Simple Harmonic oscillator with forcing + damping

(1) $$m\frac{d^2s}{dt^2} = -ks - r\frac{ds}{dt} + F_0\cos\omega t$$

- $m\frac{d^2s}{dt^2}$: mass * acceleration
- $-ks$: restoring force
- $-r\frac{ds}{dt}$: damping force
- $F_0\cos\omega t$: periodic <u>external</u> forcing

Eqn (1) is a statement of Newton's law. The R.H.S. is the sum of <u>all</u> forces acting on the oscillator.

Let us divide out the mass & set $\frac{k}{m} = \omega_0^2$; $\frac{r}{m} = 2\gamma$; $\frac{F_0}{m} = f_0$

(2) $$\therefore \quad \frac{d^2s}{dt^2} = -\omega_0^2 s - 2\gamma\frac{ds}{dt} + f_0\cos\omega t$$

To solve this, we will guess a solution. We know the harmonic oscillator has periodic oscillations of "s" as a function of time. We also know the damping force will slow down & stop these oscillations so that $s \to 0$ eventually if all <u>external</u> forces are switched off. (ie, if $f_0 = 0$).

(3) $$\therefore \text{ Guess } \quad s(t) = e^{-\beta t}(A\cos\alpha t + B\sin\alpha t)$$

A, B, α, β are unknown quantities which we need to determine in terms of known quantities.

Plugging (3) into (2) we get :-

$\boxed{\text{i}}$ $$\frac{ds}{dt} = -\beta e^{-\beta t}(A\cos\alpha t + B\sin\alpha t) + \alpha e^{-\beta t}(B\cos\alpha t - A\sin\alpha t)$$

$\boxed{\text{ii}}$ $$\frac{d^2s}{dt^2} = \beta^2 e^{-\beta t}(A\cos\alpha t + B\sin\alpha t) - \alpha^2 e^{-\beta t}(A\cos\alpha t + B\sin\alpha t) - 2\alpha\beta e^{-\beta t}(B\cos\alpha t - A\sin\alpha t)$$

(4) $\therefore e^{-\beta t}\left[\beta^2(A\cos\alpha t+B\sin\alpha t)-\alpha^2(A\cos\alpha t+B\sin\alpha t)-2\alpha\beta(B\cos\alpha t-A\sin\alpha t)\right]$

$$= -\omega_0^2 e^{-\beta t}\left[A\cos\alpha t+B\sin\alpha t\right]$$
$$-2\gamma e^{-\beta t}\left[-\beta(A\cos\alpha t+B\sin\alpha t)+\alpha(B\cos\alpha t-A\sin\alpha t)\right]$$
$$+f_0\cos\omega t$$

* Next let us set all "sin" terms equal to each other & all "cos" terms equal to each other

(5) $\therefore$ (i) $\cos\alpha t\, e^{-\beta t}\left[(\beta^2-\alpha^2+\omega_0^2)A \underset{-2\alpha\beta B}{-} 2\gamma(\beta A-\alpha B)\right]=f_0\cos\omega t$

(ii) $\sin\alpha t\, e^{-\beta t}\left[(\beta^2-\alpha^2+\omega_0^2)B \underset{+2\alpha\beta A}{-} 2\gamma(\beta B+\alpha A)\right]=0$

→ $\boxed{\text{NO FORCING}}$

* If $f_0=0$; we see that $\cos\alpha t\, e^{-\beta t}$ can be cancelled from the L.H.S. of 5(i) & $\sin\alpha t\, e^{-\beta t}$ " " " from the LHS. of 5(ii).

(6) $\therefore$ (i) $(\beta^2-\alpha^2+\omega_0^2-2\gamma\beta)A = (2\alpha\beta-2\gamma\alpha)B$

(ii) $-(2\alpha\beta-2\gamma\alpha)A = (\beta^2-\alpha^2+\omega_0^2-2\gamma\beta)B$

Dividing (i) by (ii) ⇒

(7) $$(\beta^2-\alpha^2+\omega_0^2-2\gamma\beta)^2+(2\alpha\beta-2\gamma\alpha)^2=0$$

($\therefore$ Each term in bracket must be zero)

Either (i) $\beta=\gamma$ and $\alpha^2=\omega_0^2-\gamma^2$

or (ii) $\alpha=0$ and $\beta^2-2\gamma\beta+\omega_0^2=0$

For weak damping, $\gamma < \omega_0$, we expect correct solution is

(8) $\boxed{\beta = \gamma \text{ and } \alpha^2 = \omega_0^2 - \gamma^2}$ ($\because$ if $\gamma = 0$, we must get $\alpha = \omega_0$)

$\therefore \beta = \gamma$ and $\alpha = \sqrt{\omega_0^2 - \gamma^2}$ Let's call this solution $s_1(t)$

(9) $$\boxed{\therefore s_1(t) = e^{-\gamma t}\left[A\cos\left(\sqrt{\omega_0^2-\gamma^2}\,t\right) + B\sin\left(\sqrt{\omega_0^2-\gamma^2}\,t\right)\right]} \quad \gamma < \omega_0$$

Here $s_1(t=0) = A$

$$\frac{ds_1}{dt}(t=0) = -\underset{\uparrow [\text{See } 3\cdot(i)]}{\beta} A + \alpha B = -\gamma A + B\sqrt{\omega_0^2 - \gamma^2}$$

(10) $\therefore$ Physically (i) $A = s_1(t=0)$

(ii) $$B = \frac{(ds_1/dt)(t=0) + \gamma s_1(t=0)}{\sqrt{\omega_0^2 - \gamma^2}}$$

→ $\therefore$ Given $s_1(t=0)$ & $\left(\frac{ds_1}{dt}\right)(t=0)$ we have found full solution.

→ $\boxed{\text{WITH FORCING}}$

If $f_0 \neq 0$ in eqn 5(i) & 5(ii), we must demand $\beta = 0$ & $\alpha = \omega$ to make sure the same $\cos\omega t$ arises on LHS & RHS of 5(i)

(11) $\therefore s_2(t) = A\cos\omega t + B\sin\omega t$ [call this $s_2(t)$]

& Eqn (5) (i) & (ii) become

(12) (i) $(-\omega^2 + \omega_0^2)A + 2\omega\gamma B = f_0$

(ii) $(-\omega^2 + \omega_0^2)B - 2\omega\gamma A = 0$

[12.(ii)] $\Rightarrow$ $$B = \frac{2\omega\gamma A}{\omega_0^2-\omega^2}$$

[12.(i)] $\Rightarrow$ $$(\omega_0^2-\omega^2)\cdot A + \frac{4\omega^2\gamma^2 A}{(\omega_0^2-\omega^2)} = f_0$$

(13)

(i). $$A = \frac{f_0\,(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2}$$

(ii) $$B = \frac{2\omega\gamma f_0}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2}$$

(14) $\therefore$ $$\boxed{s_2(t) = \frac{f_0(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2}\cos\omega t + \frac{2\omega\gamma f_0}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2}\sin\omega t}$$

This solution has no arbitrary constants to be fixed by initial conditions. It is only the steady state solution & not the full solution.

NOTE: It is easy to see that the full solution is of the form

$$s_{full}(t) = s_1(t) + s_2(t)$$

Check: $$\frac{d^2 s_{full}}{dt^2} + \omega_0^2 s_{full} + 2\gamma\frac{ds_{full}}{dt}$$

$$= \left(\frac{d^2 s_1}{dt^2} + \omega_0^2 s_1 + 2\gamma\frac{ds_1}{dt}\right)$$

this is zero!! $$+ \left(\frac{d^2 s_2}{dt^2} + \omega_0^2 s_2 + 2\gamma\frac{ds_2}{dt}\right)$$

$$= \frac{d^2 s_2}{dt^2} + \omega_0^2 s_2 + 2\gamma\frac{ds_2}{dt} = f_0\cos\omega t$$

* Clearly S_{full} and S_2 satisfy the <u>same</u> differential equation. But now, $S_{full}(t)$ has 2 arbitrary constants which depend on initial conditions via $S_1(t)$

* Note $S_1(t \to \infty) = 0$, so that for long times, only $S_2(t)$ survives. Hence we call

$$\left.\begin{array}{l} S_1(t) : \text{ transient solution} \\ S_2(t) : \text{ steady state solution} \end{array}\right\}$$

Resonant energy absorption :-

* To compute the energy absorbed by the oscillator from the external forcing, let us look at the work done by the external force in a short interval of time dt.

$$dW = f(t)\,ds \Rightarrow \frac{dW}{dt} = f(t)\frac{ds}{dt}$$

$$\text{power absorbed} \rightarrow \frac{dW}{dt} = f_0 \cos\omega t \cdot \frac{ds}{dt}$$

Energy absorbed in one cycle of the external force is :-

(15) $$\Delta E = \int_0^{2\pi/\omega} \frac{dW}{dt}\,dt = f_0 \cdot \int_0^{2\pi/\omega} \cos\omega t \cdot \left(\frac{ds}{dt}\right) dt$$

But in steady state, $\frac{ds}{dt} = \frac{ds_2}{dt}$

$$\therefore \Delta E = f_0 \cdot \int_0^{2\pi/\omega} dt \cdot \cos\omega t \left[\frac{f_0(\omega_0^2-\omega^2)(-\sin\omega t)\omega}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2} + \frac{f_0 \cdot 2\omega\gamma \cdot \omega\cos\omega t}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2} \right] dt$$

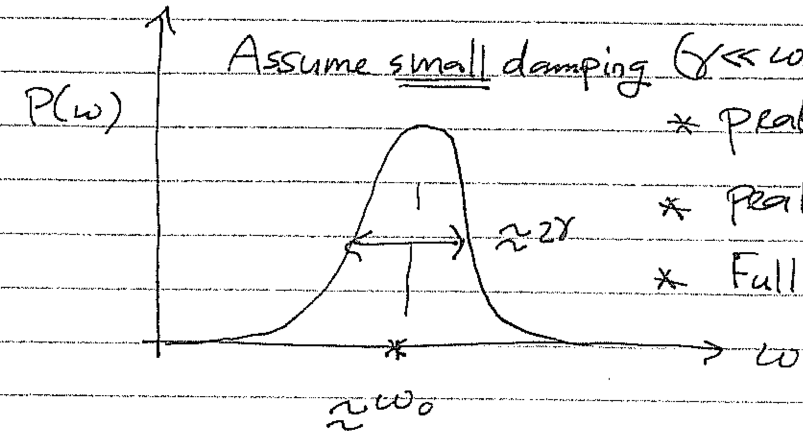
(16)
(i) $\int_0^{2\pi/\omega} \sin\omega t \cos\omega t \, dt = 0$ (show)

(ii) $\int_0^{2\pi/\omega} \cos^2\omega t \, dt = \frac{\pi}{\omega}$ (show)

(17) $$\therefore \Delta E = \frac{f_0^2 \cdot 2\omega^2\gamma \cdot (\pi/\omega)}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2}$$

$$\frac{\text{Average Energy}}{\text{unit time}} = \frac{\Delta E}{2\pi/\omega} = \frac{f_0^2 \cdot \omega^2\gamma}{(\omega_0^2-\omega^2)^2+(2\omega\gamma)^2}$$

* This average energy per unit time exhibits a resonance effect [call this "power": $P(\omega)$]



Assume small damping ($\gamma \ll \omega_0$)

* peak power $\approx \frac{f_0^2}{4\gamma}$
* peak frequency $\approx \omega_0$
* Full width at Half max $\approx 2\gamma$