Entanglement Entropy in Spacetime: Covariance and Discreteness

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Overview

Entropy

Entropy is the measure of the information one may glean from a system.

Given some distribution of possible states, each with a probability p_i, the entropy is defined as

$$S = -k_B \sum_i p_i \ln(p_i), \tag{1}$$

Entropy is also intimately connected with classical thermodynamics

$$\Delta S = \int_{\text{process}} \frac{dE + PdV}{T} \tag{2}$$

and can be used to probe internal structure at inaccessible scales

- Hydrogen doesn't have $S = 0$ at $T = 0$ (Giaque '30)
- Ice doesn't have $S = 0$ at $T = 0$ (Pauling '35)
- Light has an entropy change consistent with quantisation in frequency (Einstein '05)

Entanglement Entropy

Entanglement is one of the classic features of a quantum mechanics, resulting from noncommutativity.

Entanglement can be detected by entanglement entropy.

It has thus found myriad uses in non-relativistic quantum mechanics:

- Phase transitions, locality, and criticality
- Metrology
- Topological order
- Thermalisation
- Quantum information and computing

In a local state, entanglement entropy follows an area law scaling, not the conventional volume law.

Relativistic Entanglement Entropy

In a relativistic setting, is is largely used as a structure to probe and build fundamental physics.

It has found a place in:

- Black hole and horizon entropy / information paradox
- Energy conditions (QNEC)
- Entropic gravity
- The emergence of spacetime
- Holography
- ER=EPR

Black Hole Entropy

From classical thermodynamics arguments, we know black holes have an entropy

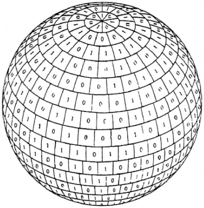
$$S = \frac{1}{4}A, \tag{3}$$

where A is the horizon area (Bekenstein '72, Hawking '74). This is unlike more familiar thermodynamic entropies, which have a volume law scaling.

In fact all horizons have an entropy, following the same area law.

Black Hole Entropy

From the scaling, we expect microstates that live on the horizon:



A Journey into Gravity and Spacetime - Wheeler '90

There have been countless proposals for the identity of the microstates responsible for this entropy, but they often lack the generality to be applied to all horizons. This has led to entanglement entropy taking its place as the leading candidate.

Nonrelativistic Entanglement Entropy

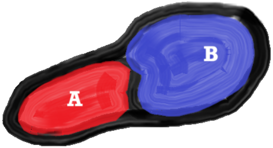
Entropy, in its von Neumann formulation, is given by

$$S = -\mathrm{Tr}(\rho \ln \rho). \tag{4}$$

Start with a pure state ρ, with $S = 0$ in a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$

Occlude (trace out) some physical degrees of freedom (states in the Hilbert space $\mathcal{H}_B$)

$$\rho_A = \mathrm{Tr}_B(\rho) \tag{5}$$



An entangled state, when the von Neumann entropy of ρ_A is calculated, has

$$S_A = -\mathrm{Tr}(\rho_A \ln \rho_A) \neq 0 \tag{6}$$

(Bombelli, Koul, Lee, Sorkin '86)

Entanglement Entropy and QFT

The quantum field is an operator valued distribution, but we will work with distribution kernels $\phi := \phi(x)$. We will work with quasifree states.

When one calculates the entanglement entropy of a vacuum state in QFT, one finds an area law scaling. The problem is that the entropy goes as

$$S = A \times \infty. \tag{7}$$

When one does the Schmidt decomposition of a state into localised states, it is written as an infinite combination

$$|\psi\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, |k\rangle_A \otimes |k\rangle_B, \tag{8}$$

and $\sum_k \lambda_k \ln \lambda_k$ does not converge.

Regularisation

This is an ultraviolet divergence, as states of infinitely high frequency contribute. However, any well motivated regularisation gives an area law

$$S \propto A \tag{9}$$

Ordinarily, one regularises by imposing a length cutoff on the Cauchy surface.

This means that the calculated entropy is frame dependent.

This was inherited from our ad hoc frame dependent regularisation.

In nature, quantum gravity should supply the regularisation. If one believes that quantum gravity is covariant, one needs a spacetime volume means of calculating entropy, such that the regularisation can actually be applied.

A Spacetime Expression for Entropy

We have a spacetime definition of entropy for quasifree states

Bosons (Sorkin 2012, JJ, Yazdi 2026):

$$\int_{\mathcal{M}} \langle\Psi|\phi(x)\phi(x')|\Psi\rangle\, h(x')\, dV' = \lambda \int_{\mathcal{M}} [\phi(x), \phi(x')]\, h(x')\, dV', \qquad (10)$$

$$S = \sum_{\lambda} \lambda \ln|\lambda|. \qquad (11)$$

Fermions (JJ, Yazdi 2026):

$$\int_{\mathcal{M}} \langle\Psi|\psi(x)\bar{\psi}(x')|\Psi\rangle\, h(x')\, dV' = \lambda \int_{\mathcal{M}} \{\psi(x), \bar{\psi}(x')\}\, h(x')\, dV', \qquad (12)$$

$$S = -\sum_{\lambda} \lambda \ln \lambda. \qquad (13)$$

ArXiv: [2602.16782]

Regularisation in QFT

We will take the scalar field in our examples, with

$$K\phi := (-\Box + m^2)\phi = 0. \tag{14}$$

For a globally hyperbolic region, there exist G^{ret} and G^{adv}, and hence $\Delta = G^{ret} - G^{adv}$, satisfying

$$K\Delta = \Delta K = 0. \tag{15}$$

We can see the image of Δ gives the full solution space.

The Peierl's relation (Peierls '52) identifies

$$i\Delta(x, x') = [\phi(x), \phi(x')] \tag{16}$$

We can form any quasifree state

$$i\Delta(x, x') \xrightarrow{\text{positivity}} \langle\Psi|\phi(x)\phi(x')|\Psi\rangle \tag{17}$$

Regularisation is control of Δ
("Spectral Density of the Causal Propagator" to appear tonight)

What Remains

With this spacetime formula, the way is open to performing regularised calculations of entanglement entropy in spacetime regions

For an actual calculation to be done, one needs to make a few choices:

- Δ
  - The regions
  - The regularisation
- $\xrightarrow{\text{positivity}}$
  - The state

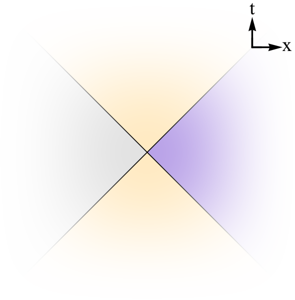
All are physical. The choice of region and state are simple.

Regularisation affects the state through its definition via Δ

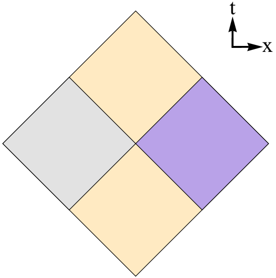
The Regions: Corner Subdiamond

As an example, we will do the simplest entanglement entropy calculation known, Rindler in 2D.

This would be



but we impose an IR cutoff (no massless Minkowski vacuum in 2D)



The State

In spacetimes with symmetries, there can exist a preferred state of some kind (e.g. the Minkowski vacuum, Rindler vacuum, Bunch-Davies vacuum)

In spacetimes without symmetries, we can use the Sorkin-Johnston vacuum. This is also the canonically chosen vacuum in the causal set, so we use it here.

It can also be constructed causal set natively in flat spacetimes.

It is known to agree with the canonical vacuum for static spacetimes (hence it would have been Minkowski with no IR cutoff)

The SJ state is defined via

$$\{G^{ret}, G^{adv}\} \longrightarrow i\Delta(x, x') \xrightarrow{\mathrm{pos}(L^2)} \langle SJ|\phi(x)\phi(x')|SJ\rangle \tag{18}$$

where positivity is w.r.t. the $L2$ inner product over the spacetime region of interest (Johnston 2009, Sorkin 2011, Afshordi, Aslanbeigi, Sorkin 2012).

Regularisation: Causal Set

The most subtle choice is the regularisation. It is the manifestation of a theory of quantum gravity.

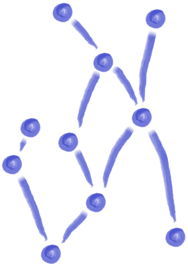
We choose a specific theory of quantum gravity, causal set theory, to perform the regularisation (Bombelli, Lee, Meyer, Sorkin '87).

In causal set theory, one essentially replaces the spacetime manifold with a directed acyclic graph.

We form a causal set from the manifold

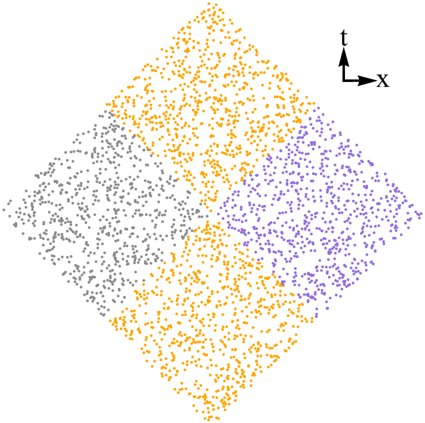
The elements are chosen according to the volume measure, and causal precedence is inherited as the order relation

To recover the continuum manifold, one uses the counting measure, while preserving the order relation.
(Makes sense because of Hawking, King, McCarthy '76, Malament '77)



Regularisation: Causal Set

This means we have the SJ state on



What we Expect

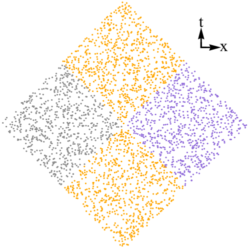
Although entanglement entropy scales with an area law, in 2 dimensions, this scaling becomes a logarithm

With an ultraviolet and infrared cutoff, we expect the universal scaling (Cardy, Calabrese 2004):

$$S = \frac{1}{6} \ln \left(\frac{l}{L_{\mathrm{UV}}} \right) + c_{\text{non-universal}} \tag{19}$$

Because we expect the (universal part of) entanglement entropy to match that calculated on the Cauchy surface

The Procedure in the Causal Set



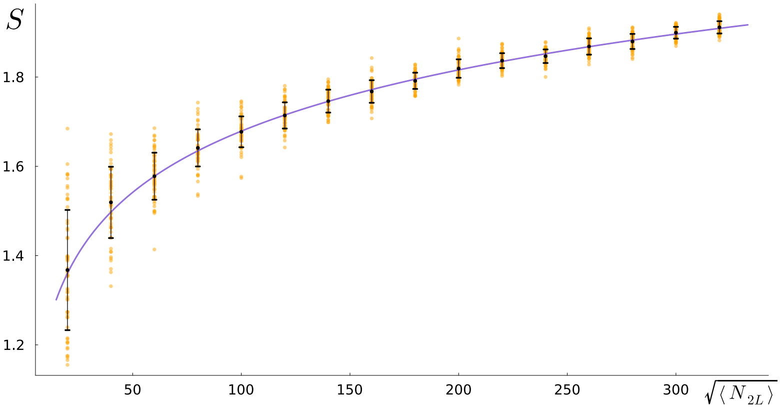
1. Form the Sorkin-Johnston $\langle SJ|\phi^i\phi^j|SJ\rangle$ on the full diamond
2. Restrict the $\langle SJ|\phi^i\phi^j|SJ\rangle$ and $i\Delta = [\phi^i, \phi^j]$ to the smaller diamond

$$\int_{\mathcal{M}} \langle\Psi|\phi(x)\phi(x')|\Psi\rangle\, h(x')\, dV' = i\lambda \int_{\mathcal{M}} \Delta(x,x')\, h(x')\, dV' \qquad (20)$$

$$S = \sum_{\lambda} \lambda \ln|\lambda|. \qquad (21)$$

3. Solve $\langle SJ|\phi^i\phi^j|SJ\rangle\, h^j = i\lambda\Delta^{ij}\, h^j$, and then insert the eigenvalues λ into the entropy formula $S = \sum \lambda \ln|\lambda|$.

Result



When $S = b \log\left(\sqrt{\langle N_{2L} \rangle}\right) + c$ was fit, $b = 0.198 \pm 0.014$ was obtained ($c = 0.767 \pm 0.08$), significantly different from the expected 0.167.
A signature of discreteness!

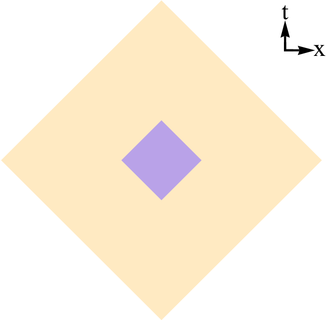
A Continuum Calculation

I'll now go over a similar calculation in the "continuum" (Saravani, Sorkin, Yazdi 2014)

Regions: A smaller subdiamond in the centre of a large diamond

State: The SJ state in the large diamond, as before

Regularisation: Spectrum Truncation



Regularisation: Spectrum Truncation

We truncate directly in the spectrum of Δ

In the continuum, there are an infinite number of eigenfunctions and eigenvalues, and so a cutoff eigenvalue is chosen

This cutoff value is related to a "minimum length" $a \sim \frac{\lambda_{min}}{L}$

Then one can express $\langle SJ|\phi(x)\phi(x')|SJ\rangle$ in the truncated eigenbasis of Δ

These are then again matrices so we can solve over the subregion

$$\langle f^i| \left(\langle SJ|\phi(x)\phi(x')|SJ\rangle\right) |f^j\rangle\, h^j = i\lambda \langle f^i|\Delta(x,x')|f^j\rangle\, h^j \tag{22}$$

and insert the eigenvalues λ into the entropy formula $S = \sum \lambda \ln|\lambda|$

What we Expect

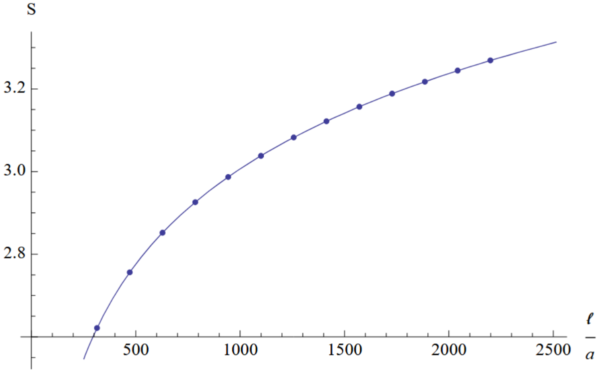
Just as before, we expect a logarithm

Again with an ultraviolet and infrared cutoff, we expect a universal scaling (Cardy, Calabrese 2004):

$$S = \frac{1}{3} \ln \left(\frac{l}{L_{\mathrm{UV}}} \right) + c_{\text{non-universal}} \tag{23}$$

Because we expect the (universal part of) entanglement entropy to match that calculated on the Cauchy surface

Result



Again $S = b\ln\left(\frac{\ell}{a}\right) + c$ was fit. $b = 0.33277$ matching the expected $1/3$. Not a trace of the difference we saw in the causal set!

On the Difference

One way we can begin to think about this difference, is via the wavelet picture.

There one considers the field as decomposed into a basis of local wavelets, such that the entanglement entropy comes from those that straddle the horizon.

Given a k_{min} and k_{max}, we find

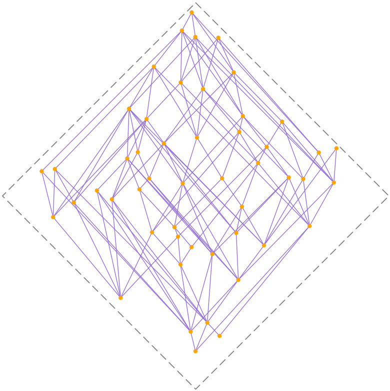
$$S \propto N_{\text{strad}} \sim \int_{k_{\text{min}}}^{k_{\text{max}}} \lambda\, A_{\partial\Sigma'} \frac{dk}{2\pi} = A_{\partial\Sigma'} \int_{\frac{2\pi}{L}}^{\frac{2\pi}{l}} \frac{dk}{k} = A_{\partial\Sigma'} \ln \frac{L}{l}. \tag{24}$$

We see that one naturally gets a proportionality factor of the area. In 2d, this is a single point.

Or is it...?

On the Difference

Consider the causal set:



Where is the Cauchy surface, and where is the boundary?

This will be encoded in Δ, but how?

Ways Forward

Given that we now have means of calculating entanglement entropy over spacetime regions, new avenues are open

There are many remaining questions:

- Will a similar difference inherited from the causal set show up in higher dimensional treatments?
- How is the difference in scaling encoded in Δ?
- Will anything be found in a fermionic calculation?

And of course:

- How else could quantum gravity regularise the calculation?

Thank you for Listening

Spectral Spacetime Entropy for Quasifree Theories - Jones, Yazdi

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