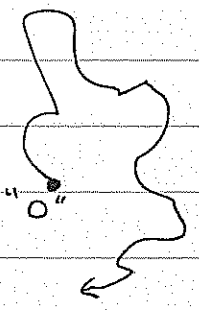
* Diffusion



Consider particle executing random walk (Brownian motion) starting from some origin "0". As time progresses, this particle will have a probability density $P(\vec{r}, t)$ of being found in the vicinity of point $\vec{r}$, at time t.

If instead of <u>one</u> particle, we release a large # of particles, then the <u>concentration</u> of particles at point $(\vec{r})$ at time t will be

$$C(\vec{r}, t) \propto P(\vec{r}, t)$$

This is because each particle is assumed to act independently of others. Knowing how $P(\vec{r}, t)$ varies in space & time, we thus learn about how concentrations evolve in time. The fact that a Brownian particle tends to move a distance $\sqrt{\langle r^2 \rangle} \propto \sqrt{t}$ tells us that the concentration which starts off being nonzero only at the origin "0" at time $t=0$, will <u>spread</u> out over a distance $\propto \sqrt{t}$ with time. This is familiar to us from how, say, a drop of ink spreads out in water.

* Diffusion Equation

In order to make quantitative progress in understanding the behavior of concentration, we seek an 'equation' that describes the evolution of the concentration.
Note that although each walker does a random walk, so we cannot hope to say anything useful about a given particle; we do know that probability distributions (or concentrations) change in a predictable fashion. This is why, unlike in Newtonian mechanics, we are not trying to write eqns for evolution of the position/velocity of a particle.

The equation governing the diffusion of concentration (or, equivalently, probability) is given by :-

3D: $\vec{r} \equiv (x, y, z)$

$$\frac{\partial c(\vec{r},t)}{\partial t} = D\left(\frac{\partial^2 c(\vec{r},t)}{\partial x^2} + \frac{\partial^2 c(\vec{r},t)}{\partial y^2} + \frac{\partial^2 c(\vec{r},t)}{\partial z^2}\right)$$

2D: $\vec{r} \equiv (x, y)$

$$\frac{\partial c(\vec{r},t)}{\partial t} = D\left(\frac{\partial^2 c(\vec{r},t)}{\partial x^2} + \frac{\partial^2 c(\vec{r},t)}{\partial y^2}\right)$$

1D: $\vec{r} \equiv x$

$$\frac{\partial c(x,t)}{\partial t} = D\,\frac{\partial^2 c(x,t)}{\partial x^2}$$

$D \equiv$ "Diffusion constant"

* Continuity eqn. & Fick's law :-

The diffusion eqn. can be obtained by combining two other eqns

1 Continuity Eqn: This tells us that total density (or rather total particle #) is conserved & the only way it changes at a point in space is when currents flow into or out of a point.

$$\boxed{\frac{\partial c(x,t)}{\partial t} = -\frac{\partial j_x(x,t)}{\partial x}} \qquad j_x \equiv \text{current density}$$

If we integrate both sides over x at a fixed time, we get

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial t} c(x,t)\,dx = -\int_{-\infty}^{\infty} \frac{\partial j_x}{\partial x}(x,t)\,dx$$

$$\therefore \frac{\partial}{\partial t} \underbrace{\int_{-\infty}^{\infty} c(x,t)\,dx}_{N(t)} = -j_x(x,t)\Big|_{-\infty}^{\infty}$$

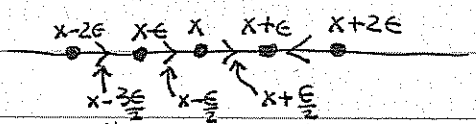
$$\therefore \frac{dN(t)}{dt} = -j_x(\infty,t) + j_x(-\infty,t)$$

$\int_{-\infty}^{\infty} c(x,t)\,dx$ gives total particle # $N(t)$. But $N(t)$ does not depend on t since particles are only moving from place to place. Their total # doesn't change: $\frac{dN}{dt} = 0$ (ie, Left Hand Side = 0).

On the right, if all particles are in some region of space, there can only be currents in that region & not at ∞. So $j_x(\infty,t) = 0 = j_x(-\infty,t)$ at any time t.

This verifies the correctness of the eqn when integrated over all space.

* But there's a different way to see the correctness of the continuity eqn. Let's consider discrete space points in 1 dimension separated by infinitesimal distances ϵ.



Now "currents" correspond to motion of particles from one point to another. We've denoted these by arrows above.

Now if we want change in concentration at point x in a small time interval δt, let's look at

$$n(x, t+\delta t) = n(x,t) + \left[I_x\left(x-\tfrac{\epsilon}{2},t\right) - I_x\left(x+\tfrac{\epsilon}{2},t\right) \right]$$

current flowing in ↑ ↑ current flowing out

$$\frac{1}{\delta t}\left[n(x,t+\delta t) - n(x,t) \right] = \left[I_x\left(x-\tfrac{\epsilon}{2},t\right) - I_x\left(x+\tfrac{\epsilon}{2},t\right) \right]$$

- In one-dimension, we can view $I_x \rightarrow j_x$ "current density"
- Also $\frac{n(x,t)}{\epsilon} = \frac{\text{Number}}{\text{unit length}} \equiv c(x,t)$

- For $\epsilon \to 0$, $\delta t \to 0$ we can set, using definition of derivatives,

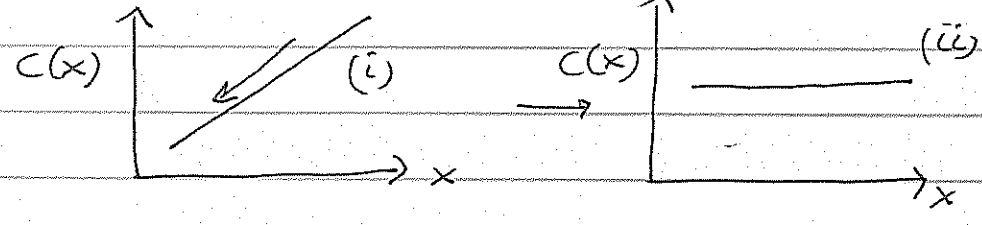
$$\frac{\partial}{\partial t} n(x,t) = -\epsilon \frac{\partial j_x(x,t)}{\partial x}$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\frac{n(x,t)}{\epsilon}\right) = -\frac{\partial j_x(x,t)}{\partial x}$$

$$\therefore \quad \boxed{\frac{\partial c(x,t)}{\partial t} = -\frac{\partial j_x(x,t)}{\partial x}}$$

2 Fick's Law :-

On experimental grounds, if we have a concentration variation as shown in (i)



which has to go to a uniform value as in (ii) because of diffusion, we expect currents have to flow in a direction opposite to the concentration gradient, ie, "downhill"

* For small concentration variations, we expect (& this is experimentally seen),

$$j_x(x,t) = -D \frac{\partial c(x,t)}{\partial x}$$

↑ "Diffusion const.".

If this is the case, we can substitute this in the continuity eqn. to get

$$\frac{\partial c(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[-D \frac{\partial c}{\partial x}(x,t)\right]$$

$$\boxed{\frac{\partial c(x,t)}{\partial t} = D \frac{\partial^2 c(x,t)}{\partial x^2}}$$ simple diffusion eqn

* At large concentration gradients, we may have more complicated current density

$$j_x(x,t) = -D \frac{\partial c(x,t)}{\partial x} - \gamma c(x,t) \frac{\partial c(x,t)}{\partial x}$$

$\{D>0,\ \gamma>0,\ c(x,t)>0\}$

This form for the current density still ensures two things

(i) currents flow opposite to density gradients

(ii) for uniform density, no currents flow.

With this,

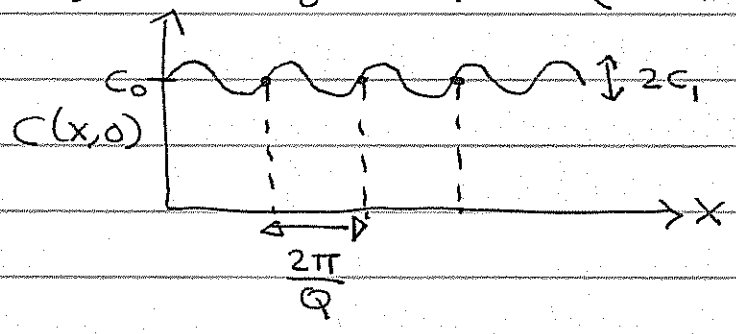
$$\frac{\partial c}{\partial t} = -\frac{\partial}{\partial x}\left[-D\frac{\partial c}{\partial x} - \gamma c \frac{\partial c}{\partial x}\right] = D\frac{\partial^2 c}{\partial x^2} + \gamma \frac{\partial}{\partial x}\left(c\frac{\partial c}{\partial x}\right)$$

... more complicated eqn.

* Simple solutions to the Diffusion Eqn :-

- Let us consider a periodic density variation at time $t=0$

$$C(x,0) = C_0 + C_1 \sin(Qx)$$



Clearly, the average density C_0 is not expected to change with time, but we might expect C_1 to be time dependent. such that $C_1 \to 0$ as $t \to \infty$. So that the density at long times simply reverts, as a result of diffusion, to its uniform value.

Substituting into the simple diffusion eqn, we set

$$C(x,t) = C_0 + C_1(t) \sin(Qx)$$

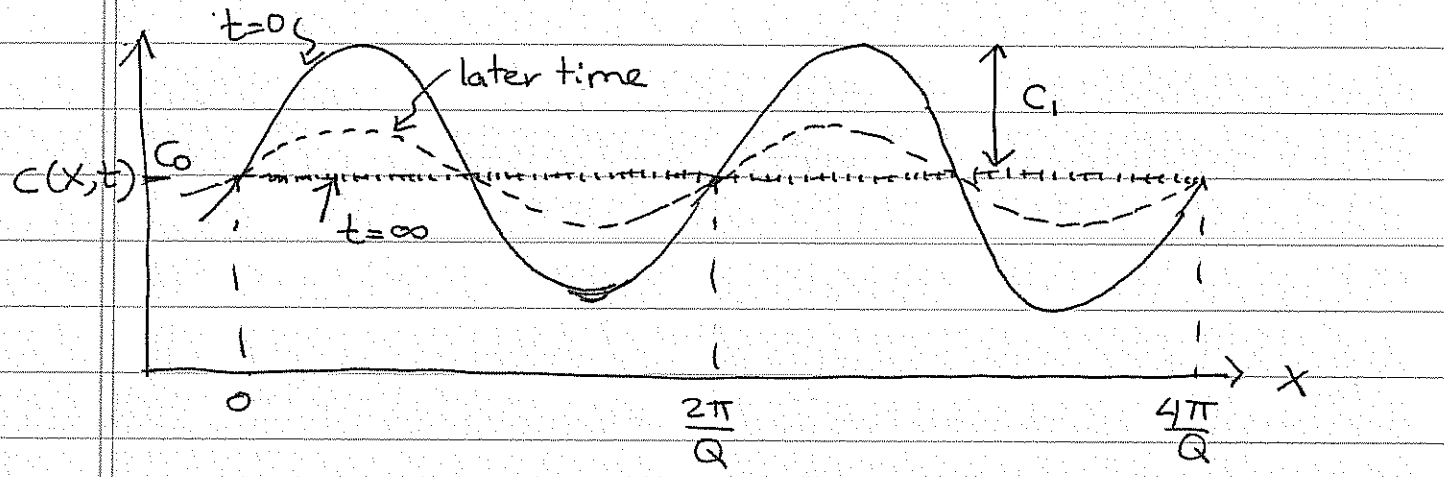
$\Rightarrow$

$$\frac{\partial C}{\partial t} = \left(\frac{\partial C_1}{\partial t}\right) \sin Qx$$

$$\frac{\partial^2 C}{\partial x^2} = -C_1(t)\, Q^2 \sin Qx$$

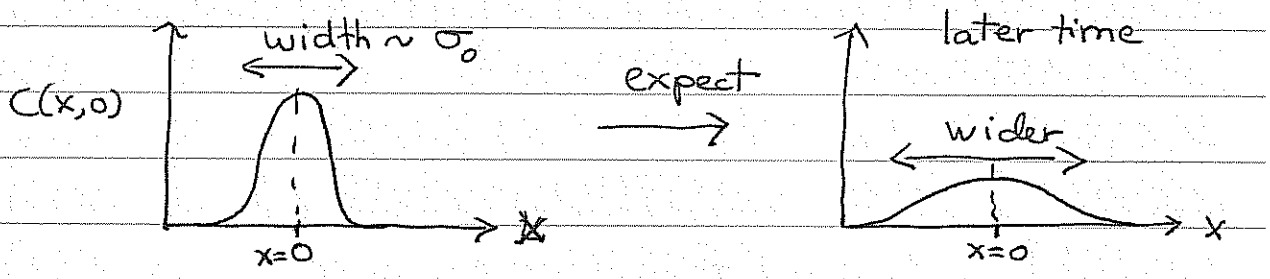
$$\therefore \quad \frac{\partial C_1}{\partial t} \sin Qx = -DQ^2 C_1(t) \sin Qx$$

$$\therefore \quad \frac{\partial C_1}{\partial t} = -DQ^2 C_1(t) \Rightarrow C_1(t) = C_1 e^{-DQ^2 t}$$



- Consider a Gaussian concentration profile initially.

$$C(x,0) = \frac{1}{\sqrt{2\pi\sigma_0^2}}\, e^{-\frac{x^2}{2\sigma_0^2}}$$



Motivated by this expectation, let's assume $\sigma_0^2 \to f(t)$ and substitute into the diff. eqn

$$C(x,t) = \frac{1}{\sqrt{2\pi f(t)}}\, e^{-\frac{x^2}{2f(t)}}$$

$$\therefore \frac{\partial C}{\partial t} = \left[\frac{-1}{2}\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{[f(t)]^{3/2}}\, e^{-\frac{x^2}{2f(t)}} + \frac{1}{\sqrt{2\pi f(t)}}\, e^{-\frac{x^2}{2f(t)}}\cdot\frac{x^2}{2f^2(t)}\right]\frac{df}{dt}$$

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{\sqrt{2\pi f(t)}} \cdot e^{-\frac{x^2}{2f(t)}} \left[\left(\frac{-2x}{2f(t)} \right)^2 - \frac{2}{2f(t)} \right]$$

$\therefore$ Putting into diff eqn, we get

$$\cdot \left[-\frac{1}{2} \frac{1}{f^{3/2}} + \frac{1}{f^{1/2}} \cdot \frac{x^2}{2f^2} \right] \frac{df}{dt} = D \left[-\frac{1}{f} + \frac{x^2}{f^2} \right] \frac{1}{f^{1/2}}$$

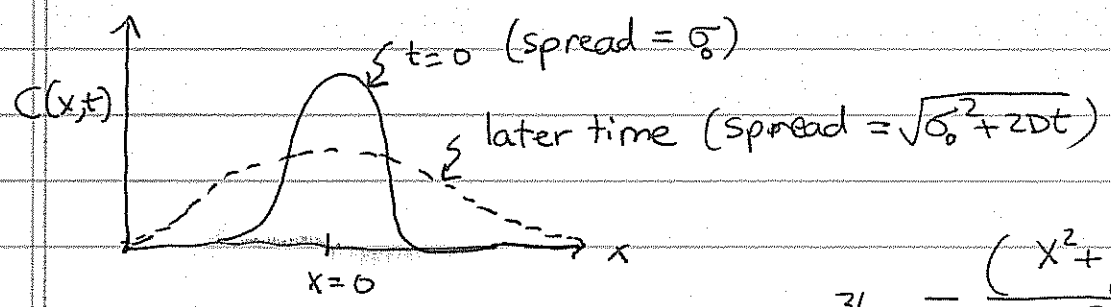
$$\therefore +\frac{1}{2} \cdot \frac{1}{f^{1/2}} \cdot \left[-\frac{1}{f} + \frac{x^2}{f^2} \right] \cdot \frac{df}{dt} = D \left[-\frac{1}{f} + \frac{x^2}{f^2} \right] \cdot \frac{1}{f^{1/2}}$$

$$\therefore \quad \frac{df}{dt} = 2D \quad \therefore \quad f(t) = 2Dt + \text{constant}$$

But at time $t=0$, $f(t=0) = \sigma_0^2$ $\therefore$

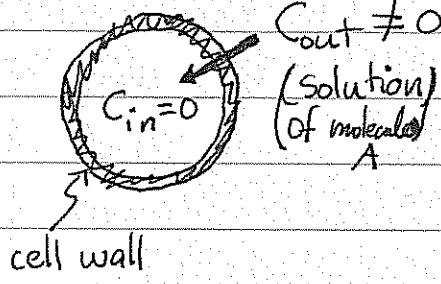
$$\therefore f(t) = \sigma_0^2 + 2Dt$$

$$\therefore \quad \boxed{C(x,t) = \frac{1}{\sqrt{2\pi(\sigma_0^2 + 2Dt)}} e^{-\frac{x^2}{2(\sigma_0^2 + 2Dt)}}}$$



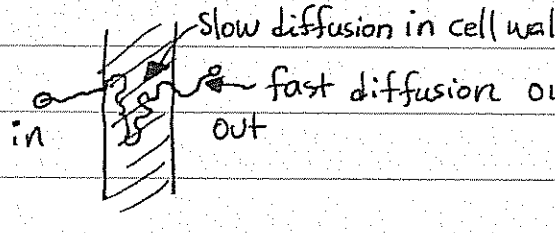
In 3D :- $$C(\vec{r},t) = \left[\frac{1}{2\pi(\sigma_0^2 + 2Dt)} \right]^{3/2} \cdot e^{-\frac{(x^2+y^2+z^2)}{2(\sigma_0^2 + 2Dt)}}$$

• Cell soaked in a solution: Consider a cell soaked in a solution of some compound. - say sugar in water - and let us ask how long it takes for the cell to 'absorb' this chemical if we view it purely as a 'diffusive' process.



← Initial state

Here the main slow-down of the intake occurs due to the cell wall, which acts as a barrier to the free motion of molecules in & out of the cell. Let D be the diffusion constant of molecule-A in the tissue of the cell wall.



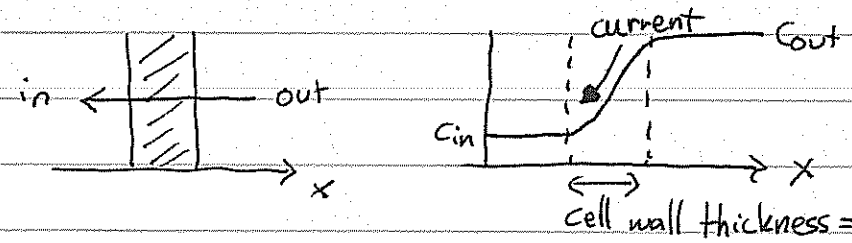
So for intake process, we assume molecules slowly diffuse into the cell but the concentration inside & outside stay uniform, taking on values C_{in} & C_{out}.

↳ (don't vary within the cell from point to point but can change in time)

Further, if the cell is small compared to the volume of the medium (solution), any uptake of molecules by the cell will change C_{in} dramatically but hardly affect C_{out}. So we will also assume $C_{in} \equiv C_{in}(t)$ but C_{out} is independent of time.

Look at a one-dimensional version first :-

From Fick's law $j = -D \frac{\partial c}{\partial x}$



For a thin cell wall, we replace $\frac{\partial c}{\partial x} \approx \frac{c_{out} - c_{in}}{\delta}$

$\therefore \quad j = -\left(\frac{D}{\delta}\right)(c_{out} - c_{in})$

$\frac{D}{\delta} \equiv$ "Permeability of cell membrane"

Thin wall → more permeable }
Larger D → more permeable }

Back to our spherical cell wall :-

$j = -\frac{D}{\delta}(c_{out} - c_{in})$

But $j \equiv$ current per unit area
{"Flowing in" ⇒ "negative current"}

∴ To get total flow/intake, $I = j \times (\text{Area})$

$= j \times 4\pi R^2$ (R = radius of cell)

$\therefore \quad I = -4\pi R^2 \cdot \left(\frac{D}{\delta}\right) \cdot (c_{out} - c_{in}(t))$

↑ depends on time

Total current going in increases total # molecules in the cell at that rate, ie $\frac{dN_{in}}{dt} = -I$

But $N_{in}(t) = c_{in}(t) \times V_{cell}$

V_{cell} = cell volume $= \frac{4}{3}\pi R^3$

$$\therefore V_{cell} \frac{d c_{in}(t)}{dt} = 4\pi R^2 \cdot \frac{D}{\delta} \cdot (C_{out} - c_{in}(t))$$

Let $C_{out} - c_{in}(t) = W(t)$

$\frac{d c_{in}(t)}{dt} = -\frac{dW}{dt}$

$$\therefore V_{cell} \cdot \frac{dW}{dt} = -4\pi R^2 \left(\frac{D}{\delta}\right) \cdot W(t)$$

Solving $W(t) = W(0)\, e^{-\frac{4\pi R^2 D}{\delta \cdot V_{cell}} t}$

$$\therefore C_{out} - c_{in}(t) = [C_{out} - c_{in}(0)]\, e^{-\frac{4\pi R^2 D}{\delta \cdot V_{cell}} t}$$

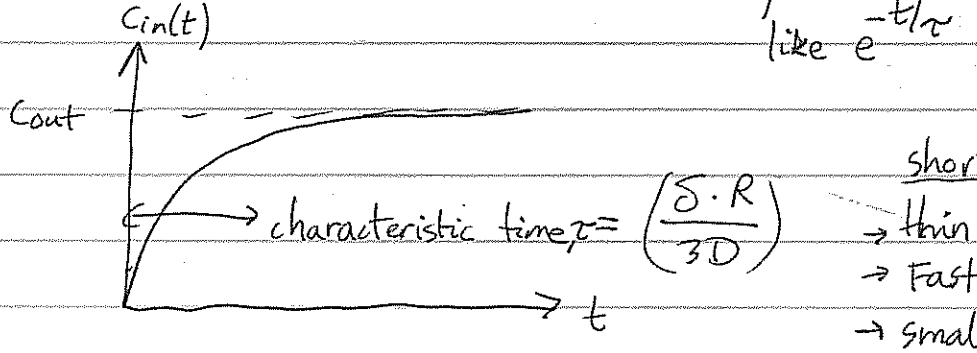
↳ zero initially $\Rightarrow c_{in}(0) = 0$

$$\therefore c_{in}(t) = C_{out}\left(1 - e^{-\frac{4\pi R^2 D}{\delta \cdot V_{cell}} \cdot t}\right)$$

$$\frac{4\pi R^2}{V_{cell}} = \frac{4\pi R^2}{4/3\,\pi R^3} = \frac{3}{R}$$

$$\therefore \boxed{c_{in}(t) = C_{out}\left(1 - e^{-\frac{3 \cdot D}{\delta \cdot R} t}\right)}$$

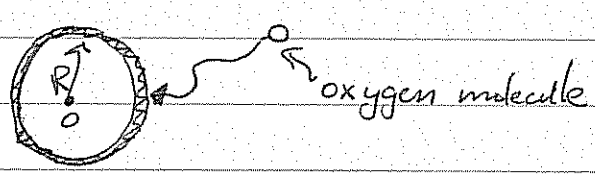
like $e^{-t/\tau}$



shorter τ if
- → thin wall (δ small)
- → Fast Diffusion (D large)
- → small cell (R small)

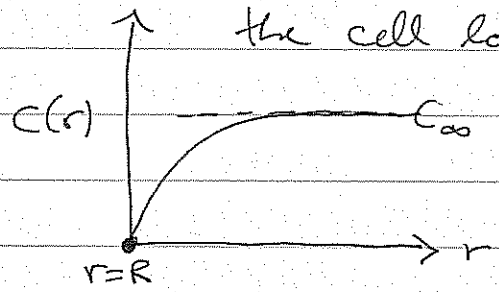
Diffusion combined with metabolism :-

Consider a bacterial cell which is swimming in oxygen rich water. Let's try & estimate, assuming oxygen molecules diffuse with diffusion constant D in water, how efficiently this bacterial cell can consume oxygen.



Let $r \equiv |\vec{r}|$ denote distance of point $\vec{r}$ from cell center

Let's assume that the moment an oxygen molecule touches the bacterial cell, it gets 'consumed' (into generating energy). This means the concentration of oxygen on cell wall $c(r=R) = 0$ whereas far from the bacterial cell wall, let C_∞ be the oxygen concentration. Schematically, the concentration outside the cell looks as follows. in the steady state.



To solve for the profile, and the rate of oxygen consumption, in the steady state

we set $c(\vec{r}, t) \equiv c(\vec{r})$ (time independent)

$$\therefore \frac{\partial c}{\partial t} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) = 0$$

In 3 dimensions, if $r^2 = x^2 + y^2 + z^2$

then, $\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)$

assuming $c(\vec{r})$ only depends on distance from center & not on direction.

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) = 0 \Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial c}{\partial r}\right) = 0$$

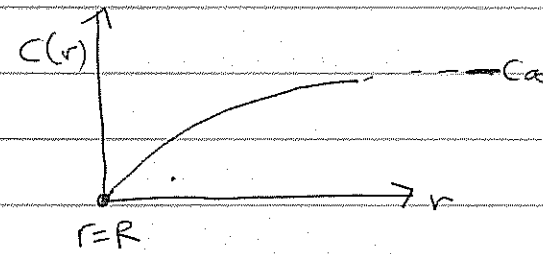
$$\therefore r^2\frac{dc}{dr} = a \quad \text{(constant unknown)}$$

$$\therefore \frac{dc}{dr} = \frac{a}{r^2} \quad \therefore c(r) = -\frac{a}{r} + b$$

We now use $\left\{\begin{array}{l} c(r=R) = 0 \\ c(r=\infty) = C_\infty \end{array}\right\}$

$$\left.\begin{array}{l} \therefore b = C_\infty \\ a = RC_\infty \end{array}\right\}$$

$$\therefore \boxed{c(r) = C_\infty\left(1 - \frac{R}{r}\right)}$$



The rate of consumption of oxygen is the

["oxygen current density"] × [Area of cell wall] ($4\pi R^2$)

But $j\big|_{r=R} = -D\frac{\partial c}{\partial r}\Big|_{r=R} = \frac{DC_\infty R}{r^2}\Big|_{r=R} = \frac{DC_\infty}{R}$

$$\therefore \text{Rate of consumption} = \frac{DC_\infty}{R} \cdot 4\pi R^2 = 4\pi DC_\infty R$$

Expect bacterial oxygen need will scale as cell volume ie.

Oxygen Need $\propto \frac{4}{3}\pi R^3$

∴ If cell gets bigger, (Need) >> Max. rate of Consumption

(Need) → prop. to R^3

Max. rate of Consumption → prop. to R.

∴ Unless some circulatory system is developed, organism will be starved of oxygen when it gets larger, no matter <u>how</u> efficiently it takes up oxygen.

* **Spread:-**

We can compute the "spread" of the concentration via (for $\sigma = 0$)

$$\text{1 Dimension: } \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$$

$$\therefore \langle x^2 \rangle = 2Dt \qquad (\text{Note: } \langle x \rangle = 0)$$

$$\text{ie } \boxed{\langle \vec{r}^2 \rangle_{1D} = 2Dt} \qquad (\vec{r} = x\hat{x})$$

$$\text{2 Dim: } \langle \vec{r}^2 \rangle = \langle x^2 + y^2 \rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x^2+y^2) \cdot \left(\frac{1}{4\pi D t}\right) e^{-\left(\frac{x^2+y^2}{4Dt}\right)} dx\,dy$$

$$\therefore \boxed{\langle \vec{r}^2 \rangle_{2D} = 4Dt}$$

$$\text{3 Dim: } \langle \vec{r}^2 \rangle_{3D} = \langle x^2 + y^2 + z^2 \rangle$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \cdot (x^2+y^2+z^2) \left(\frac{1}{4\pi D t}\right)^{3/2} * e^{-(x^2+y^2+z^2)/4Dt}$$

$$\boxed{\langle \vec{r}^2 \rangle_{3D} = 6Dt}$$