VECTOR COHERENT STATE THEORY: A POWERFUL TOOL FOR SOLVING ALGEBRAIC PROBLEMS IN PHYSICS

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VCS theory is perhaps the simplest and most effective way known for computing the matrix elements of a Lie algebra. It is a mathematical tool that noone who is serious about using algebraic methods in physics should be without. It encorporates the mathematical theories of induced representations and geometric quantization in a physically intuitive manner that makes it easy to construct the explicit representations of a desired Lie algebra in a chosen basis in a systematic manner. Its practical utility has been confirmed in numerous applications.

1 Introduction

VCS theory was developed in nuclear physics to provide practical and efficient ways to do calculations with non-trivial algebraic models. It was designed for use with the symplectic model [1] on which the microscopic theory of nuclear collective motion is founded [2]. However, it subsequently proved capable of solving numerous other problems in physics and the mathematics it employs [3].

VCS theory is a synthesis[4,5]of the powerful mathematical theories of induced representations[6] and geometric quantization[7,8]. It is accessible to physicists and provides the explicit results they need. In addition to the standard (reducible) representations of induced-representation theory , it gives the explicit matrices of irreducible representations required for applications of symmetry in physics.

It is shown here how VCS theory is used to construct the representations of su(3) in an su(2) basis [4], the representations of su(3) in an so(3) basis [9,10], and the generic representations of so(5) in an so(3) basis [11,12]. The theory has been applied to many other Lie algebras and superalgebras (cf. ref. [4] for a list of early references) and to the computation of SU(3) Clebsch-Gordan coefficients [13]. As a theory of quantization [5], VCS theory relates the classical and quantal representations of an algebraic model and provides the maps between them. It resolves the problem with Dirac's canonical theory of quantization. It provides a physical perspective on the methods of *geometric quantization* and simple ways to implement the prescriptions of that theory. Moreover, the vector generalizations of coherent state theory provide quantizations of systems with intrinsic gauge degrees of freedom [5]. Unfortunately, there is no space to discuss these many applications here.

2 Scalar coherent state representations

Definition (COHERENT STATES): If T is a representation of a Lie group G on a Hilbert space $\mathbb{H}$ and $|0\rangle$ is a fixed state in $\mathbb{H}$, then the states

$$\{|g\rangle = T^{\dagger}(g)|0\rangle\ ,\ g \in G\} \tag{1}$$

are called coherent states [14].

If the representation T is irreducible, then the coherent states span the Hilbert space $\mathbb{H}$ for this irrep. Thus, any state $|\psi\rangle \in \mathbb{H}$ is defined by its overlaps with a set of coherent states in $\mathbb{H}$.

Definition (COHERENT STATE WAVE FUNCTIONS): If $|0\rangle$ is a fixed state in the Hilbert space $\mathbb{H}$ for a representation T of a group G, then any $|\psi\rangle \in \mathbb{H}$ can be represented by a coherent state wave function Ψ, i.e., a function over G with values

$$\Psi(g) = \langle g|\psi\rangle = \langle 0|T(g)|\psi\rangle\,, \quad g \in G\,. \tag{2}$$

Suppose, for example, that $\hat{R}$ is an irrep of SO(3)[a] of angular momentum L and $|0\rangle \equiv |LK\rangle$. Then a state $|LM\rangle$ has coherent state wave function Ψ_{LM} defined as a function of Euler angles by

$$\Psi_{LM}(\Omega) = \langle LK|\hat{R}(\Omega)|LM\rangle = \mathcal{D}^L_{KM}(\Omega)\,. \tag{3}$$

Depending on the choice of the fixed state $|0\rangle$ it is generally sufficient to specify a state $|\psi\rangle$ by giving the values of its coherent state wave function at a subset of elements of G. In the above example, if $|0\rangle \equiv |L0\rangle$ then the state $|LM\rangle$ has coherent state wave function Ψ_{LM} defined over an SO(2)\SO(3) coset (the sphere) by

$$\Psi_{LM}(\theta,\varphi) = \langle L0|e^{i\hat{L}_y\theta}e^{i\varphi\hat{L}_z}|LM\rangle = \sqrt{\frac{2L+1}{4}}\,Y_{LM}(\theta,\varphi)\,. \tag{4}$$

Definition (COHERENT STATE REPRESENTATION): With coherent state wave functions defined by eq. (2), the coherent state representation $\Gamma(X)$ of an infinitesimal generator X of the group G is defined by

$$[\Gamma(X)\Psi](g) = \langle g|T(X)|\psi\rangle = \langle 0|T(g)T(X)|\psi\rangle\,, \quad g \in G\,. \tag{5}$$

For example, the group SU(2) has coherent state wave functions

$$\Psi_{jm}(z) = \langle j, m{=}-j|e^{z\hat{J}_-}|jm\rangle\,, \quad m = -j, \ldots, +j\,, \tag{6}$$

and the definition (5) gives

$$\Gamma(J_-) = \frac{d}{dz}\,, \quad \Gamma(J_0) = z\frac{d}{dz} - j\,, \quad \Gamma(J_+) = 2jz - z^2\frac{d}{dz}\,. \tag{7}$$

This is the well-known Dyson representation of su(2).

3 Vector coherent state representations

The above shows that a good choice of the fixed state $|0\rangle$ can result in a simple coherent state representation. We now show that by choosing a vector space of intrinsic states rather than a single state, much more simplification is achieved and the theory becomes much more powerful and versatile.

Definition (VCS WAVE FUNCTIONS): If $B = \{\xi_\nu \equiv |\nu\rangle\}$ is an orthonormal basis for a fixed subspace $U \subset \mathbb{H}$ of the Hilbert space for an irrep T of a group G and

[a] We use upper case symols for the group and lower case for its Lie algebra.

$N \subset G^c$ *is a subset of elements of the complex extension of* G *such that the coherent states*

$$\{|\nu(z)\rangle = T^\dagger(z)|\nu\rangle\,,\ z\in N\,,\ |\nu\rangle\in B\} \tag{8}$$

span the Hilbert space $\mathbb{H}$, *then any state* $|\psi\rangle\in\mathbb{H}$ *can be represented by a VCS wave function* Ψ, *with vector values in* U *given by*

$$\Psi(z) = \sum_\nu \xi_\nu\,\langle\nu(z)|\psi\rangle = \sum_\nu \xi_\nu\,\langle\nu|T(z)|\psi\rangle\,,\quad z\in N\,. \tag{9}$$

Definition (VCS REPRESENTATION): With VCS wave functions defined by eq. (9), the VCS representation $\Gamma(X)$ *of an infinitesimal generator* X *of the group* G *is defined by*

$$[\Gamma(X)\Psi](z) = \sum_\nu \xi_\nu\,\langle\nu(z)|T(X)|\psi\rangle = \sum_\nu \xi_\nu\,\langle\nu|T(z)T(X)|\psi\rangle\,,\quad z\in N\,. \tag{10}$$

4 VCS representations of su(3) in an su(2) basis

The su(3) algebra is a subalgebra of traceless Hermitian complex linear combinations of a set of matrices $\{C_{ij}\}$ with entries

$$(C_{ij})_{kl} = \delta_{ik}\delta_{jl} \tag{11}$$

and commutation relations

$$[C_{ij}, C_{kl}] = \delta_{jk}C_{il} - \delta_{il}C_{kj}\,. \tag{12}$$

The complex extension of su(3) is spanned by the matrices

$$e_2 = C_{13}\,,\quad e_3 = C_{12}\,, \tag{13}$$

$$H_1 = C_{11} - \tfrac{1}{2}(C_{22}+C_{33})\,,\quad H_2 = C_{22}-C_{33}\,,\quad e_1 = C_{23},\quad f_1 = C_{32}\,, \tag{14}$$

$$f_2 = C_{31}\,,\quad f_3 = C_{21}\,. \tag{15}$$

These matrices are associated with the root vectors of the root diagram for su(3) shown in fig. 1. The horizontal root vectors define a u(2)⊂su(3) subalgebra.

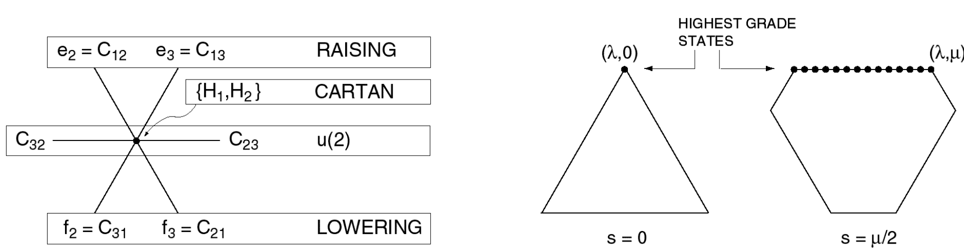

Figure 1. The root diagram for su(3) showing a u(2) subalgebra and complementary sets of raising and lowering operators. Also shown are outlines of the weight diagrams for irreps of highest weight $(\lambda, 0)$ and (λ, μ) and their highest grade states.

For an irrep T of highest weight $(\lambda, 0)$, it is best to choose the highest weight state as the fixed state for a scalar coherent state representation. This choice results in considerable simplification because the highest weight state is annihilated by the raising operators $\hat{e}_2 = T(C_{12})$, $\hat{e}_3 = T(C_{13})$, and the su(2) operators $\hat{H}_2$, $\hat{C}_{23}$, $\hat{C}_{32}$; it satisfies the equations

$$\hat{H}_1|0\rangle = \lambda|0\rangle\,, \quad \hat{H}_2|0\rangle = \hat{C}_{23}|0\rangle = \hat{C}_{32}|0\rangle = \hat{e_2}|0\rangle = \hat{e_3}|0\rangle = 0\,. \tag{16}$$

The Hilbert space for the representation T of highest weight $(\lambda, 0)$ is then spanned by the states $\{|z\rangle = e^{z_2^* \hat{f}_2 + z_3^* \hat{f}_3}|0\rangle\}$ for a suitable range of a pair of complex variables z_2 and z_3.

For a general irrep T of highest weight $(\lambda\mu)$, the states that are annihilated by the $\hat{e}_2$ and $\hat{e}_3$ raising operators

$$U = \big\{|\phi\rangle \in \mathbb{H} \mid \hat{e}_2|\phi\rangle = \hat{e}_3|\phi\rangle = 0\big\} \tag{17}$$

are not also annihilated by elements of the u(2) $\subset$ su(3) subalgebra. However, they span a u(2)–invariant subspace $U \subset \mathbb{H}$ of *highest grade* states. Moreover, if $\{\xi_\nu \equiv |s\nu\rangle\}$ is an orthornormal basis for U indexed by ν, then the Hilbert space for the su(3) representation T is spanned by the states $\{e^{z_2^*\hat{f}_2 + z_3^*\hat{f}_3}|s\nu\rangle\}$ for a suitable range of the complex variables z_2 and z_3. Thus, any state $|\psi\rangle$ in the Hilbert space is represented by the VCS wave function

$$\Psi(z) = \sum_\nu \xi_\nu \langle s\nu|e^{\hat{z}}|\psi\rangle\,, \quad \hat{z} = z_2\hat{e}_2 + z_3\hat{e}_3\,. \tag{18}$$

An element X of the su(3) algebra is then represented as a linear operator $\Gamma(X)$ on the VCS wave functions that is defined by

$$[\Gamma(X)\Psi](z) = \sum_\nu \xi_\nu \langle s\nu|e^{\hat{z}}\hat{X}|\psi\rangle = \sum_\nu \xi_\nu \langle s\nu|\hat{X}(z)e^{\hat{z}}|\psi\rangle\,, \tag{19}$$

where

$$\hat{X}(z) = e^{\hat{z}}\hat{X}e^{-\hat{z}} = \hat{X} + [\hat{z}, X] + \tfrac{1}{2}[\hat{z}, [\hat{z}, \hat{X}]]\,. \tag{20}$$

Explicit expressions for the $\Gamma(X)$ operators are obtained by first observing that $\hat{X}(z)$ is an element of $su(3)$ and that

$$\textstyle\sum_\nu \xi_\nu \langle s\nu|\hat{f}_i e^{\hat{z}}|\psi\rangle = 0\,, \quad \sum_\nu \xi^\nu \langle s\nu|\hat{e}_i e^{\hat{z}}|\psi\rangle = \partial_i \Psi(z)\,, \tag{21}$$

$$\textstyle\sum_\nu \xi_\nu \langle s\nu|\hat{H}_1 e^{\hat{z}}|\psi\rangle = (\lambda + s)\Psi(z)\,, \quad \sum_\nu \xi_\nu \langle s\nu|\hat{H}_2 e^{\hat{z}}|\psi\rangle = 2\hat{s}_0\Psi(z)\,, \tag{22}$$

$$\textstyle\sum_\nu \xi_\nu \langle s\nu|\hat{C}_{23} e^{\hat{z}}|\psi\rangle = \hat{s}_+\Psi(z)\,, \quad \sum_\nu \xi_\nu \langle s\nu|\hat{C}_{32} e^{\hat{z}}|\psi\rangle = \hat{s}_-\Psi(z)\,, \tag{23}$$

where $\hat{s}_0$ and $\hat{s}_\pm$ are intrinsic spin operators defined such that

$$\hat{s}_0\xi_\nu = \nu\,\xi_\nu\,, \quad \hat{s}_\pm\,\xi_\nu = \sqrt{(s \mp \nu)(s \pm \nu + 1)}\,\xi_{\nu\pm1}\,, \tag{24}$$

with $s = \mu/2$. It follows that

$$\Gamma(H_1) = \lambda + s - \tfrac{3}{2}\hat{n}\,, \quad \Gamma(H_2) = 2(\hat{s}_0 + \hat{j}_0)\,, \tag{25}$$

$$\Gamma(C_{23}) = \hat{s}_+ + \hat{j}_+\,, \quad \Gamma(C_{32}) = \hat{s}_- + \hat{j}_-\,, \quad \Gamma(e_i) = \partial_i\,, \tag{26}$$

$$\textstyle\Gamma(f_2) = [\lambda - \hat{s}_0]z_2 - \hat{s}_+z_3 - z_2\sum_i z_i\partial_i\,, \tag{27}$$

$$\textstyle\Gamma(f_3) = [\lambda + \hat{s}_0]z_3 - \hat{s}_-z_2 - z_3\sum_i z_i\partial_i\,, \tag{28}$$

where

$$\hat{n} = \sum_i z_i \partial_i \,, \quad \hat{j}_0 = \tfrac{1}{2}(z_2\partial_2 - z_3\partial_3)\,, \quad \hat{j}_+ = z_2\partial_3\,, \quad \hat{j}_- = z_3\partial_2\,. \tag{29}$$

It is seen that all the operators are simply expressed in terms of the elements $\hat{s}_i$ and $\hat{j}_i$ of two su(2) algebras, one of which of which is regarded as an intrinsic spin. The most complicated operators in the set are $\Gamma(f_1)$ and $\Gamma(f_2)$. However, their matrix elements are easily determined by expressing them in the form

$$\Gamma(f_i) = [\hat{\Lambda}, z_i]\,, \tag{30}$$

where

$$\hat{\Lambda} = (\lambda + s)\hat{n} - \tfrac{1}{2}\hat{n}(\hat{n}-1) - 2\hat{\mathbf{s}}\cdot\hat{\mathbf{j}}\,. \tag{31}$$

The expressions suggest defining orthonormal basis states for the su(3) irrep in the su(2)–coupled form

$$\psi_{jJM}(z) = K_{jJ}\,[\xi \otimes \varphi_j(z)]_{JM}\,, \tag{32}$$

where

$$\varphi_{jm}(z) = \frac{(z_2)^{j+m}(z_3)^{j-m}}{\sqrt{(j+m)!(j-m)!}}\,, \quad m = -j, \ldots, +j\,, \tag{33}$$

and the norm factors $\{K_{jJ}\}$ remain to be determined. It is seen that $\Gamma(H_1)$, $\Gamma(H_2)$ and $\hat{\Lambda}$ are diagonal in this basis with eigenvalues given by

$$\Gamma(H_1)\,\psi_{jJM} = (\lambda + s - 3j)\,\psi_{jJM}\,, \quad \Gamma(H_2)\,\psi_{jJM} = 2M\,\psi_{jJM}\,, \tag{34}$$

$$\hat{\Lambda}\,\psi_{jJM} = \Omega(sjJ)\,\psi_{jJM}\,, \tag{35}$$

and

$$\Omega(sjJ) = 2(\lambda+s)j + s(s+1) - j(j-2) - J(J+1)\,. \tag{36}$$

The operators $\Gamma(C_{23})$ and $\Gamma(C_{32})$ are simply the su(2) raising and lowering operators

$$\Gamma(C_{23}) = \hat{J}_+ = \hat{s}_+ + \hat{j}_+\,, \quad \Gamma(C_{32}) = \hat{J}_- = \hat{s}_- + \hat{j}_-\,, \tag{37}$$

with the usual su(2) actions

$$\hat{J}_\pm \psi_{jJM} = \sqrt{J \mp M)(J \pm M + 1)}\;\psi_{jJ,M\pm 1}\,. \tag{38}$$

The matrix elements of ∂_i and z_i can be evaluated explicity for the given basis wave functions. Since the z_i are components of an su(2) spin–1/2 tensor, the result is conveniently expressed in terms of reduced matrix elements. With some Racah recoupling, we obtain

$$\begin{aligned}\langle sjJ\|\hat{e}\|s, j+\tfrac{1}{2}, J'\rangle &= -\sqrt{(2j+1)(2j+2)(2J+1)(2J'+1)}\;W(\tfrac{1}{2}jJ's : j+\tfrac{1}{2}J)\\ &\quad\times \frac{K_{j+1/2,J'}}{K_{jJ}}\,,\end{aligned} \tag{39}$$

$$\begin{aligned}\langle s, j+\tfrac{1}{2}, J'\|\hat{f}\|sjJ\rangle &= (-1)^{J'-J+\frac{1}{2}}\,\langle sjJ\|\hat{e}\|s, j+\tfrac{1}{2}, J'\rangle\\ &\quad\times \left(\frac{K_{jJ}}{K_{j+1/2,J'}}\right)^2\;[\tfrac{1}{2}(2\lambda+\mu) + J(J+1) - J'(J'+1) - j + \tfrac{3}{4}]\,,\end{aligned} \tag{40}$$

Thus, by setting

$$\left(\frac{K_{j+1/2,J'}}{K_{jJ}}\right)^2 = \tfrac{1}{2}(2\lambda+\mu) + J(J+1) - J'(J'+1) - j + \tfrac{3}{4}\,, \tag{41}$$

we obtain the reduced matrix elements of a unitary representation with j and J running over all integer and half-odd integer values for which the K_{jJ} coefficients are non-zero.

5 Representation of $su(3)$ in an $so(3)$ basis

For applications in nuclear physics, one needs the su(3) representations in an angular momentum basis. They are easily constructed in coherent state theory as a result of the well-known observation [?]:

If $|0\rangle$ is a highest weight state for an su(3) irrep, then the rotated states

$$\{\hat R(\Omega)|\lambda\mu\rangle; \Omega\in SO(3)\} \tag{42}$$

span the Hilbert space $\mathbb{H}$ of this irrep.

Suppose the Hilbert space $\mathbb{H}$ has an orthonormal basis of angular-momentum coupled states $\{|\alpha LM\rangle\}$. Then, these states are represented by coherent state wave functions of the form

$$\Psi_{\alpha LM}(\Omega) = \langle\lambda\mu|\hat R(\Omega)|\alpha LM\rangle = \sum_K \langle\lambda\mu|\alpha LK\rangle\, \mathcal{D}^L_{KM}(\Omega)\,. \tag{43}$$

An element X of the $su(3)$ Lie algebra then has coherent state representation as a linear operator $\Gamma(X)$ on the coherent state wavefunctions, defined by

$$[\Gamma(X)\Psi](\Omega) = \langle\lambda\mu|\hat R(\Omega)\hat X|\psi\rangle = \langle\lambda\mu|\hat X(\Omega)\hat R(\Omega)|\psi\rangle\,, \tag{44}$$

where (with $\tilde\Omega$ denoting the transpose of Ω)

$$\hat X(\Omega) = \hat R(\Omega)\hat X\hat R(\tilde\Omega)\,. \tag{45}$$

In an angular-momentum basis, the su(3) algebra is spanned by the angular momentum and quadrupole operators with components given in terms of the root vectors shown in fig. 1 by

$$L_0 = -\mathrm{i}(C_{23}-C_{32})\,,\quad L_\pm = \mathrm{i}(e_3-f_3)\pm(e_2-f_2)\,, \tag{46}$$

$$Q_0 = 2H_1\,,\quad Q_{\pm1} = \mp\sqrt{\tfrac{3}{2}}\,[e_2+f_2\pm\mathrm{i}(e_3+f_3)]\,,$$

$$Q_{\pm2} = \sqrt{\tfrac{3}{2}}\,[H_2\pm\mathrm{i}(C_{23}+C_{32})]\,. \tag{47}$$

From the definition (44), the coherent state representation of a quadrupole operator is given by

$$\begin{aligned}[\Gamma(Q_m)\Psi_{\kappa LM}](\Omega) &= \langle\lambda\mu|\hat R(\Omega)Q_m|\kappa LM\rangle \\ &= \sum_\nu \langle\lambda\mu|\hat Q_\nu|\kappa LK\rangle\, \mathcal{D}^2_{\nu m}(\Omega)\mathcal{D}^L_{KM}(\Omega)\,.\end{aligned} \tag{48}$$

The matrix elements $\langle\lambda\mu|\hat{Q}_\nu|\kappa LK\rangle$ are inferred from the expansions (46) and (47) and the identities

$$\langle\lambda\mu|\hat{L}_0|\alpha LK\rangle = K\,\langle\lambda\mu|\hat{L}_0|\alpha LK\rangle\,, \tag{49}$$

$$\langle\lambda\mu|\hat{L}_\pm|\alpha LK\rangle = \sqrt{(L\mp K)(L\pm K+1)}\,\langle\lambda\mu|\alpha L, K\pm 1\rangle\,, \tag{50}$$

$$\langle\lambda\mu|\hat{H}_1|\alpha LK\rangle = \tfrac{1}{2}(2\lambda+\mu)\langle\lambda\mu|\alpha LK\rangle\,, \quad \langle\lambda\mu|\hat{H}_2|\alpha LK\rangle = \mu\langle\lambda\mu|\alpha LK\rangle\,, \tag{51}$$

$$\langle\lambda\mu|\hat{C}_{32}|\alpha LK\rangle = \langle\lambda\mu|\hat{f}_i|\alpha LK\rangle = 0\,. \tag{52}$$

One finds [10] that, if

$$\Psi_{\alpha LM} = \sum_K a_K(\alpha L)\,\mathcal{D}^L_{KM}\,, \tag{53}$$

then

$$[\Gamma(Q)\otimes\Psi_{\alpha L}]_{L'M} = \sum_{\kappa\kappa'} M^{(L'L)}_{\kappa'\kappa} a_\kappa(\alpha L)\,\mathcal{D}^{L'}_{\kappa' M} \tag{54}$$

with

$$\begin{aligned} M^{(L'L)}_{\kappa'\kappa} = {} & \delta_{\kappa',\kappa}\left[(2\lambda+\mu+3)+\delta_{K1}\sigma_{L'L}-\tfrac{1}{2}L'(L'+1)+\tfrac{1}{2}L(L+1)\right](L\kappa,20|L'\kappa) \\ & +\delta_{\kappa',\kappa+2}\sqrt{\tfrac{3}{2}}\,(\mu-\kappa)(L\kappa,22|L'\kappa+2) \\ & +\delta_{\kappa',\kappa-2}\sqrt{\tfrac{3}{2}}\,(\mu+\kappa)(L\kappa,2-2|L'\kappa-2)\,, \end{aligned} \tag{55}$$

and

$$\sigma_{L'L} = \tfrac{1}{2}(\mu+1)(-1)^{\lambda+L}\times\begin{cases} -\frac{3L(L+1)}{3-L(L+1)} & \text{for } L'=L \\ L+1 & \text{for } L'=L+1 \\ -L & \text{for } L'=L-1 \\ -1 & \text{for } L'=L\pm 2\,. \end{cases} \tag{56}$$

Thus, one needs only a table of Clebsch-Gordan coefficients to obtain explicit matrix elements of the su(3) quadrupole operators in any given basis of $a(\alpha L)$ vectors. However, to obtain the matrix elements of a unitary representation, one needs an orthonormal basis. An orthonormal basis is constructed as follows.

A set of vectors $\{a(\alpha L)\}$ whose components are the expansion coefficients $\{a_K(\alpha L)\}$ of an orthonormal basis $\{|\alpha LM\rangle\}$ is now constructred by use of the following three theorems:

Theorem 1 (Elliott): *An SU(3) irrep of highest weight $(\lambda\mu)$ contains a sequence of SO(3) states of angular momenta*

$$L = \begin{cases} \lambda+K, \lambda+K-1, \ldots, K & \textit{for } K\neq 0 \\ \lambda, \lambda-2, \ldots, 0 \text{ or } 1 & \textit{for } K=0 \end{cases} \tag{57}$$

with K running over the range

$$K = \mu, \mu-2, \ldots 0 \text{ or } 1\,. \tag{58}$$

Theorem 2: *If Ψ_1 is a wave function from an irrep $(\lambda_1\mu_1)$ and Ψ_2 is a wave function from an irrep $(\lambda_2\mu_2)$, then Ψ defined by $\Psi(\Omega) = \Psi_1(\Omega)\Psi_2(\Omega)$ is a wave function belonging to the irrep $(\lambda_1+\lambda_2, \mu_1+\mu_2)$.*

Thus, we can easily build up a non-orthonormal basis starting from the wave functions for the $(1,0)$ and $(0,1)$ irreps:

$$\Psi^{(1,0)}_{01M}(\Omega) \propto \mathcal{D}^1_{0M}(\Omega)\,, \quad \Psi^{(0,1)}_{11M}(\Omega) \propto \mathcal{D}^1_{1M}(\Omega) + \mathcal{D}^1_{-1,M}(\Omega)\,. \tag{59}$$

Theorem 3: *A state $|\alpha LM\rangle$ which is an eigenfunction of the scalar operator $[L\otimes Q\otimes L]_0$ is characterized by an eigenvector of the matrix $M^{(LL)}$, i.e., a vector $a_K(\alpha L)$ satisfying the equation*

$$\sum_{K'} M^{(LL)}_{KK'} a_{K'}(\alpha L) = a_K(\alpha L)\,. \tag{60}$$

Moreover, a set of vectors $\{a(\alpha L)\}$ which are all eigenvectors of the corresponding $\{M^{(LL)}\}$ matrices define a set of orthogonal states with coherent state wave functions given by eqn. (53).

Proof: The theorem follows from the observation that

$$[L\otimes Q\otimes L]_0|\alpha LM\rangle \propto [\Gamma(Q)\times\Psi_{\alpha L}]_{LM} \tag{61}$$

and the observation that, for an SU(3) irrep, there are no multiplicities of $L=0$ states for which $[\Gamma(Q)\times\Psi_{\alpha L}]_{LM}$ vanishes. Orthogonality of states of different L and/or different M follows automatically from the transformation properties of the states under $SO(3)$ rotations. □

Thus, for each of the L values in eq. (57), one can diagonalize the corresponding $\{M^{(LL)}\}$ matrix to obtain the $\{a(\alpha L)\}$ eigenvectors for a set of orthogonal states. It then remains only to normalize these vectors such that the reduced matrix elements, defined by the Wigner-Eckart theorem in an orthonormal basis by

$$[\Gamma(Q)\times\psi_{\alpha L}]_{L'M} = \frac{1}{\sqrt{2L'+1}}\sum_\beta \psi_{\beta L'M}\,\langle\beta L'\|Q\|\alpha L\rangle\,, \tag{62}$$

satisfy the hermiticity relationship of a unitary representation

$$\langle\beta L'\|Q\|\alpha L\rangle = (-1)^{L-L'}\langle\alpha L\|Q\|\beta L'\rangle^*\,. \tag{63}$$

For a multiplicity-free representation, i.e., one for which the α label is redundant, everything can be done analytically as shown explicitly in ref. [10]. For example, for the multiplicity free $(\lambda,0)$ irreps the reduced matrix elements are given by

$$\langle L\|Q\|L\rangle = \sqrt{2L+1}\,(L0,20|L0)\,(2\lambda+3)\,, \tag{64}$$

$$\langle L+2\|Q\|L\rangle = \sqrt{2L+1}\,(L0,20|L+2,0)\,\big[4(\lambda-L)(\lambda+L+3)\big]^{\frac{1}{2}}\,. \tag{65}$$

It is seen that the sequence of angular momentum states with $L=0,2,4,\ldots$ or $L=1,3,5,\ldots$ terminates with $L=\lambda$ in accordance with the branching rule (57). More details of the procedure are given in refs. [9,10,3].

VCS theory gives analytical asymptotic expressions for matrix elements of su(3) in an so(3) basis that become accurate as the dimension of the representation becomes large. This is because the su(3) algebra has the rotor model algebra as a contraction limit.

6 Representations of so(5) in an so(3) basis

I conclude by indicating that construction of the generic representations of so(5) in an so(3) basis, is also straightforward by VCS methods. A computer code for implimenting the construction, written by Peter Turner [12], and will soon be generally available.

Constructions have been given for the so-called one-rowed representations by several authors (reviewed in ref. [11]). These are the representations that appear in the space of a single particle in a five-dimensional harmonic oscillator and in the IBM1 version of the Interacting Boson Model [15]. The generic two-rowed representations occur for two or more particles in a five-dimensional oscillator and in the neutron-proton IBM2 version of the Interacting Boson Model.

It was shown a while ago by Hecht and myself [11] that the one-rowed so(5) representations are constructed simply and systematically by the methods outlined in section V for su(3). For the solution of this problem, we did not need the full power of VCS theory; the coherent state wave functions were simple scalar functions. However, for the generic representations, vector-valued coherent state wave functions are needed.

Parallels with the su(3) representation theory can be seen by comparison of the root diagrams for the two algebras, shown respectively in figs. 1 and 2. Both Lie algebras are of rank two and their irreps can be labelled by highest weights $(\lambda\mu)$. In both cases the corresponding highest weight state $|\lambda\mu\rangle$ for an irrep is an eigenstate of two mutually orthogonal Cartan operators $\hat{H}_1$ and $\hat{H}_2$ with eigenvalues

$$\hat{H}_1|\lambda\mu\rangle = \tfrac{1}{2}(2\lambda+\mu)\,|\lambda\mu\rangle\,, \quad \hat{H}_2|\lambda\mu\rangle = \mu\,|\lambda\mu\rangle\,. \tag{66}$$

Moreover, for both Lie algebras, there are subspaces of highest grade states that are annihilated by a subset of raising operators and carry irreps of the 'horizontal' u(2) algebras shown in figs. 1 and 2. Thus, for so(5), as for su(3) we can define an orthonormal basis $\{\xi_\nu \equiv |s\nu\rangle; \nu = -s,\dots,+s\}$ of highest grade states for a u(2) irrep of spin $s = \mu/2$. Constructing an so(5) irrep in an so(3) basis by inducing from this u(2) irrep using VCS theory is now made possible by the following observation.

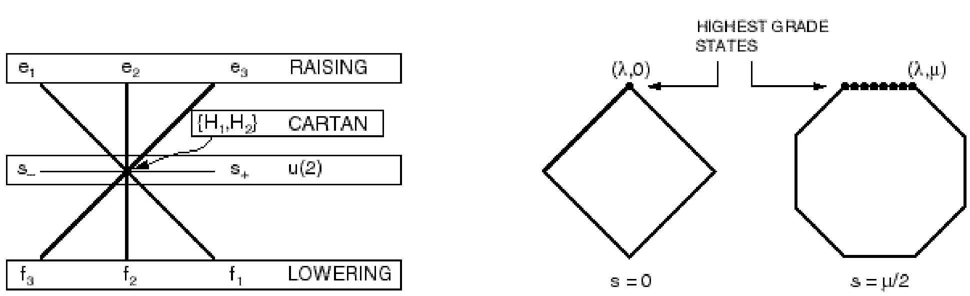

Figure 2. The root diagram for so5 showing a u(2) subalgebra and complementary sets of raising and lowering operators. Also shown are outlines of the weight diagrams for irreps of highest weight $(\lambda,0)$ and (λ,μ) and their highest grade states.

Observation: *Provided the Cartan subalgebra for so(5) is chosen such that it contains no component of the angular momentum, then the set of states* $\{\hat{R}(\Omega)|s\nu\rangle; \Omega \in \mathrm{SO}(3)\}$ *spans the Hilbert space for the so(5) irrep of highest weight* $(\lambda\mu)$.

It follows that, if the highest grade states $\{|s\nu\rangle\}$ are assigned wave functions $\{\xi_\nu\}$ then a state $|\psi\rangle$ of the so(5) irrep of highest weight $(\lambda\mu)$ is defined by the set of overlaps

$$\Psi(\Omega) = \sum_\nu \xi_\nu \langle s\nu|\hat{R}(\Omega)|\psi\rangle \, . \tag{67}$$

Thus, the vector- valued function Ψ over the group $SO(3)$ is a VCS a wave function for the state $|\psi\rangle$. The rest of the construction follows the general prescription for such VCS representations.

7 Discussion

The above fairly detailed outline of the su(3) irreps in both su(2) and so(3) basis give the essential principles underlying VCS theory. The application to so(5) is an indication of how the theory can be applied systematically to the construction of representations by straighforward systematic methods that have traditionally proved challenging. The limitations of the theory have yet to be discovered.

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