EMBEDDED REPRESENTATIONS AND QUASI-DYNAMICAL SYMMETRY*

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This presentation explains why models with a dynamical symmetry often work extraordinarily well even in the presence of large symmetry breaking interactions. A model may be a caricature of a more realistic system with a "quasi-dynamical" symmetry. The existence of quasi-dynamical symmetry in physical systems and its significance for understanding collective dynamics in complex nuclei is explained in terms of the precise mathematical concept of an "embedded representation". Examples are given which exhibit quasi-dynamical symmetry to a remarkably high degree. Understanding this unusual symmetry and why it occurs, is important for recognizing why dynamical symmetries appear to be much more prevalent than they would otherwise have any right to be and for interpreting the implications of a model's successes. We indicate when quasi-dynamical symmetry is expected to apply and present a challenge as to how best to make use of this potentially powerful algebraic structure.

1. Introduction

I intended to talk about vector coherent state theory. However, several examples shown by others of what Jerry Draayer appropriately referred to as an *adiabatic coherent mixing* of representations, prompted me to change my topic to a description of the mathematical structure and physical significance of this potentially powerful and physically useful concept.

When a simple model is successful at describing a physical system, there is a temptation to infer that the model has a corresponding degree of reality. However, it is easy to be misled. This concern led us to investigate why systems frequently appear to hold onto a dynamical symmetry in spite of strong symmetry-breaking interactions. This is particular evident in systems which exhibit a Landau second order transition from a phase with

*Work supported by the Natural Sciences and Engineering Research Council of Canada.

one apparent symmetry to a phase with a different symmetry. The outcome was the discovery of quasi-dynamical symmetry[1,2,3].

2. What is quasi-dynamical symmetry?

It is well known that states of different, but equivalent irreps, of a Lie algebra (or Lie group) can mix coherently to form new irreps. For example, if $\{|\alpha LM\rangle\}$ are states of angular momentum L and z-component M, with α distinguishing different states of the same angular momentum, then the states

$$\{|\Psi_{\kappa LM}\rangle = \sum_\alpha C_{\kappa\alpha}|\alpha LM\rangle,\ M = -L, \ldots, +L\} \tag{1}$$

span another (equivalent) so(3) irrep of angular momentum L. What is remarkable is that, for some Lie algebras, there are linear combinations of states from similar, but inequivalent, irreps that actually form a basis for an irrep of the Lie algebra. Such an irrep is called an *embedded representation*. They may seem like bizarre mathematical oddities but, in fact, embedded representations are common in physics and underlie the *adiabatic separation of variables*. We say that a model has a quasi-dynamical symmetry if its states span a so-called *embedded representation* of a Lie algebra[2].

Definition: If $\mathbb{H}$ is the Hilbert space for a (generally reducible) representation U of a Lie algebra $\mathfrak{g}$ and $\mathbb{H}_0 \subset \mathbb{H}$ is a subspace then, if the matrix elements of $\mathfrak{g}$ between states lying in $\mathbb{H}_0$ are equal to those of a representation U_0 of $\mathfrak{g}$, then U_0 is said to be an embedded representation.

Subrepresentations and linear combinations of equivalent irreps are trivial examples of embedded representations. Non-trivial examples are found for semi-direct sum Lie algebras of the rotor model kind (semi-direct sums with Abelian ideals). Other Lie algebras contract to this kind of algebra in large quantum number limits and, consequently, have very good approximations to embedded representations. The su(3) and symplectic model algebras are examples of the latter. This is important for the microscopic theory of collective motion because, although spin-orbit and other residual interactions break the dynamical symmetries of the su(3) and symplectic models, they mix representations in a highly coherent way that preserves the algebraic structures of these models as quasi-dynamical symmetries. This was predicted to happen as an algebraic expression of an adiabatic separation of rotational and intrinsic degrees of freedom[1] according to the Born-Oppenheimer approximation. Thus, it is exciting to discover how extraordinarily good quasi-dynamical symmetry is in practical situations.

3. The rigid rotor algebra as a quasi-dynamical symmetry of the soft-rotor model

Without vibrational degrees of freedom, the soft-rotor model is not an algebraic model. It nevertheless has a quasi-dynamical symmetry given by the dynamical symmetry of the (less realistic) rigid-rotor model.

A spectrum generating algebra for a rigid-rotor model[4] is spanned by three angular momentum operators and five quadrupole moments. The angular momenta span an so(3) subalgebra; the quadrupole moments commute among themselves as elements of an Abelian subalgebra and transform under rotations as components of a rank two spherical tensor. This algebra, known as rot(3), has irreducible unitary representations characterized by rigid intrinsic quadrupole shape parameters β and γ, related to the rotational invariants by

$$[Q \otimes Q]_0 \propto \beta^2 \,, \quad [Q \otimes Q \otimes Q]_0 \propto \beta^3 \cos 3\gamma \,. \tag{2}$$

Rigid-rotor irreps have basis wave functions expressible in the language of coherent state theory in the form

$$\Psi^{(\beta,\gamma)}_{KLM}(\Omega) = \langle \beta, \gamma | R(\Omega) | KLM \rangle \,. \tag{3}$$

In the physical world, there is no such thing as a truly rigid rotor. Real rotor wave functions, have intrinsic wave functions that are linear superpositions of rigid-rotor intrinsic wave functions with vibrational fluctuations;

$$\Phi_{KLM}(\Omega) = \int \psi(\beta,\gamma) \, \langle \beta, \gamma | R(\Omega) | KLM \rangle \, dv(\beta,\gamma) \,. \tag{4}$$

Due to Coriolis and centrifugal forces, an intrinsic wave function $\psi(\beta,\gamma)$ will generally change with increasing angular momentum. However, if the rotational dynamics is adiabatic relative to the intrinsic vibrational dynamics, then $\psi(\beta,\gamma)$ will be independent of L as assumed in the standard (soft) nuclear rotor model; the rigid-rotor algebra is then an exact quasi-dynamical symmetry for the soft rotor. This is clear from the fact that the matrix elements between states of a soft-rotor model band are given by

$$\begin{aligned} \langle \Phi_{K'L'M'} | Q_\nu | \Phi_{KLM} \rangle &= \langle \beta \cos\gamma \rangle \int \mathcal{D}^{L'}_{K'M'}(\Omega) \mathcal{D}^2_{0\nu}(\Omega) \mathcal{D}^L_{KM}(\Omega) \, d\Omega \\ &+ \frac{1}{\sqrt{2}} \langle \beta \sin\gamma \rangle \int \mathcal{D}^{L'}_{K'M'}(\Omega) \big[\mathcal{D}^2_{2\nu}(\Omega) + \mathcal{D}^2_{-2,\nu}(\Omega) \big] \mathcal{D}^L_{KM}(\Omega) \, d\Omega \,, \end{aligned} \tag{5}$$

which is precisely the expression of the rigid-rotor model albeit with the rigidly-defined values of $\beta\cos\gamma$ and $\beta\sin\gamma$ replaced by their average values.

Note that there is no way to distinguish the states of a soft-rotor band from those of a rigid-rotor band without considering states of other bands. This is because an embedded irrep is mathematically a genuine representation of the rot(3) algebra; it is simply realized in a way that may seem contrived from a mathematical perspective but which is natural and very physical for a nuclear physicist. Moreover, it is useful to extract the essence of this simple structure because of its less-than-obvious implications for other dynamical symmetries which have rotor and vibrator contractions.

4. Effects of the spin-orbit interaction in the SU(3) model

In molecular physics, one can find near-rigid-rotor spectra of orbital angular momentum states weakly coupled by a spin-orbit interaction to the spins of the atomic electrons. In nuclear physics the spin-orbit interaction is much stronger. However, far from destroying the rotational structure of odd nuclei, the spin is usually strongly coupled to the rotor and participates actively in the formation of strongly-coupled rotational bands. Indeed, in the Nilsson model, one includes the spin-degrees of freedom explicitly in constructing unified model intrinsic states.

It is important to recognize that it is not the spin-orbit interaction that works against strong coupling; it is the Coriolis force. In other words, both a strong rotationally-invariant interaction between the spin and spatial degrees of freedom and adiabatic rotational motion (meaning weak centrifugal and Coriolis forces) are important for strong coupling. Thus, it was anticipated[5] that a spin-orbit interaction might well modify the predictions of a simple su(3) model and even mix its irreps strongly. But, the underlying su(3) structure should nevertheless remain discernable and even be indistinguishable from strongly-coupling rotor model predictions in the limit of large-dimensional representations. In other words, the mixing of su(3) irreps should be highly coherent as expected for an embedded representation and give low-angular momentum states of the form

$$\Phi_{LM} = \sum_{\lambda\mu} C_{\lambda\mu K} \Psi_{\lambda\mu KLM} \tag{6}$$

with $C_{\lambda\mu K}$ coefficients essentially independent of L. Calculations[3], cf. Fig. 1, confirm this to a high degree of accuracy. (Note that, since the SU(3) model does not include Coriolis interactions, the quasi-dynamical symmetry for this sitation becomes exact in the $\lambda + \mu \to \infty$ rotor limit.)

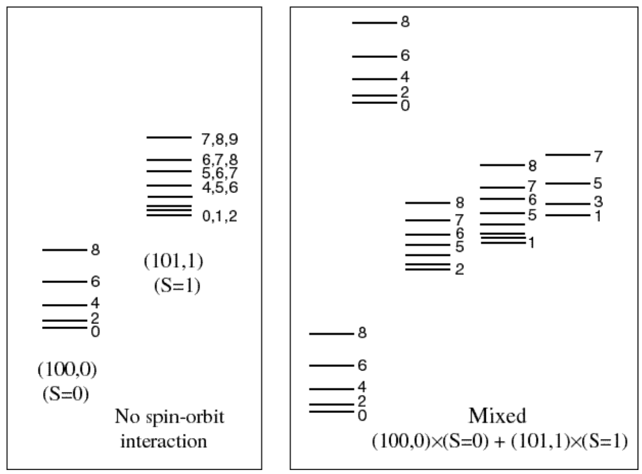

Figure 1. The figure shows two SU(3) irreps: a (100,0) irrep with spin $S = 0$ and a (101,1) irrep coupled to states of spin $S = 1$. The right figure shows the result of mixing these two irreps with a strong spin-orbit interaction In spite of the ground and a beta-like vibrational band being very strongly mixed (essentially 50-50) the resulting bands would be indistinguishable by experiment from pure su(3) bands. (The first calculations of this type were carried out by Rochford[3] for lower-dimensional irreps.)

5. SU(3) quasi-dynamical symmetry and major shell mixing

We know, from Nilsson model calculations, that major shell mixing is essential for a reasonable microscopic description of rotational states. We also know that, while the symplectic model does not adequately account for the spin-orbit and short-range interactions, it contains the rigid-rotor and quadrupole vibrational algebras as subalgebras and, consequently, does well as regards the long-range rotational correlations. Thus, on the basis of many preliminary investigations, we are confidant that the symplectic algebra, $sp(3, \mathbb{R})$, should be an excellent quasi-dynamical symmetry for a realistic microscopic theory of nuclear rotational states. At this time, I show results which demonstrate that, within a quite large symplectic model irrep with a Davidson interaction, both the su(3) and rigid-rotor algebras are extraordinarily good quasi-dynamical symmetries.

Fig. 2 shows the spectrum of ^{166}Er fitted with three models[7]: the su(3), symplectic, and rigid-rotor models, The fitted results are barely distinguishable; they are equally successful at fitting the lower levels and E2 transitions and equally unsuccessful at taking account of centrifugal stretching effects. In the symplectic model case, this is due to the Davidson potential.

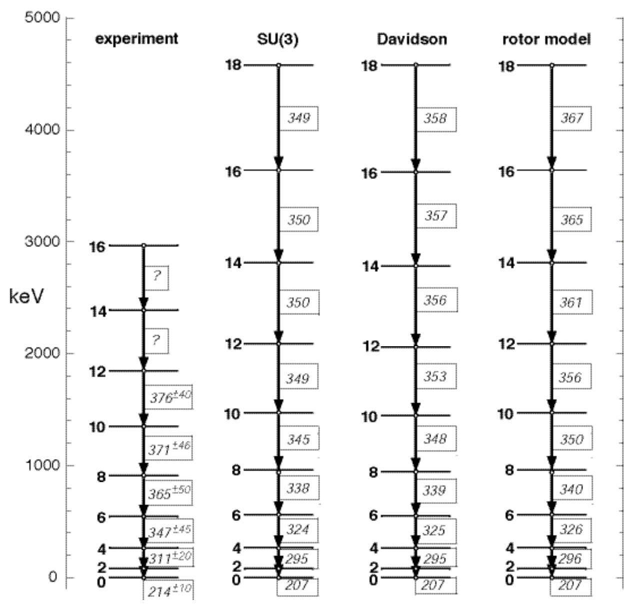

Figure 2. Fits to the ground state band of ^{166}Er with the SU(3), symplectic, and rigid-rotor models[7].

Fig. 3 shows what the symplectic model wave functions look like in an su(3) basis. They exhibit an extraordinary degree of coherence; i.e., the cofficients are independent of angular momenta for a large range of values and indicate the goodness of su(3) as a quasi-dynamical symmetry.

6. SU(3) quasi dynamical symmetry for a model with pairing interactions

Finally, we investigated what happens in a model that includes both pairing and $Q \cdot Q$ interactions with a Hamiltonian of the form

$$H(\alpha) = H_0 + (1-\alpha)V_{\rm su(2)} + \alpha V_{\rm su(3)}\,, \tag{7}$$

where $V_{\rm su(2)} = -G\hat{S}_+\hat{S}_-$ is an su(2) quasi-spin pairing interaction and $V_{\rm su(3)} = -\chi Q \cdot Q$ is an su(3) interaction. When α is zero or one, H is easily diagonalized because of its respective su(2) and su(3) dynamical symmetries. However, for intermediate values of α, diagonalization of H is a notoriously difficult problem because of the incompatible nature of su(2) and su(3); they are incompatible in the sense[8] that, within a given harmonic oscillator shell model space, the only space that is invariant under

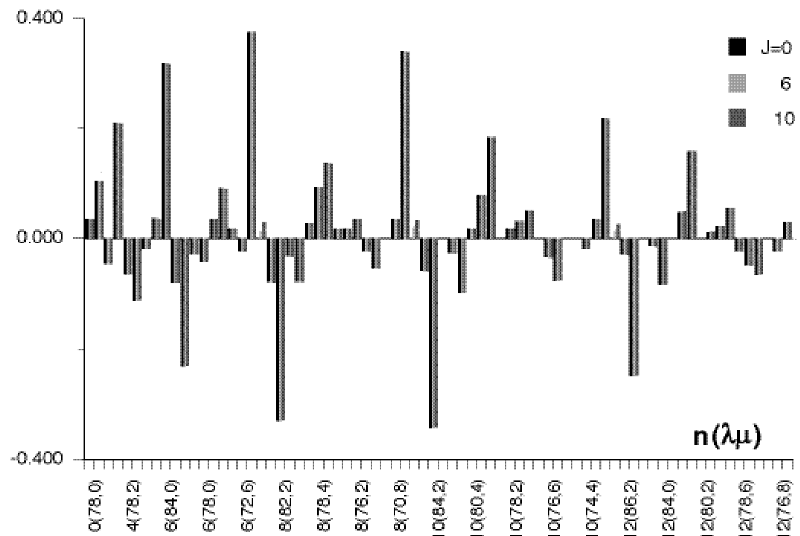

Figure 3. Expansion coefficients of symplectic-Davidson model wave functions in a multi-shell SU(3) basis covering 12 major harmonic oscillator shells[7].

both su(2) and su(3) is essentially the whole $S = T = 0$ subspace.

We therefore considered a model having a unitary symplectic dynamical symmetry, usp(6) (the smallest Lie algebra that contains both quasispin su(2) and su(3) as subalgebras) and generated large-dimensional usp(6) irreps by artificially considering particles of large pseudo spin[9,10].

The lowest energy states of $J = 0, \ldots, 8$ are shown in Fig. 4. The results exhibit a phase transition at a critical value of $\alpha \approx 0.6$ that becomes increasingly sharp as the number of particles is increased. However, the system does not flip from an su(2) to an su(3) dynamical symmetry at the critical point. In fact, it undergoes a second order phase transition in which the su(3) symmetry above the critical point is a quasi-dynamical symmetry. This is seen by looking at the extraordinary coherence of the wave functions shown for four values of α in Fig. 5. When $\alpha = 1$ (not shown) the wave functions, of course, belong to a single su(3) irrep but, for smaller values of $\alpha > 0.6$, they straddle large numbers of su(3) irreps with expansion coefficients that are essentially independent of angular momentum, as characteristic of a quasi-dynamical symmetry.

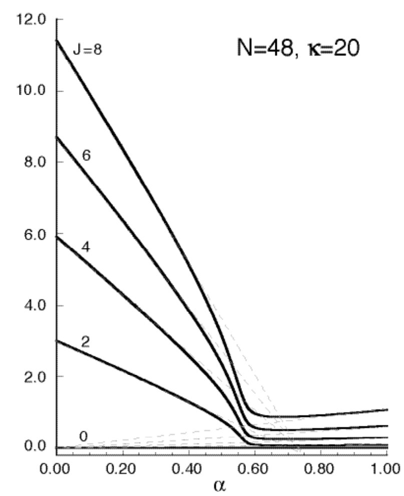

Figure 4. Energy levels of the Hamiltonian (7) as a function of α (taken from ref.[10]).

7. Concluding remarks

What destroys rotational bands is not the residual interactions. It is the Coriolis and centrifugal forces. Thus, we can expect quasi-dynamical symmetry to be a characteristic of any realistic description of rotational states.

I conjecture that quasi-dynamical symmetry will prove essential for a realistic microscopic theory of the rotational states observed in nuclei and other many-body systems. My belief that this will be the case is a response to the fundamental question: why do physical many-body systems exhibit rotational bands? In spite of huge efforts to separate the variables of a many-body system into subsets of intrinsic and collective variables, the fact remains that the separation of collective dynamics is fundamentally due to the adiabaticity of collective motions (as understood long ago by the architects of the collective models). Thus, after years of grappling with the complexity of realizing collective states in microscopic terms, the conclusion emerges that unless we give the adiabatic principle a central place in the theory, there is no way we will ever succeed. The remaining question is: just how do we do this?

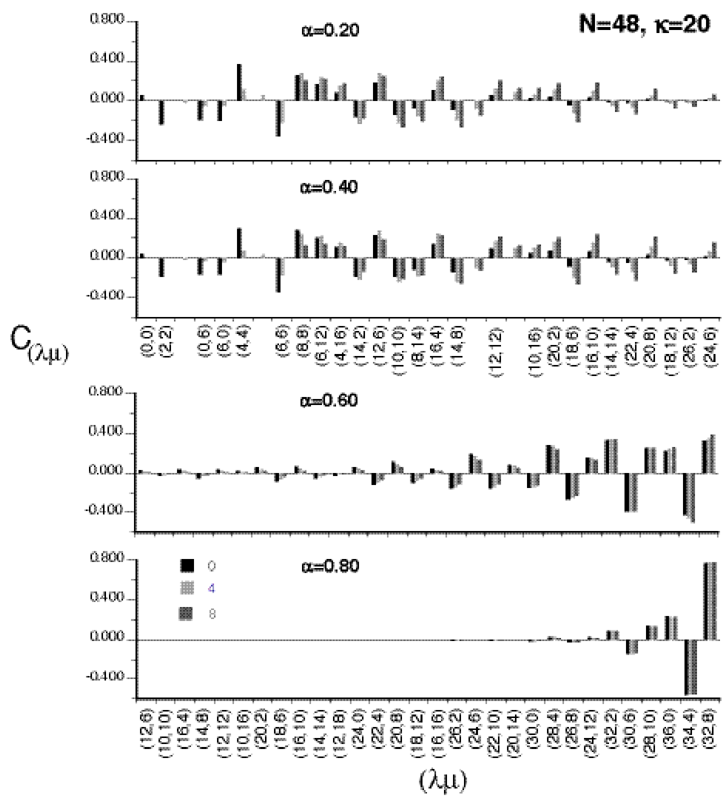

Figure 5. Eigenfunctions of the Hamiltonian (7) for four values of α shown as histograms in an SU(3) basis (taken from ref.[10]).

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