COMPETITION BETWEEN THE PAIRING AND ALIGNED COUPLING SCHEMES*

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1. Introduction

It is a pleasure to celebrate Ikuko Hamamoto's coming of age and the fiftieth anniversary of the Bohr–Mottelson papers[1] that have guided the research of most of us here during our academic careers. I plan to talk about a problem that I learned about in Copenhagen in 1962.

Two competing coupling schemes in nuclear physics are aligned coupling and pair coupling. The former leads to deformed states and rotational bands. The latter leads to pair coupling and nuclear superconductivity. These coupling schemes suggested early on that a simple model Hamiltonian

$$H = \sum_j \varepsilon_j a^\dagger_{jm} a_{jm} - \chi Q \cdot Q - G S_+ S_- \tag{1}$$

containing a sum of "independent-particle", "quadrupole", and "pairing" interactions could explain the broad systematics of much of low-energy nuclear collective structure. Although the separate terms of this Hamiltonian are easily handled, e.g., within the space of a single harmonic-oscillator shell, the diagonalization of H is virtually impossible, except by approximate methods or numerical computations in relatively small spaces. This is because the pairing and quadrupole interactions are associated with incompatible symmetry groups.

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To understand the nature of incompatible symmetries and how to handle them, I will first review some new insights we have recently obtained with the aligned and pair coupling schemes and then discuss a model with competing pairing and quadrupole degrees of freedom. A new and potentially powerful concept of *quasi-dynamical symmetry* emerges

2. Aligned coupling

2.1. *Three algebraic rotor models*

It is known that a Hamiltonian with a pure $Q \cdot Q$ interaction is analytically solvable if the quadrupole tensor Q is replaced by a tensor $\mathcal{Q}$ whose matrix elements are equal to those of Q within the space of a single harmonic shell and zero otherwise. This is because the Hamiltonian

$$\hat{H}_{\rm su3} = -\tfrac{1}{2}\chi \hat{\mathcal{Q}} \cdot \hat{\mathcal{Q}} \,, \tag{2}$$

is a rotationally-invariant quadratic in the elements of the SU(3) Lie algebra spanned by three components of angular momentum $\{\hat{L}_k\}$ and the five components $\{\hat{\mathcal{Q}}_\nu\}$ of the quadrupole tensor $\mathcal{Q}$. For a suitable $(\lambda, 0)$ irrep, and a value of χ adjusted to fit the lowest energy levels of the ground state rotational band of the ^{168}Er nucleus, the SU(3) model[?] gives the results shown in fig. 1. The energy levels of a rigid (axially symmetric) rotor with Hamiltonian

$$\hat{H}_{\rm rotor} = \frac{\hbar^2}{2\Im}\hat{\mathbf{L}}^2 \tag{3}$$

are also shown and, of course, they are indistinguishable from those of the SU(3) model.

The SU(3) and rotor models have two adjustable parameters each: λ and χ for SU(3); the moment of inertia $\Im$ and an intrinsic quadrupole moment $\bar{Q}_0$ for the rotor. In the rotor model, quadrupole matrix elements are given by

$$\langle L' \| Q \| L \rangle = \sqrt{2L+1}\, (L0, 20|L'0)\, \bar{Q}_0 \tag{4}$$

and in the SU(3) model by[?]

$$\langle L \| \mathcal{Q} \| L \rangle = \sqrt{2L+1}\, (L0, 20|L0)\, (2\lambda + 3) \,, \tag{5}$$

$$\langle L+2 \| \mathcal{Q} \| L \rangle = \sqrt{2L+1}\, (L0, 20|L+2, 0)\, [4(\lambda - L)(\lambda + L + 3)]^{1/2} \,. \tag{6}$$

Fits to the E2 transitions of ^{168}Er using these the two models are shown in the figure and seen to be experimentally indistinguishable. (Details of the SU(3) model as applied to heavy nuclei are given in ref. [?].)

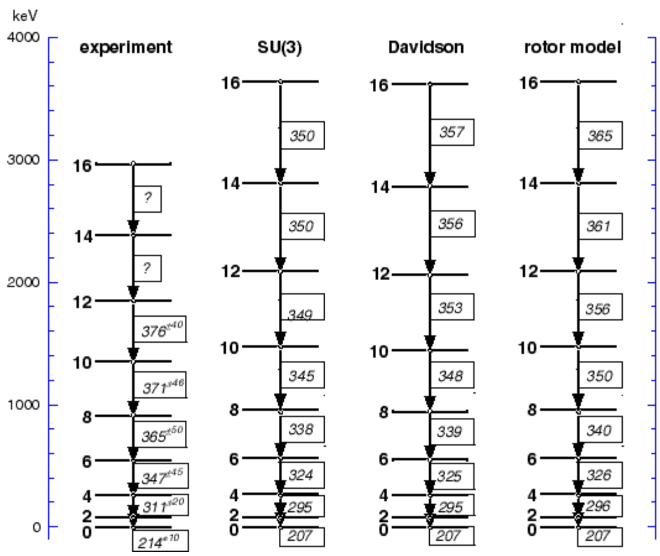

Figure 1. Energies and E2 transition rates for the ground state band of ^{168}Er described by the SU(3) model, the symplectic model with a Davidson potential, and the rigid rotor model.

The low-energy states of ^{168}Er can also be described by the symplectic model[3] with a Hamiltonian

$$H = \sum_n \frac{p_n^2}{2m} + \tfrac{1}{2} m\omega \textstyle\sum_n r_n^2 + V(Q)\,, \tag{7}$$

where the potential

$$V(Q) = \chi\Big(Q \cdot Q - \frac{\varepsilon}{Q \cdot Q}\Big) \tag{8}$$

is adapted[4] from the Davidson[5] potential of molecular physics. The energy levels and E2 transition rates for this Hamiltonian are essentially identical to those of the rigid rotor and SU(3) models. The three rotor models are experimentally indistinguishable.

2.2. *Algebraic relationships of the three rotor models*

The symplectic Sp(3,ℝ) model[3] has an algebra of observables spanned, in the context of the nuclear shell model, by the operators

$$Q_{ij} = \sum_n^A x_{ni} x_{nj}\,, \quad K_{ij} = \sum_n^A p_{ni} p_{nj}\,, \quad S_{ij} = \sum_n^A (x_{ni} p_{nj} + p_{nj} x_{ni})\,, \tag{9}$$

where x_{ni} and p_{ni} are the Cartesian position and momentum coordinates of the n'th nucleon, with n running from 1 to A, for a mass number A nucleus, and $i = 1$, 2, or 3. The algebra of observables for the rigid rotor model is the subalgebra spanned by the angular momenta and quadrupole operators

$$L_k = S_{ij} - S_{ji}\,, \quad Q_{ij}\,, \tag{10}$$

and the SU(3) algebra is the subalgebra spanned by

$$L_k = S_{ij} - S_{ji} \quad \mathcal{Q}_{ij} = Q_{ij} + K_{ij}\,. \tag{11}$$

These subalgebra relationships make it possible to express the wave functions of the symplectic model in either a rigid rotor or SU(3) basis and thereby attempt to understand the successes of the latter two models.

A rigid rotor representation is characterized by precisely defined intrinsic quadrupole moments $\{\bar{Q}_\nu\}$, whereas the intrinsic states of a soft rotor, like the symplectic model, have a distribution of quadrupole moments about the minimum of the potential, as illustrated qualitatively in fig. **??**. Thus, if $|\bar{Q}; L\rangle$ are rigid rotor model states, the states of a soft rotor are linear superpositions

$$|\text{soft rotor}; L\rangle = \int \psi(\bar{Q})\, |\bar{Q}; L\rangle\, \mathrm{d}^2\bar{Q}\,. \tag{12}$$

Furthermore, to the extent that centrifugal stretching effects are negligible, the function $\psi(\bar{Q})$ is independent of L. The remarkable fact, is that the quadrupole matrix elements of a soft rotor with wave functions of this form are identical to those of a rigid rotor representation. We then say that the rigid rotor algebra is a *quasi-dynamical symmetry* for such a soft rotor.

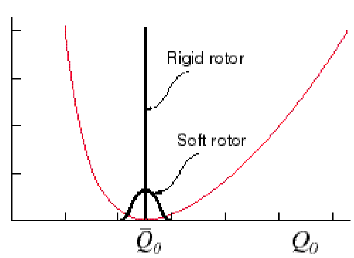

Figure 2. Wave functions for a rigid rotor and for a symplectic model rotor with a Davidson potential.

Expansion of the symplectic-Davidson model states in an SU(3) basis,

$$|\sigma LM\rangle = \sum_{n\lambda\mu} C^{n\lambda\mu}_{\sigma KL}\, |(n\lambda\mu)KLM\rangle\,, \tag{13}$$

shows that SU(3) is also a remarkably good quasi-dynamical symmetry for this soft rotor model. This is demonstrated by the fact that coefficients of this expansion, shown for $L = 0, \ldots, 10$ in fig. **??**, are not only independent of M, as they must be for a rotationally-invariant Hamiltonian, but are also essentially independent of K and L. The implication is that the results of the symplectic model calculation should be the same as for the SU(3) model with an average value of the SU(3) irreps appearing in the above expansion. This relationship evidently parallels the relationship between the rigid and soft rotor models.

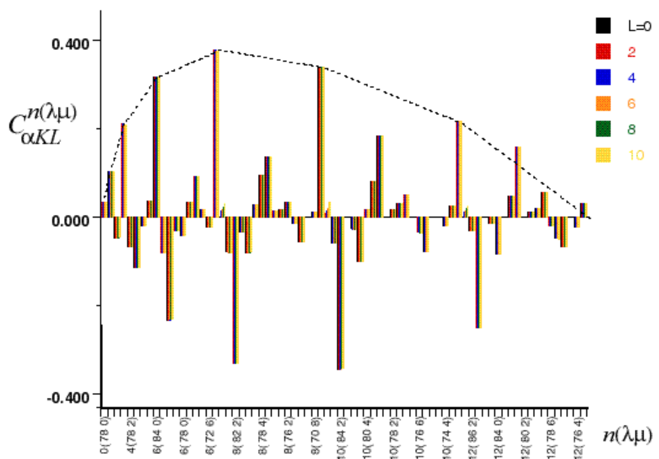

Figure 3. Wave functions for the symplectic-Davidson model calculation in a U(3) basis.

3. Pair coupling

3.1. *Seniority*

The simple pairing force

$$V_{\text{pairing}} = -G\hat{S}_+\hat{S}_-$$

is just one of many interactions that conserve seniority. In fact, for nucleons of a single type (all protons or all neutrons), the number of independent rotationally-invariant two-body interactions that mix seniority in a $(j)^n$ configuration is $[(2j-3)/6]$, the integer part of $(2j-3)/6$. Thus, for $j = 7/2$ all single-j interactions conserve seniority, and for $9/2 \leq j \leq 13/2$

all but one conserve seniority. We have also found[?] that the most general Hamiltonian for a single j shell nucleus is expressible

$$H = H_0 - \chi Q \cdot Q\,, \tag{14}$$

where H_0 is seniority-conserving. Fits to the low-energy levels of ^{214}Ra, for example, both with a seniority conserving Hamiltonian, for which the algebra observables is USp(10), and with the extra $Q \cdot Q$ interaction, for which the algebra of observables is U(10), are shown in fig. **??**. It is found that seniority is a remarkably good quantum number for this singly-closed shell nucleus. However, when multiple j shells are involved, particularly for doubly-open shell nuclei, seniority becomes strongly mixed by the $Q \cdot Q$ and other components of the two-body interaction.

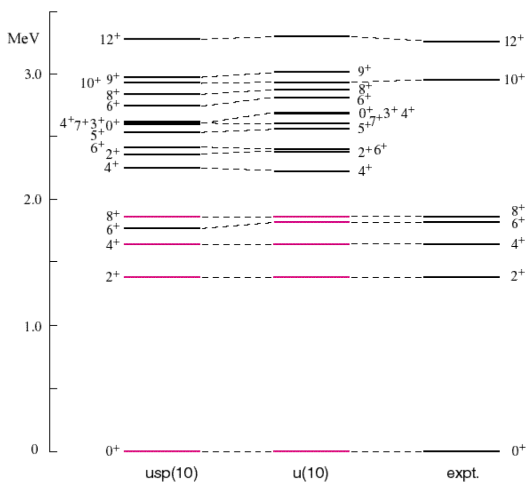

Figure 4. Energy levels of ^{214}Ra fitted to the lowest $J = 0$, 2, 4, and 8 levels by a seniority conserving interaction (USp(10) dynamical symmetry) and by a seniority non-conserving interaction, with an extra $Q \cdot Q$ interaction which also fits the energy of the lowest $J = 6$ state (U(10) dynamical symmetry).

3.2. *Multilevel pairing models*

Consider first the Hamiltonian for $2j+1$ particles in two levels of the same spin j:

$$H = \sum_j \varepsilon_j \hat{n}_j - G \sum_{ij} \hat{S}^i_+ \hat{S}^j_- . \tag{15}$$

The ratio of the number of nucleons in the upper to the lower level is shown for the ground state of this Hamiltonian in fig. **??**. The results of a numerical diagonalization are seen to be poorly reproduced by the BCS approximation but become increasingly better as the value of j increases. In contrast, a number-projected quasi-particle approximation does exceedingly well. It is found that, for this model[?], a second-order phase transition occurs in the $j \to \infty$ limit at a critical value $G_{\rm crit}$ of the interaction strength.

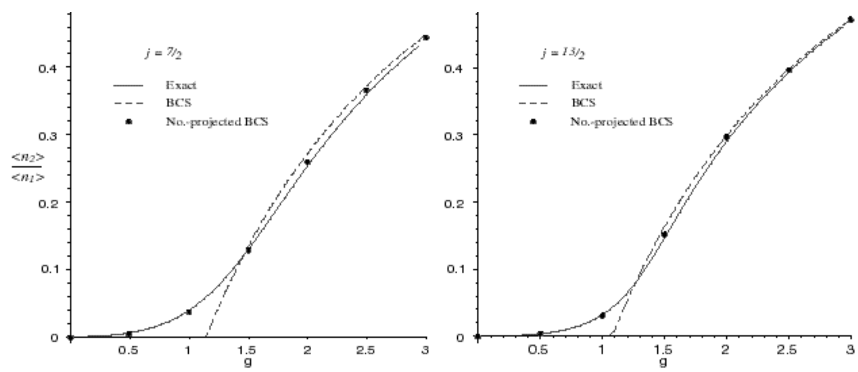

Figure 5. Ratio of the number of nucleons in the upper to the lower level for the ground state of the two-level Hamiltonian (??) plotted as a function of $g = G/G_{\rm crit}$.

For many levels of different spins, the pure pairing model can be solved because of an SU(2)$\oplus$SU(2)$\oplus \cdots$ dynamical algebra. For a more general interaction, especially one that mixes multishell seniority, the complexity of the problem becomes intractable. However, the number-projected quasiparticle approximation remains viable. Thus, it is of considerable interest to know that there is an elegant algebraic method for solving this problem using the fact that number-projected multishell wave functions are well known as Schur functions in the theory of orthogonal functions.[?] Unfortunately, there isn't space to present the theory here.

4. A pairing plus quadrupole model

We now consider a simple model with incompatible pairing plus quadrupole interactions

$$H(\alpha) = H_0 + (1-\alpha)V_{\rm SU(2)} + \alpha V_{\rm SU(3)} \tag{16}$$

acting within the space of a single harmonic oscillator shell. When $\alpha = 0$ or 1, this model is exactly solvable having SU(2) quasispin and SU(3) dynamical symmetries, respectively. However, for $0 < \alpha < 1$, the eigenvalue equations are generally too complicated to solve because there is no simple subgroup that contains both SU(2) and SU(3). Thus, to assess the nature of the problem, we considered[?] a model system in which SU(2) and SU(3) are embedded in the simplest algebraic structure that could contain both of them, namely the compact symplectic algebra Usp(6). The eigenvalues of the Hamiltonian obtained in this way are shown as a function of α in fig. **??**.

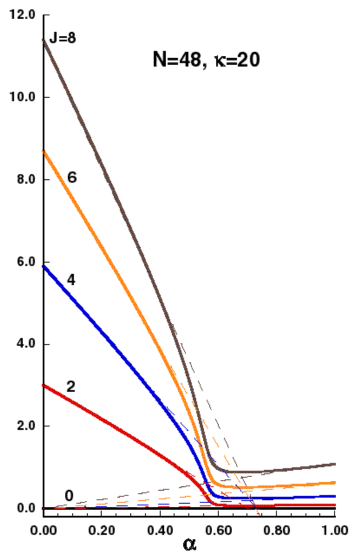

Figure 6. Energy levels of the pairing plus quadrupole Hamiltonian (**??**) as a function of α.

The results of this model are very insightful. First, we find that the model exhibits a second-order phase transition which becomes increasingly sharp as the particle number (48 for the results shown) increases. However, what is most instructive is the structure of the wave functions in an SU(3) basis, as a function of α. The expansion coefficients are shown in fig. **??**. For $\alpha = 1$, the states are those of a single SU(3) irrep (32,8). For values of $\alpha \lesssim 0.58$, the wave functions are complicated in an SU(3) basis. However, for $\alpha \geq 0.6$ the amplitudes for all L values become essentially equal. According to the definition given in section 2.2, the model suddenly acquires an SU(3) quasidynamical symmetry. This results in a very subtantial reduction in the effort required to compute the spectrum. In the first place, if one has a solution for the $L = 0$ states, one has a very good approximation to the solutions for the $L > 0$ states of the ground state band. And, in the second

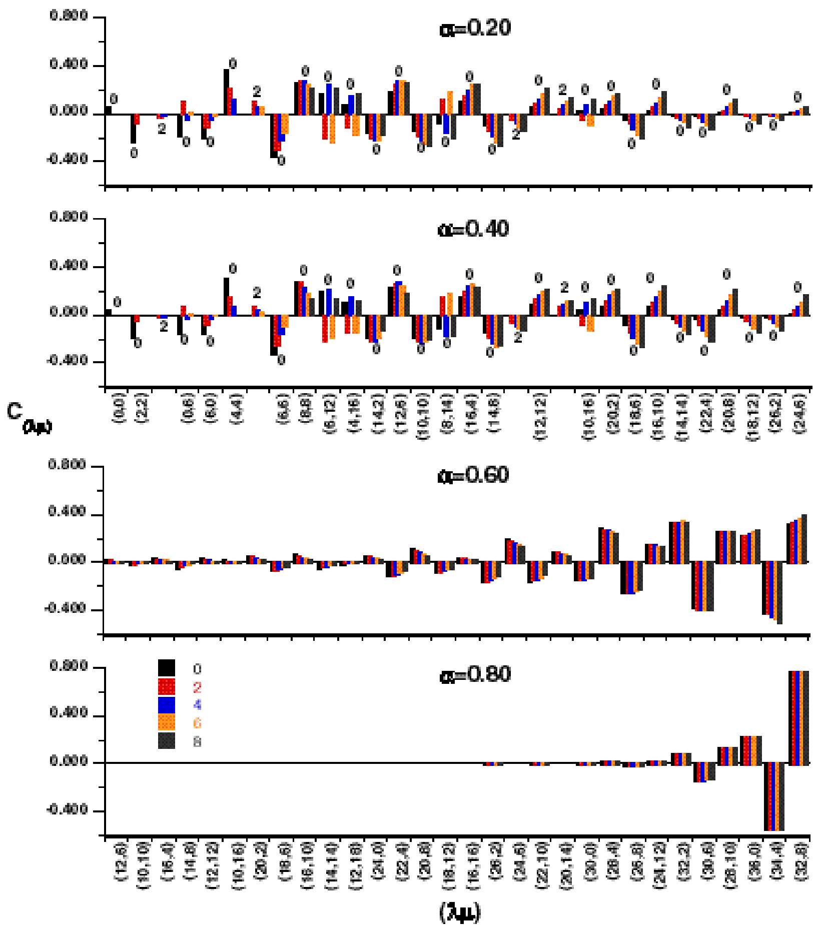

Figure 7. Eigenstates of the pairing plus quadrupole Hamiltonian (??) in an SU(3) basis.

place, one can make good meaningful first guesses for the amplitudes of the SU(3) coefficients for an iterative solution of the eigenvalue equations by a method of successive approximations.

5. Conclusions

We have only given a solution to the pairing plus quadrupole problem in an unrealistic situation. And we have certainly not given a solution to the general problem of a Hamiltonian with components of incompatible symmetry.

However, we have gained valuable insights into the nature of the problem. In particular, we have identified the concept of a quasidynamical symmetry, both of the rigid rotor and SU(3) types, as a characteristic of a soft rotor whose quadrupole shape fluctuations are caused either by centrifugal forces or residual pairing interactions.

In this brief review, I have outlined the appearence of highly coherent mixings of rigid rotor and SU(3) irreps in two models. In fact, the idea of a quasi-dynamical symmetry was conceived on purely physical grounds, and phrased mathematically in terms of an embedded representation, before the model examples to illustrate its occurence were constructed[?]. The underlying idea is that rotational states are seen in nuclei only when the rotational motions are adiabatic relative to other internal degrees of freedom. In such a situation, a rotating intrinsic frame of reference is close to being an inertial frame in as much as the Coriolis and centrifugal forces are negligible. Moreover, in the adiabatic limit, any residual interactions which are rotationally invariant and not functions of the angular momentum, can have strong effects on the intrinsic structure of a rotational nucleus. Moreover, whatever structure emerges should be the same for all states of a rotational band for which the angular momentum and, hence, the inertial forces are sufficiently small. These ideas lead naturally to the concept of quasi-dynamical symmetry.

References

1. A. Bohr, *Mat. Fys. Medd. Dan. Vid. Selsk.* **26** no. 14 (1952) 1; A. Bohr and B.R. Mottelson, *Mat. Fys. Medd. Dan. Vid. Selsk.* **27** no. 16 (1952) 1.
2. J. P. Elliott, *Proc. Roy. Soc.* A**245**, 128, 562 (1958).
3. D. J. Rowe, M. G. Vassanji, and J. Carvalho, *Nucl. Phys.* **A504**, 76 (1989).
4. M. Jarrio, J. L. Wood, and D. J. Rowe, *Nucl. Phys.* **A528**, 409 (1991).
5. G. Rosensteel and D. J. Rowe, *Phys. Rev. Lett.* **38**, 10 (1977); *Ann. Phys.* (N.Y.) **126**, 343 (1980); D. J. Rowe, *Rep. Prog. Phys.* **48**, 1419 (1985).
6. D. J. Rowe and C. Bahri, *J. Nucl. Phys.* A**31**, 4947 (1998); C. Bahri and D. J. Rowe, *Nucl. Phys.* **A662**,125 (2000).
7. P.M. Davidson, *Proc. Roy. Soc.* **135**, 459 (1932).
8. D. J. Rowe and G. Rosensteel, *Phys. Rev. Lett.* **87**,172501 (2001); G. Rosensteel and D. J. Rowe, *Phys. Rev.* C (in press).
9. D. J. Rowe, *Nucl. Phys.* **A691**, 691 (2001).
10. D.J. Rowe, C. Bahri and W. Wijesundera, *Phys. Rev. Lett.* **80**, 4394 (1998); C. Bahri, D.J. Rowe and W. Wijesundera, *Phys. Rev.* **C58**, 1539 (1998).
11. D.J. Rowe, P. Rochford and J. Repka, *J. Math. Phys.* **29**, 572 (1988).