Let's Make a Quantum Deal!

Weak measurements: subensembles from tunneling to Hardy’s Paradox

• First: some more on tunneling times, by way of motivation...
• How does one discuss subensembles in quantum mechanics?
  • Weak measurement
  • How can the spin of a spin-1/2 particle be found to be 100?
  • How can a particle be in two places at once?
  • Where is a particle when it's in the forbidden region?
• What do these weak measurements say about tunneling?

And more on weak measurement to come (next time, presumably):
• The 3-box problem
• Weak measurements shed light on Hardy's Paradox as well
• The which-path controversy
• Bohm trajectories

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Too many tunneling times!

Various "times":

- group delay
- "dwell time"
- Büttiker-Landauer time
  (critical frequency of oscillating barrier)
- Larmor times (three different ones!)

et cetera...

Questions which seem unambiguous classically may have multiple answers in QM – in other words, different measurements which all yield "the time" classically need not yield the same thing in the quantum regime.
In particular: in addition to affecting a pointer, the particle itself may be affected by it.
Relationships between times

\[ \tau_g = \hbar \frac{\partial}{\partial E} \text{arg}(t) \]  
(\text{the group delay})

\[ \tau_y = -\hbar \frac{\partial}{\partial V_0} \text{arg}(t) = \tau_d \rightarrow \tau_s \text{ in WKB limit} \]

\[ \tau_z = -\hbar \frac{\partial}{\partial V_0} \ln|t| \rightarrow \tau_{BL} \text{ in opaque limit} \]

\[ \tau_c = i\hbar \frac{\partial}{\partial V_0} \ln t \]
\[ = \tau_y - i\tau_z. \]
(\text{the “complex time” derived from a Feynman path integral seems to unify the Larmor times, and its norm is Büttiker’s } \tau_x... \text{ why? and what is a complex time?})

NOTE: in the classical regime, speed depends only on E-V_0; thus, \( \tau_g \) and \( \tau_y \) (and even \( \tau_d \)) are automatically the same – not always so in QM.
What is this measurement?

A few things to note:

• This -µ · B interaction is a von Neumann measurement of B (which in turn stands in for whether or not the particle is in the region of interest)

• Since B_z couples to σ_z, the pointer is the conjugate variable (precession of the spin about z) — Note that this measurement is thus just another interference effect, as the precession angle φ is the phase difference accumulated between ↓ and ↑.

• We want to know the outcome of this von Neumann measurement only for those cases where the particle is transmitted.

• "Being transmitted" doesn't commute with "being under the barrier"; is it valid to even ask such post-selected questions? If so, how can you do so without first collapsing the particle to be under the barrier?

• Note: this Larmor precession could not determine for certain whether or not the particle had been in the field, or for how long; only on a large ensemble can the precession angle be measured to better accuracy than 180°.
Can we talk about what goes on behind closed doors?

(“Postselection,” as we’ll see later on in the term, is the big new buzzword in QIP... but how should one describe post-selected states?)
Predicting the past?

Standard recipe of quantum mechanics:
1. Prepare a state \( |i> \) (by measuring a particle to be in that state; see 4)
2. Let Schrödinger do his magic: \( |i> \Rightarrow |f> = U(t) |i> \), deterministically
3. Upon a measurement, \( |f> \Rightarrow \) some result \( |n> \), randomly
4. Forget \( |i> \), and return to step 2, starting with \( |n> \) as new state.

Aharonov’s objection (as I read it):
No one has ever seen any evidence for step 3 as a real process; we don’t even know how to define a measurement.
Step 2 is time-reversible, like classical mechanics.
Why must I describe the particle, between two measurements (1 & 4) based on the result of the first, propagated forward, rather than on that of the latter, propagated backward?
Conditional measurements
(Aharonov, Albert, and Vaidman)

Prepare a particle in $|i\rangle$ ... try to "measure" some observable $A$ ...
postselect the particle to be in $|f\rangle$

Does $\langle A \rangle$ depend more on $i$ or $f$, or equally on both?
Clever answer: both, as Schrödinger time-reversible.
Conventional answer: $i$, because of collapse.

Reconciliation: measure $A$ "weakly."
Poor resolution, but little disturbance.

"weak values"
A (von Neumann) Quantum Measurement of A

Well-resolved states
System and pointer become entangled

Initial State of Pointer

Final Pointer Readout

$H_{int} = gAp_x$

System-pointer coupling

Decoherence / "collapse"
Large back-action
**A Weak Measurement of A**

**Poor resolution on each shot.**
Negligible back-action (system & pointer separable)

**Strong:** \[ |\Psi\rangle_s \phi_p(x) \rightarrow \sum_i c_i |\psi_i\rangle_s \phi_p(x - g\alpha_i) \]

**Weak:** \[ |\Psi\rangle_s \phi_p(x) \rightarrow |\Psi\rangle_s \phi_p(x - g\langle A_s \rangle) \]
Since $X$(pointer) is very uncertain, $P_x$(pointer) *may* be very well-defined --

then this Hamiltonian no longer looks “noisy” to the system (which couples to it through $A$)
By the same token, no single event provides much information...

Initial State of Pointer

Final Pointer Readout
By the same token, no single event provides much information...

But after many trials, the centre can be determined to arbitrarily good precision...
Bayesian Approach to Weak Values

\[ \langle A \rangle_{wk} = \sum_j a_j P(j|i, f), \]

\[ P(A_i|f) = \frac{\langle \text{Proj}(f) \text{Proj}(A_i) \rangle}{\langle \text{Proj}(f) \rangle}. \]

\[ P(A|B) = \frac{P(A \& B)}{P(B)}. \]

\[ P(A) = \langle \text{Proj}(A) \rangle = \langle |A\rangle \langle A| \rangle = \langle \psi|A\rangle \langle A|\psi \rangle = |\langle A|\psi \rangle|^2. \]

\[ P(A \& B) = \langle \text{Proj}(B) \text{Proj}(A) \rangle = \langle \psi|B\rangle \langle B|A\rangle \langle A|\psi \rangle. \]

\[ \langle A \rangle_f = \sum_i a_i \frac{\langle \text{Proj}(f) \text{Proj}(A_i) \rangle}{\langle \text{Proj}(f) \rangle} = \frac{\langle f \rangle \langle f|A \rangle}{\langle f \rangle \langle f|f \rangle}. \]

\[ \langle A \rangle_{fi} = \frac{\langle i| \langle f \rangle \langle f|A \rangle |i \rangle}{\langle i| \langle f \rangle \langle f|f \rangle |i \rangle} = \frac{\langle f|A \rangle |i \rangle}{\langle f|f \rangle |i \rangle}. \]

\[ A_w = \frac{\langle f|A|i \rangle}{\langle f|i \rangle}. \]

Note: this is the same result you get from actually performing the QM calculation (see A&V).
Some nice properties

In many respects, they behave in a more intuitive fashion than do wave functions themselves. For example, in [26] it is pointed out that if a weak measurement of an operator \( R + S \) is made on a particle after it is prepared in an eigenstate of \( R \) with eigenvalue \( r \) and before it is detected in an eigenstate of \( S \) with eigenvalue \( s \), the result will be simply \( r + s \). This holds whether or not \( R \) and \( S \) commute and even if \( r + s \) is outside the eigenvalue spectrum of \( R + S \); hence such a simple rule could not be obeyed by standard quantum measurements (ones that are precise, or "strong," enough to disturb the time evolution between \( r \) and \( s \)). More generally, weak values are noncontextual and additive; \( \langle R + S \rangle_{f_i} = \langle R \rangle_{f_i} + \langle S \rangle_{f_i} \). When averaged over an orthonormal set of final states, they reproduce the usual expectation value, since \( P(A) = \sum_f P(f) P(A|f) \). These conditional probabilities are also easily shown to obey a chain rule, \( P(A \& B|f) = P(B|f) P(A|B) \). In sum, there are many reasons to ascribe a certain level of reality to these conditional probability distributions.

From PRA 52, 32 (1995)
How the Result of a Measurement of a Component of the Spin of a Spin-$\frac{1}{2}$ Particle Can Turn Out to be 100

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We have found that the usual measuring procedure for preselected and postselected ensembles of quantum systems gives unusual results. Under some natural conditions of weakness of the measurement, its result consistently defines a new kind of value for a quantum variable, which we call the weak value. A description of the measurement of the weak value of a component of a spin for an ensemble of preselected and postselected spin-$\frac{1}{2}$ particles is presented.

\[ |\tilde{i}\rangle \equiv \{ |\uparrow\rangle + \epsilon |\downarrow\rangle \} / \sqrt{1 + |\epsilon|^2} \]

\[ \{ |\epsilon \uparrow\rangle \} / \sqrt{1 + |\epsilon|^2} \]

\[ \text{Re} \ \epsilon / (1 + |\epsilon|^2) \]

\[ S_y |\tilde{i}\rangle = 1 \]

\[ \langle S_y \rangle_w = \frac{1 + |\epsilon|^2}{2 \text{Re} \ \epsilon} \to \infty \]

Ritchie, Story, & Hulet 1991

Very rare events may be very strange as well.
Weak measurement & tunneling times

How does this apply to tunneling?

- Prepare $|1\rangle$.
- Post-select $|1\rangle$ (or $|1c\rangle$).
- Ask where the particle was at intermediate times.

$$P(x) = |\psi(x)|^2 = \langle 4|x\rangle\langle x|4\rangle$$

The probability of being at $x$ is just the expectation value of the projector onto $x$. 
Conditional probability distributions

Bayes’s thm. \( \Rightarrow P(x|\text{trans}) = \frac{P(x \& \text{trans})}{P(\text{trans})} \)

\[= \frac{\langle \text{lt}\rangle\langle t\|x\rangle\langle x\|l \rangle}{\langle \text{lt}\rangle\langle t\|l \rangle} \]

\[= \frac{\langle \text{lt}\rangle\langle t\|x\rangle\langle x\|l \rangle}{\langle \text{lt}\rangle\langle t\|l \rangle} = \frac{\langle \text{lt}\rangle\langle t\|x\rangle\langle x\|l \rangle}{\langle \text{lt}\rangle\langle t\|l \rangle} \]

Precisely A&V’s result. \( P(x|\text{trans}) = \frac{1}{T} \psi_T^*(x) \psi_T(x) \).

- We can write the prob. distrib. of either trans. or refl. particles, as a function of time.
- We can integrate over time + over the barrier to obtain a total “conditional dwell time.”
- But: these results are complex.
A problem...

These expressions can be complex.

Much like early tunneling-time expressions derived via Feynman path integrals, et cetera.
A solution...

But consider a quantum-mechanical stopwatch.

\[ \psi(x) \sim e^{-(x-t)^2/4\sigma^2} \]

some inevitable uncertainty

\[ \imath \text{ complex } \Rightarrow \psi \sim e^{-(x-\text{Re } t)^2/4\sigma^2} \imath x \text{ Im } t/2\sigma^2 \ldots \]

hand shifts by \text{Re } t

picks up momentum of \text{Im } t/2\sigma^2

This is precisely the meaning of weak (or conditional) measurements.

\[ \text{Re } \imath t \text{ describes clock hand's position shift (e.g., Larmor precession).} \]

\[ \text{Im } \imath t \text{ describes back-action (e.g., spin aligning with } B) \]

For large \sigma, back-action vanishes, but position shift of hand remains constant.
Conditional $P(x)$ for tunneling
What does this *mean* practically?
Interesting observations

• The conditional distribution reproduces $\psi_i$ at early times and $\psi_f$ at late times (of course?); it does not support the notion we’d expect from classical waves, that transmission is only from the (causal) leading edge of the wavepacket...

• Like the Feynman-path times of Sokoloski et al., the weak-measurement formalism yields a complex times, whose real & imaginary parts are the Larmor times – but unlike either earlier approach, it provides a clear physical interpretation.

• We now understand why the two different Larmor times are so intimately related:
The evolution of a spin on the Bloch sphere during a Larmor-time measurement

Büttiker’s $\tau_y$ is the *real* part of the weak value
(the expected shift in pointer position);
his $\tau_z$ is the *imaginary* part of the weak value
(the surprising [?] back-action, & shift on pointer mom.)
Impossible: *all* pure states of a spin-1/2 look like the $|+1/2>$ eigenstate along some direction in real space (and have equal uncertainties in the two directions orthogonal to it).
But what if instead of spin-1/2, we allowed higher spin... e.g., the \( m=0 \) state of a spin-1, 2, 3, etc.? 

\[ m=0 \text{ state of higher spin} \]

"Infinitely weak" if no sensitivity to rotation

\[ \text{Invariant under } Bz. \]

\[ \text{no ability to "align" with } z. \]
And the intermediate case...

Spm-squeezed state.
Increase $\Delta S_y$ (pointer pos'n)
Decrease $\Delta S_z$ (pointer mom).
Rotation unchanged, just harder to detect.
Back-action reduced.
What to conclude about tunneling times?

There is a physical difference between the two Larmor times. One has a fixed physical effect regardless of initial pointer state; the importance of the other (back-action) depends on that state.

Also to note: the popular “Büttiker-Landauer” time turns out to be related not to the dwell- or group-like real part, but to the back-action-related imaginary part. Why? Because it measures not the effect of the particle on a clock, but how sensitive the particle is to back-action from the clock. (And don’t be too surprised – recall that $\tau_{\text{BL}}$ is like $\tau_{\text{S}}$ except for a factor of $i$!)

Open issues:
• it appears transmitted particles really sample more of the barrier than reflected ones, and yet never the middle (in the sense of the real times). Can this be observed?
• it doesn’t appear that you can really say that transmitted particles originate any earlier in the w.p. than the others. Interesting...
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