account for this structure using quasinormal modes.

Warmup: Scalar Determinant on BTZ

The DHS formula was

$$Z = \prod_{k,l\in\mathbb{Z}} \prod_{z_l} \left(\omega_{|k|,l} + i z_l\right)^{-1/2}$$

It is often more useful to convert to an integral representation using

$$\log x = -\int_0^\infty \frac{dt}{t} e^{-xt}$$

Using that identity term by term we get an integral representation of DHS

$$\log Z = \int_0^\infty \frac{dt}{2t} \coth\frac{\pi t}{\beta} \chi(t) \,, \qquad \chi(t) = \sum_l \sum_{z_l} e^{i(l\Omega_H - z_l)t}$$

This quantity $\chi(t)$ is sometimes called the "quasinormal character." If you expand the $\coth\frac{\pi t}{\beta}$ in exponentials, it does the sum over the Matsubara's.

The QNM spectrum for a scalar in BTZ is [Birmingham, Sachs, Solodukhin '02]

$$\omega_{nl}^L = l - 2\pi i T_L \left(2n + \Delta\right) \quad \text{and} \quad \omega_{nl}^R = -l - 2\pi i T_R \left(2n + \Delta\right)$$

The "left branch" looks a lot like the lightly damped QNM we have in Kerr, while the "right branch" doesn't seem to have much to do with extremality.

Now we want to calculate the quasinormal mode character

$$\chi(t) = \sum_{n=0}^{\infty} \sum_{l \in \mathbb{Z}} \left(e^{i\left(l\Omega_H - \omega_{nl}^L\right)t} + e^{i\left(l\Omega_H - \omega_{nl}^R\right)t} \right)$$

Using the relations

$$\Omega_H - 1 = -\frac{2\pi T_L}{r_+}\,, \qquad \Omega_H + 1 = \frac{2\pi T_R}{r_+}$$

we can write this as

$$\chi(t) = \frac{e^{-2\pi T_L \Delta t}}{1 - e^{-4\pi T_L t}} \sum_{l \in \mathbb{Z}} e^{-il\frac{2\pi T_L}{r_+}t} + \frac{e^{-2\pi T_R \Delta t}}{1 - e^{-4\pi T_R t}} \sum_{l \in \mathbb{Z}} e^{il\frac{2\pi T_R}{r_+}t}$$

Now we use Poisson summation

$$\sum_{l\in\mathbb{Z}} e^{\pm i2\pi l\tau} = \sum_{p\in\mathbb{Z}} \delta\left(\tau - p\right)$$

to exchange the angular momentum for a dual variable

$$\chi(t) = \frac{e^{-2\pi T_L \Delta t}}{1 - e^{-4\pi T_L t}} \frac{r_+}{T_L} \sum_{p\in\mathbb{Z}} \delta\left(t - \frac{p r_+}{T_L}\right)$$
$$+ \frac{e^{-2\pi T_R \Delta t}}{1 - e^{-4\pi T_R t}} \frac{r_+}{T_R} \sum_{p\in\mathbb{Z}} \delta\left(t - \frac{p r_+}{T_R}\right)$$

and use the delta functions to do the integral. We get

$$\log Z = \sum_{p=1}^{\infty} \frac{1}{p} \frac{q_L^{\frac{p\Delta}{2}}}{1 - q_L^p} \frac{q_R^{\frac{p\Delta}{2}}}{1 - q_R^p}\,, \qquad q_{L/R} \equiv e^{-(2\pi)^2 T_{L/R}}$$

So we have

$$\log Z = \sum_{p=1}^{\infty} \frac{1}{p} \frac{q_L^{\frac{p\Delta}{2}}}{1-q_L^p} \frac{q_R^{\frac{p\Delta}{2}}}{1-q_R^p}\,, \qquad q_{L/R} \equiv e^{-(2\pi)^2 T_{L/R}}\,.$$

Expand the denominators using the geometric series, and do the $\sum_p$ using $\sum_p \frac{x^p}{p} = -\log(1-x)$. We recover the known formula

$$Z = \prod_{n_L, n_R = 0}^{\infty} \frac{1}{1 - q_L^{\frac{\Delta}{2}+n_L} q_R^{\frac{\Delta}{2}+n_R}}$$

The same set of steps will allow us to derive the graviton determinant, with two important changes:

- we must include ghost determinants
- for spinning fields, a finite subset of QNM do not continue to regular solutions and must be excluded from the DHS product

Graviton determinant from QNM

The graviton determinant in a locally AdS_3 geometry can be expressed as the ratio of two determinants

$$Z_{\mathrm{grav}} = \frac{\det\left(-\nabla^2_{(1)} + 2\right)^{1/2}}{\det\left(-\nabla^2_{(2)} - 2\right)^{1/2}}$$

The determinant on the bottom is for a field with $s = 2, \Delta = 2$ while the one on top is $s = 1, \Delta = 3$.

First, the naive application of the DHS formula. The (s, Δ) QNM spectrum

$$\omega_{nl}^{\Delta,s,L,\mp} = l - 2\pi i T_L \left(2n + \Delta \mp s\right) \qquad \omega_{nl}^{\Delta,s,R,\mp} = -l - 2\pi i T_R \left(2n + \Delta \pm s\right)$$

A spin-s field has two independent degrees of freedom, so there are 4 branches instead of two for the scalar.

We are going to apply DHS to the numerator and the denominator

$$\log Z = \int_0^\infty \frac{dt}{2t} \frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}} \left(\chi_{[\Delta=2,s=2]}(t) - \chi_{[\Delta=3,s=1]}(t)\right)$$

So we need to compute a difference of two QNM characters. There are huge cancellations because the QNM of the two operators are related

$$\omega_{nl}^{2,2,L,+} = \omega_{nl}^{3,1,L,+}\,, \qquad \omega_{nl}^{2,2,R,-} = \omega_{nl}^{3,1,R,-}\,, \qquad \forall n = 0,1,\cdots\,, \quad l \in \mathbb{Z}$$

and

$$\omega_{n+1,l}^{2,2,L,-} = \omega_{n,l}^{3,1,L,-}\,, \qquad \omega_{n+1,l}^{2,2,R,+} = \omega_{n,l}^{3,1,R,+}\,, \qquad \forall n = 0,1,\cdots\,, \quad l \in \mathbb{Z}\,.$$

This comes from

$$\omega_{nl}^{\Delta,s,L,\mp} = l - 2\pi i T_L\left(2n+\Delta\mp s\right) \qquad \omega_{nl}^{\Delta,s,R,\mp} = -l - 2\pi i T_R\left(2n+\Delta\pm s\right)$$

The two branches that contribute are actually totally undamped modes

$$\omega_{n=0,l}^{2,2,L,-} = \ell \,, \qquad \omega_{n=0,l}^{2,2,R,+} = -\ell \,.$$

and we can easily calculate the difference of the characters

$$\chi_{[\Delta=2,s=2]}(t) - \chi_{[\Delta=3,s=1]}(t) = \frac{r_+}{T_L} \sum_{p\in\mathbb{Z}} \delta\left(t - \frac{pr_+}{T_L}\right) + \frac{r_+}{T_R} \sum_{p\in\mathbb{Z}} \delta\left(t - \frac{pr_+}{T_R}\right)$$

Do the integral

$$\log Z_{\mathsf{naive}} = \sum_{p=1}^{\infty} \frac{1}{p} \frac{1 - q_L^p q_R^p}{\left(1 - q_L^p\right)\left(1 - q_R^p\right)}$$

and expand the geometric series. We get a formula that is not quite right

$$\log Z_{\mathsf{naive}} = \sum_{p=1}^{\infty} \frac{1}{p} \left(1 + \sum_{n_L=1}^{\infty} q_L^{n_L p} + \sum_{n_R=1}^{\infty} q_R^{n_R p}\right)$$

We don't want the constant term, and the sums should start at $n = 2$.

Excluded modes

What happened? The DHS formula assumed that the Euclidean continuation of a QNM wavefunction with Matsubara frequency is a regular Euclidean eigenvector with zero eigenvalue.

This is always true for scalar fields. It is not always true for spinning fields. The $(L,-)$ and $(R,+)$ modes with $n < s - |k|$ are not square integrable [Datta, David '11; Castro, Keeler, Szepietowski '17; Grewal, Law, Parmentier '22]

So we have to exclude them from the DHS product, they cannot actually lead to poles. The excluded contributions are

$$\int_0^\infty \frac{dt}{2t} \sum_{k=-1}^{1} \sum_{n=0}^{1-|k|} \sum_{l\in\mathbb{Z}} e^{-\frac{2\pi|k|}{\beta}t} \left(e^{i\left(l\Omega_H - \omega_{nl}^{2,2,L,-}\right)t} + e^{i\left(l\Omega_H - \omega_{nl}^{2,2,R,+}\right)t} \right)$$
$$- \int_0^\infty \frac{dt}{2t} \sum_{l\in\mathbb{Z}} \left(e^{i\left(l\Omega_H - \omega_{n=0,l}^{3,1,L,-}\right)t} + e^{i\left(l\Omega_H - \omega_{n=0,l}^{3,1,R,+}\right)t} \right)$$

Following the usual steps we find we have to subtract the contribution

$$\log Z_{sing.} = \sum_{p=1}^{\infty} \frac{1}{p} \left(1 + q_L^p + q_R^p\right)$$

The naive determinant was

$$\log Z_{\text{naive}} = \sum_{p=1}^{\infty} \frac{1}{p} \left(1 + \sum_{n_L=1}^{\infty} q_L^{n_L p} + \sum_{n_R=1}^{\infty} q_R^{n_R p}\right)$$

So we lose the constant term, and the sums start at $n = 2$. That gives the expected determinant (modular transform of the CFT character).

Lessons so far:

- Only a tiny subset of the graviton QNM are needed to calculate the determinant due to cancellations with ghosts (probably true for Kerr)
- The only graviton QNM that contribute are totally undamped $\omega = \pm\ell$ That is unlikely to be true for Kerr
- The exclusion of certain modes for spinning fields is crucial for $T^{3/2}$

Which QNM contribute the $T^{3/2}$?

We had four branches of QNM for the graviton

$$\omega_{nl}^{\Delta,s,L,\mp} = l - 2\pi i T_L \left(2n + \Delta \mp s\right) \qquad \omega_{nl}^{\Delta,s,R,\mp} = -l - 2\pi i T_R \left(2n + \Delta \pm s\right)$$

Almost none of them contribute to the determinant for any temperature.

$$\omega_{n=0,l}^{2,2,L,-} = \ell \, , \qquad \omega_{n=0,l}^{2,2,R,+} = -\ell \, .$$

The ones that do have no imaginary part. You can check that it is actually the "right branch" that is responsible for the $T^{3/2}$.

$$\log Z^L = \frac{1}{2} \sum_p \frac{1}{p} \frac{1 + q_R^p}{1 - q_R^p} \qquad \log Z^R = \frac{1}{2} \sum_p \frac{1}{p} \frac{1 + q_L^p}{1 - q_L^p}$$

Since we know which modes account for the scaling, we can look for a shortcut in the product representation.

The naive product form of DHS using the undamped right moving modes is

$$(Z_{\text{right}})^2 = \prod_{k,l\in\mathbb{Z}} \frac{1}{2\pi|k|T - il\Omega_H - il} = \prod_{k,l\in\mathbb{Z}} \frac{1}{2\pi|k|\frac{2T_LT_R}{T_L+T_R} - \frac{2ilT_R}{T_L+T_R}}$$

We are actually supposed to exclude $k = 0, \pm 1$ from this product so

$$Z_{\text{right}} = \prod_{k>1}\prod_{l\in\mathbb{Z}} \frac{1}{2\pi k\frac{2T_LT_R}{T_L+T_R} - \frac{2ilT_R}{T_L+T_R}}$$

Infinite constants of the form $\prod_{l\in\mathbb{Z}} \frac{1}{A}$ where A is independent of l, can be absorbed into field redefinitions/local counterterms. So we have

$$Z_{\text{right}} = \prod_{k>1}\prod_{l\in\mathbb{Z}} \frac{1}{2\pi kT_L - il}$$

The low-T limit of this product exhibits $T^{3/2}$ scaling. To see this explicitly we separate off the $l = 0$ term so that the product becomes

$$Z_{\text{right}} = \left[\prod_{k>1} \frac{1}{2\pi kT_L}\right]\left[\prod_{k>1}\prod_{l>0} \frac{1}{l^2 + (2\pi kT_L)^2}\right]$$

Wrapping up

We derived the low temperature behavior of the black hole partition function using a throat calculation

$$Z_{BH} \sim T^{3/2} e^{S_0}$$

The result resolved some old questions about cold black holes, but the source of the effect was subtle.

In particular, the spectrum of fluctuations in the throat is very different from the spectrum of fluctuations in the full black hole geometry.

We derived the $T^{3/2}$ behavior from the full near-extremal BTZ geometry using the DHS formula, focusing on the aspects of the QNM spectrum responsible for the scaling.

Along the way we derived the spectral density for the BTZ black hole, which allowed for an explicit derivation of the DHS formula without assumptions.