DEVELOPMENTS OF ALGEBRAIC COLLECTIVE MODELS AND SECOND-ORDER PHASE TRANSITIONS[*†]

D. J. ROWE

Deparment of Physics, University of Toronto,
Toronto, Ontario, M5S 1A7, Canada
E-mail: rowe@physics.utoronto.ca

This talk focuses on three topics: the development of a program to determine SO(5) spherical harmonics and SO(5) Clebsch-Gordan coefficients; efficient ways to do collective model calculations in an $SU(1,1) \times SO(5) \supset U(1) \times SO(3)$ basis; and quasi-dynamical symmetry in an IBM second-order phase transition.

1. Introduction

Attempts to understand phase transitions have profited considerably from the study of models with symmetry. Landau stated that two phases of matter with different symmetries (which cannot change continuously from one to the other) must be separated by a line of transition. Consider a system with control parameter α which is in a phase with a symmetry group G_1 when $\alpha = 0$ and in a phase with symmetry group G_2 when $\alpha = 1$. The question then is what happens when α is varied continuously from 0 to 1? It often transpires that the model exhibits a second-order phase transition from a phase characterized by one symmetry to a phase characterized by the other. However, closer examination reveals that, in the phase characterized by the G_1 symmetry, the symmetry of the system is increasingly distorted by the forces that favour the competing phase, as α is increased, until a point comes at which it can be distorted no further and a rapid change occurs to a phase dominated by the G_2 symmetry. A complementary behaviour may be observed when the critical point is approached from the other side. The distorted symmetries, called quasi-

[*]Work supported by the Natural Sciences and Engineering Research Council of Canada.
[†]Published in Proceedings of the 8th International Spring Seminar on Nuclear Physics, Paestum, Italy, May 23-27, 2004, (ed. A. Covello, World Scientific, 2005)

dynamical symmetries, have an elegant expression in the language of group theory and lead to new concepts in representation theory of considerable significance for understanding why models with symmetries are often more successful in practice than they apparently have any right to be. Such phase transitions have been examined in a variety of models.[1] A review of quasi-dynamical symmetry has been given in Ref.[2]

2. Bases for the hydrodynamic collective model

Standard basis functions are given by eigenfunctions of the harmonic vibrator Hamiltonian

$$\hat{H} = \frac{1}{2B}\nabla^2 + \frac{1}{2}B\omega^2\beta^2\,, \tag{1}$$

where $\beta^2 = q \cdot q$ is the squared length of the quadrupole tensor. This Hamiltonian is U(5), SO(5), and SO(3) invariant and an element of an $\mathrm{SU}(1,1) \times \mathrm{SO}(5)$ spectrum generating algebra. These basis functions

$$\Psi_{nv\alpha LM}(\beta,\gamma,\Omega) = R_{nv}(\beta)\mathcal{Y}_{v\alpha LM}(\gamma,\Omega)\,, \tag{2}$$

reduce the subgroup chain

$$\begin{array}{ccccccc} \mathrm{SU}(1,1) & \times & \mathrm{SO}(5) & \supset & \mathrm{U}(1) & \times & \mathrm{SO}(3) \supset \mathrm{SO}(2) \\ v & & & & \alpha & v & L \qquad M \end{array} \tag{3}$$

The SU(1,1) $\supset$ U(1) beta wave functions are well known in terms of generalized Laguerre polynomials. A basis of SO(5) $\supset$ SO(3) $\supset$ SO(2) wave functions can be written down immediately (cf. Ref. [3]) in the form

$$\Phi_{tKLM}(\gamma,\Omega) = f_{tKL}(\gamma)\left[\mathcal{D}^L_{KM}(\Omega) + (-1)^L\mathcal{D}^L_{-K,M}(\Omega)\right], \tag{4}$$

where the functions $\{f_{tKL}\}$ are simple polynomials in $\cos\gamma$ and $\sin\gamma$ and K is an even integer. This basis is then orthonormalized sequentially to give SO(5) spherical harmonics that satisfy the familiar inner product

$$\int \mathcal{Y}^*_{v\alpha LM}(\gamma,\Omega)\mathcal{Y}_{v'\alpha'L'M'}(\gamma,\Omega)\,\sin 3\gamma\,\mathrm{d}\gamma\,\mathrm{d}\Omega = \delta_{vv'}\delta_{\alpha\alpha'}\delta_{LL'}\delta_{MM'}\,. \tag{5}$$

The integrals needed for this procedure are evaluated analytically although a computer is used to keep track of the results. The methods we use[3] make build on many of the results of Chacón *et al.*[4]

Having determined a set of SO(5) spherical harmonics, it is straightforward to compute the SO(5) CG coefficients of relevance to the collective

model and the IBM1 by evaluating the integrals

$$\int \mathcal{Y}^*_{v_3\alpha_3 L_3 M_3}(\gamma,\Omega)\mathcal{Y}_{v_2\alpha_2 L_2 M_2}(\gamma,\Omega)\mathcal{Y}_{v_1\alpha_1 L_1 M_1}(\gamma,\Omega)\,\sin 3\gamma\,\mathrm{d}\gamma\,\mathrm{d}\Omega \propto (v_1\alpha_1 L_1 M_1, v_2\alpha_2 L_2 M_2|v_3\alpha_3 L_3 M_3)\,. \quad (6)$$

CG coefficients for the couplings $v \otimes 1 \to v'$ are tabulated in Ref.[3]

3. A more efficient basis for deformed nuclei

With a basis of SO(5)-coupled wave functions, it is possible to diagonalize a general collective model Hamiltonians. However, for well-deformed nuclei, a large number of spherical vibrator basis states are needed for accurate results.

As shown by Elliott *et al.*[5], beta wave functions that are much closer to those of a deformed nucleus are given by eigenfunctions of the Hamiltonian

$$\hat{H}(\beta_0) = -\frac{1}{2B}\nabla^2 + \frac{1}{2}B\omega^2\left(\beta^2 + \frac{\beta_0^4}{\beta^2}\right)\,, \quad (7)$$

where β_0 is a suitably chosen parameter. The potential for this Hamiltonian has a minimum value when $\beta = \beta_0$. As shown in Refs.[6,7], this Hamiltonian also defines a basis for the collective model that reduces the subgroup chain $\mathrm{SU}(1,1) \times \mathrm{SO}(5) \supset \mathrm{U}(1) \times \mathrm{SO}(3)$, where the SU(1,1) algebra is spanned by operators (with β now expressed in harmonic oscillator units)

$$\hat{X}_1 = \tfrac{1}{4}\left(-\nabla^2 - \beta^2 + \tfrac{\beta_0^4}{\beta^2}\right)\,, \quad (8)$$

$$\hat{X}_2 = \tfrac{1}{4}\left(q\cdot\nabla + \nabla\cdot q\right)\,, \quad (9)$$

$$\hat{X}_3 = \tfrac{1}{4}\left(-\nabla^2 + \beta^2 + \tfrac{\beta_0^4}{\beta^2}\right)\,, \quad (10)$$

that satisfy the commutation relations

$$[\hat{X}_1, \hat{X}_2] = -i\hat{X}_3\,, \quad [\hat{X}_2, \hat{X}_3] = i\hat{X}_1\,, \quad [\hat{X}_3, \hat{X}_1] = i\hat{X}_2\,. \quad (11)$$

The energy-level spectrum for the Hamiltonian (7) is given by[5,6]

$$E_{\nu v} = (2\nu + \lambda_v)\hbar\omega, \quad \lambda_v = 1 + \sqrt{\left(v + \tfrac{3}{2}\right)^2 + \beta_0^4} \quad (12)$$

and the corresponding wave functions are again known in terms of generalized Laguerre polynomials.

As an example of the kind of calculation that can be done with the above-defined basis wave functions, Fig. 1 shows the energy-level spectrum and E2 transition rates obtained by diagonalizing the Hamiltonian

$$\hat{H}(\beta_0) = -\frac{1}{2B}\nabla^2 + \frac{1}{2}B\omega^2\left(\beta^2 + \frac{\beta_0^4}{\beta^2}\right) - \chi\cos 3\gamma \quad (13)$$

with a single beta wave function. Results obtained to a similar accuracy in a spherical vibrator basis require of the order of 100 basis wave functions. Details of the calculation are given in Ref.[7]

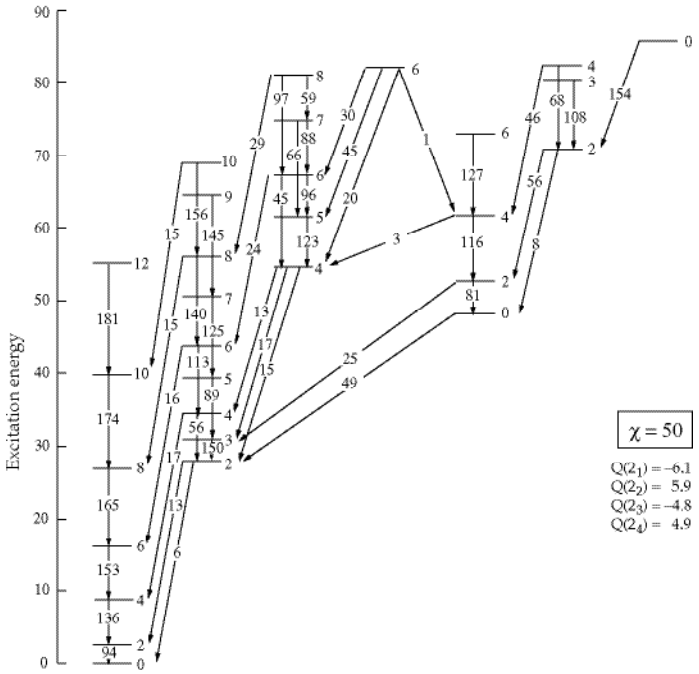

Figure 1. The low-energy spectrum and B(E2) transition rates calculated for the Hamiltonian (7) in Ref.[7]

4. The U(5) to O(5) phase transition in the IBM

I now come to the main subject of this talk which is to review the evolution of the states of a system as it progresses from a phase with one dynamical symmetry to another with variation of a control parameter. We have studied several such systems and the results are remarkably similar.[1] Here I focus on a system of N interacting bosons having two states: a lower-energy s-boson state of angular momentum $L = 0$ and a higher-energy d-boson state of angular momentum $L = 2$. This model was developed for use in nuclear physics[8] but is of much wider interest.

Consider the Hamiltonian

$$\hat{H}(\alpha) = (1-\alpha)\hat{n} + \frac{\alpha}{N}\hat{S}_+\hat{S}_- \,, \tag{14}$$

where $\hat{n}$ is the d-boson number operator and

$$\hat{S}_+ = \tfrac{1}{2}(d^\dagger \cdot d^\dagger - s^\dagger s^\dagger), \quad \hat{S}_- = \tfrac{1}{2}(d \cdot d - ss)\,. \tag{15}$$

are the raising and lowering operators of an SU(1,1) Lie algebra. The Hamiltonian $\hat{H}(\alpha = 0)$ has eigenstates that reduce the subgroup chain

$$\mathrm{U(6)} \supset \mathrm{U(5)} \supset \mathrm{O(5)} \supset \mathrm{SO(3)} \supset \mathrm{SO(2)} \tag{16}$$

whereas eigenstates of $\hat{H}(\alpha = 1)$ reduce the chain

$$\mathrm{U(6)} \supset \mathrm{O(6)} \supset \mathrm{O(5)} \supset \mathrm{SO(3)} \supset \mathrm{SO(2)}\,. \tag{17}$$

Moreover, $\hat{H}(\alpha)$ is easy to diagonalize for arbitrary values of α, because it is an element of an SU(1,1)+SU(1,1) Lie algebra. Thus, it is possible to follow the progression of its eigenstates as a function of the control parameter α.

The low energy-level spectra for $N = 20$ and $N = 40$ are shown in Fig. 2. It is seen that the system appears to hold onto its U(5) symmetry as α

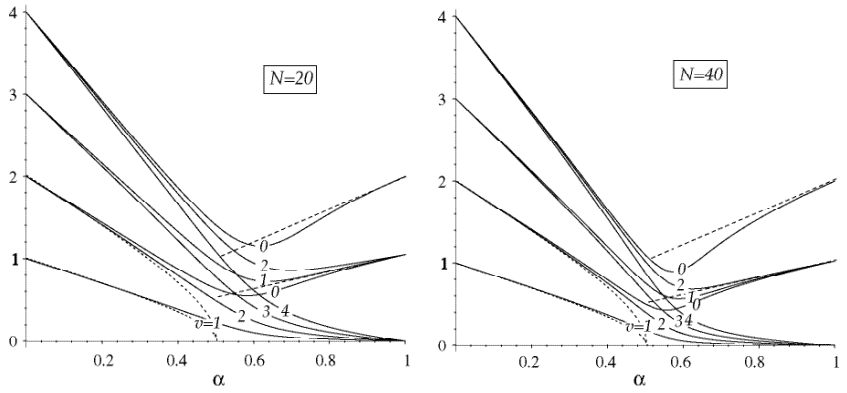

Figure 2. Spectrum of energy levels for $N = 20$ and 40 shown as a function of α for the Hamiltonian $\hat{H}(\alpha)$. Precise numerically computed energies are shown as continuous lines. The dotted lines are the results of an RPA calculation, for $\alpha < 0.5$, and the shifted harmonic approximation, for $\alpha > 0.5$.

increases until it approaches a transition region from below and similarly to holds onto its O(6) symmetry as it approaches the transition region from above. It is also seen that the transition region shrinks as N increases

and, as evidenced by other calculations not shown, it approaches a singular critical point $\alpha_c = 0.5$ as $N \to \infty$.

However, a detailed inspection of the wave function shows the U(5) $\subset$ U(6) symmetry to be badly broken, well before α enters the transition region; thus, the persistent symmetries are really quasi-symmetries. For $\alpha < 0.5$, the quasi-U(5) symmetry can be understood in terms of the Random Phase Approximation. The ground state is an s-boson condensate at $\alpha = 0$. In the RPA, the ground state becomes a quasi-s-boson condensate when $\alpha \neq 0$ in which pairs of s bosons are replaced by zero-coupled d boson pairs. The RPA predictions for excitation energies and E2 transition rates are shown in Figs. 2 and 3 which, respectively, show that the RPA

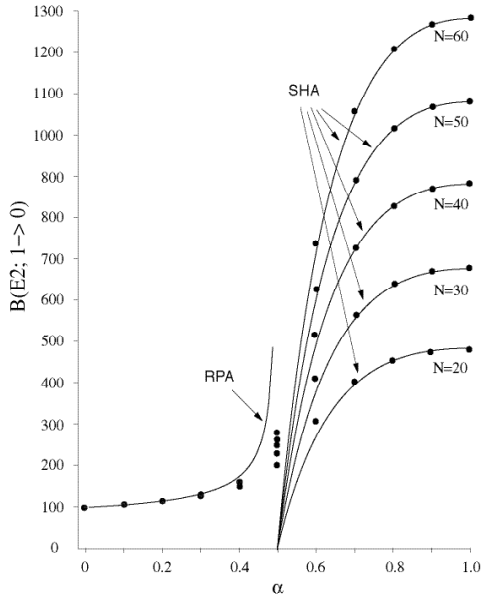

Figure 3. B(E2) transition rates for decay of the first excited $v = 1$ state to the ground state for various values of N. The continuous lines for $\alpha < 0.5$ are for the RPA and those for $\alpha > 0.5$ are for the shifted harmonic approximation (SHA).

excitation energies collapse and the E2 transition rate from the first excited state to the ground state diverges as $\alpha \to \alpha_c$. However, for the values of N shown, the RPA is a very good approximation in the region $0 \leq \alpha \lesssim 0.35$. Moreover, it becomes increasing accurate for all $\alpha < 0.5$ as N increases. The important observation for present purposes is that the RPA shows

the existence of an effective Hamiltonian and effective quadrupole moment operators

$$\hat{H}_1^{\rm eff}(\alpha) = \sqrt{(1-\alpha)(1-2\alpha)}\,\hat{n}, \quad \hat{q}_m^{\rm eff} = e^{\rm eff}(\alpha)\hat{q}_m\,, \tag{18}$$

which are idential to those of the $\alpha = 0$ limit to within α-dependent normalization factors. Thus, the results of the RPA are indistinguishable from those of an effective IBM with U(5) dynamical symmetry.

Similar results hold for $\alpha > 0.5$. The coefficients in the expansion

$$\Psi_{\nu v=0} = \sum_n C_{\nu n}\varphi_{nv=0} \tag{19}$$

of the ground and first excited states of SO(5) seniority $v = 0$ are shown in the U(5) basis for $N = 60$ in Fig. 4 for two α values. The remarkable

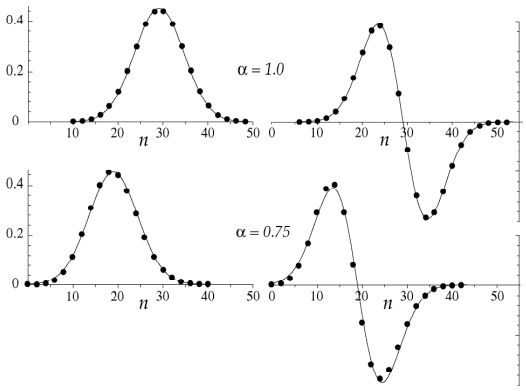

Figure 4. Coefficients of the lowest and first excited states of seniority $v = 0$ of the Hamiltonian (7) for $N = 60$ and $\alpha = 1.0$ and 0.75. It is seen that the wave functions just reach the lower $n = 0$ boundary when $\alpha = 0.75$.

fact is that the coefficients are given very precisely for large N by harmonic oscillator wave functions for $\alpha = 1.0$. Morever, the added term $(1-\alpha)\hat{n}$ in the Hamiltonian behaves as a Lagrange multiplier and simply shifts the centroid of each wave function to a smaller mean value of n but otherwise leaves it unchanged until the shifted wave function reaches the $n = 0$ boundary (n cannot take negative values). This is the point at which a shifted (coherent state) harmonic oscillator approximation starts to break down. It can be shown that the centroid of a wave functions is shifted to $n = 0$ at $\alpha = 0.5$ but because of its width, the wings of a wave function reach the $n = 0$ boundary for higher values of α for finite values of N;

the width of a harmonic oscillator wave function goes to zero as $N \to \infty$. It can be shown that, when the SHA (shifted harmonic approximation) is valid, the properties of the N-boson system described by the Hamiltonian $\hat{H}(\alpha)$ are reproduced accurately by an effective Hamiltonian and effective quadrupole moment operators

$$\hat{H}_2^{\rm eff}(\alpha) = \frac{\alpha}{N}\hat{S}_+\hat{S}_-, \quad \hat{q}_m^{\rm eff} = \left[1 - \left(\frac{1-\alpha}{\alpha}\right)\right]^{\frac{1}{2}} \hat{q}_m \tag{20}$$

which are indistinguishable from those of an effective IBM with O(6) dynamical symmetry. However, O(6) is only a quasi-dynamical symmetry of the original N-boson model.

5. Concluding remarks

The above model analysis of a phase transition shows many properties that have been observed in several similar systems that are of wide physical significance. One is an explanation of why models with symmetry are often much more successful, than could reasonably be expected, even when there are known to be relatively strong symmetry-breaking interactions. The apparent persistence of symmetry is a wide spread phenomenon with a physically natural interpretation in terms of quasi-dynamical symmetry and the corresponding mathematical concept of embedded representations[2]. An understanding of quasi-symmetry and why and when it occurs, is of particular importance for understanding what successful models can really teach us about the systems they represent.

References

1. H. Chen, J.R. Brownstein and D.J. Rowe, Phys. Rev. **C42** 1422 (1990); H. Chen, T. Song and D.J. Rowe, Nucl. Phys. A**582** 181 (1995); D.J. Rowe, C. Bahri and W. Wijesundera, Phys. Rev. Lett. **80** (1998) 4394; C. Bahri, D.J. Rowe, and W. Wijesundera, Phys. Rev. C **58**, 1539 (1998).
2. D.J. Rowe, "Embedded representations and quasi-dynamical symmetry" in *'Computational and Group Theoretical Methods in Nuclear Physics* Proc. Symp. in Honor of Jerry P. Draayer's 60th Birthday (Eds. O. Castanos, J. Escher, J. Hirsch, S. Pittel, and G. Stoicheva. World Scientific, Singapore, 2004).
3. D.J. Rowe, P. Turner and J. Repka, *J. Math. Phys.* **45**, 2761 (2004).
4. E. Chacón, M. Moshinsky, and R.T. Sharp, J. Math. Phys. **17**, 668 (1976); E. Chacón and M. Moshinsky, J. Math. Phys. **18**, 870 (1977).
5. J.P. Elliott, J.A. Evans and P. Park, Phys. Lett. **169**B, (1986) 309; J.P. Elliott, P. Park and J.A. Evans, Phys. Lett. **171**B (1986) 145; cf. also S.G. Rohoziński, J. Srcbrny, and K. Horbaczewska, Z. Phys. **268**, 401 (1974).

6. D.J. Rowe and C. Bahri, J. Phys. A: Math. Gen. **31**, 4947 (1998).
7. D.J. Rowe, *Nucl. Phys.* A**735**, 372 (2004).
8. A. Arima and F. Iachello, Ann. Phys. **99**, 253 (1976); **111**, 201 (1978); O. Scholten, A. Arima and F. Iachello, Ann. Phys. **115**, 325 (1978).