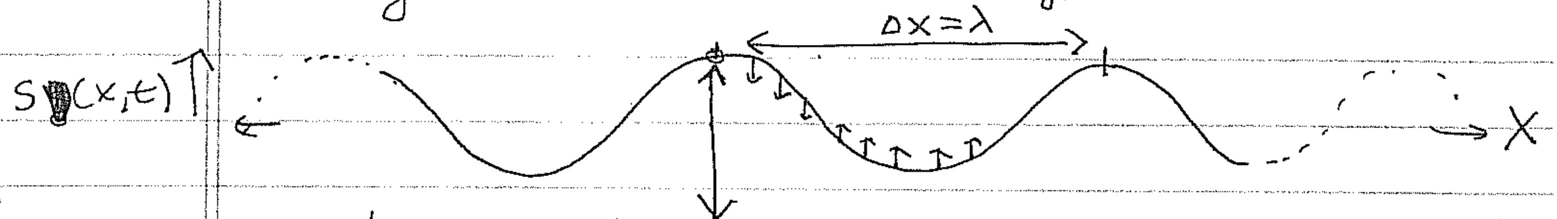
Waves on a string :

Next consider a string set in oscillatory motion by plucking it when it is under tension. Example of this is in a guitar string, except a guitar string is forced to be stationary at its ends.



Such a string has two periodicities :-

① If we look at a snapshot as shown above, there is a "space periodicity" of the crests (or the maxima) & the minima. This periodicity is called the "wavelength" λ of the wave.

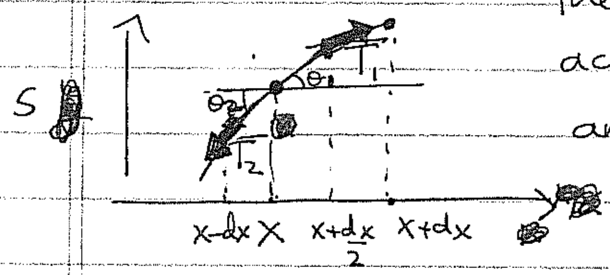
② Every point on the string executes periodic motion in time. This is the "temporal period" T (which is just called the "Period").

- Forces on the string element :-

We can consider a snapshot of the string as shown, which is a plot $s(x,t)$ of the vertical displacement s of the string at a given point x at a given time t.

Denote tension by N below?

Ignoring gravity which plays only a minor role, the forces on a small element of this string are resolved as follows.



They are just tensions $\overline{T_1}$ & $\overline{T_2}$ acting at $x+\frac{dx}{2}$ and $x-\frac{dx}{2}$ for an element of length dx centered at x.

Net upward force on the string element is obtained via

$$\overline{T_1}\sin\theta_1 \equiv \overline{T}\left(x+\frac{dx}{2}\right)\cdot\sin\theta_1$$

↑ "tension at point $x+\frac{dx}{2}$"

$$\overline{T_2}\sin\theta_2 \equiv \overline{T}\left(x-\frac{dx}{2}\right)\sin\theta_2$$

& $$dF = \overline{T_1}\sin\theta_1 - \overline{T_2}\sin\theta_2$$
$$\approx \overline{T}(\sin\theta_1 - \sin\theta_2)$$

(assuming tension $\overline{T}$ is roughly const)

($T_0 \equiv$ period, $\overline{T} \equiv$ tension) {sorry for the bad notation.}

Now $\sin\theta \approx \tan\theta \approx \frac{dS}{dx}$ (for small angles)

$$\therefore \sin\theta_1 - \sin\theta_2 = \left(\frac{dS}{dx}\right)_{x+\frac{dx}{2}} - \left(\frac{dS}{dx}\right)_{x-\frac{dx}{2}}$$

$$\approx dx\cdot\left(\frac{d^2S}{dx^2}\right)$$

$$\therefore\ dF = \overline{T}\left(\frac{d^2S}{dx^2}\right)\cdot dx$$

is the force acting on the string element at x.

By Newton's law, this force will cause an acceleration

element mass $= \mu\, dx$ (μ = mass per unit length)

acceleration $= \dfrac{d^2 s}{dt^2}$

$\therefore$ (mass) $\times$ (acceleration) $=$ Force $\Rightarrow$

$$T\left(\frac{d^2 s}{dx^2}\right) . dx = (\mu\, dx)\left(\frac{d^2 s}{dt^2}\right)$$

$$\therefore \boxed{\frac{d^2 s}{dt^2} = \frac{T}{\mu}\left(\frac{d^2 s}{dx^2}\right)}$$

Note: all derivatives become partial derivatives actually for the string, once X is not a fixed location.

This is the equation for the motion of the string.

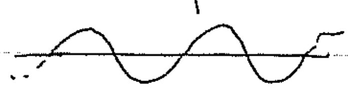
- Note: Dimensionally, tension T = Force

  $\mu = \dfrac{\text{mass}}{\text{unit length}}$

  $\dfrac{d^2 s}{dt^2} \equiv \dfrac{\text{Length}}{(\text{Time})^2}$; $\dfrac{d^2 s}{dx^2} \equiv \dfrac{1}{(\text{Length})}$

  $\therefore \left(\frac{\partial^2 s}{\partial t^2}\right)/\left(\partial^2 s/\partial x^2\right) \rightarrow$ dimension of $(\text{Length})^2/(\text{Time})^2 \rightarrow (\text{Velocity})^2$

Consider periodic solutions of the type we drew earlier



Simplest function which has

$$\left.\begin{array}{l}\text{space period} = \lambda \\ \text{time period} = T_0\end{array}\right\} \Rightarrow \cos\left(\frac{2\pi X}{\lambda_0} - \frac{2\pi t}{T_0}\right)$$

Note: sending $X \to X + \lambda_0$ or $t \to t + T_0$ leaves this unchanged

Let us set $s(x,t) = A\cos\left(\frac{2\pi x}{\lambda_0} - \frac{2\pi t}{T_0}\right)$ in the string equation $\Rightarrow$

$$\frac{\partial^2 s}{\partial x^2} = -A\left(\frac{2\pi}{\lambda_0}\right)^2 \cdot \cos\left(\frac{2\pi x}{\lambda_0} - \frac{2\pi t}{T_0}\right)$$

$$\frac{\partial^2 s}{\partial t^2} = -A\left(\frac{2\pi}{T_0}\right)^2 \cos\left(\frac{2\pi x}{\lambda_0} - \frac{2\pi t}{T_0}\right)$$

$$-A\left(\frac{2\pi}{T_0}\right)^2 \cos\left(\frac{2\pi x}{\lambda_0} - \frac{2\pi t}{T_0}\right) = \left(\frac{T}{\mu}\right)\left(-A\left(\frac{2\pi}{\lambda_0}\right)^2\right)\cos\left(\frac{2\pi x}{\lambda_0} - \frac{2\pi t}{T_0}\right)$$

$$\therefore \quad \frac{1}{T_0^2} = \left(\frac{T}{\mu}\right) \cdot \frac{1}{\lambda_0^2}$$

$$\therefore \quad \boxed{\frac{\lambda_0}{T_0} = \left(\frac{T}{\mu}\right)^{\frac{1}{2}}}$$

Lessons ① simple periodic form is a solution to the string eqn $\Rightarrow$ string can oscillate in this fashion governed by Newton's law.

② String: $\left(\frac{\text{Wavelength}}{\text{Period}}\right) = \left(\frac{\text{Tension}}{\text{Mass per unit length}}\right)^{1/2}$

$\therefore$ If we double λ_0, we will also double the period for fixed tension on a given wire. λ_0 & T_0 are not independent.