Entangled Granular Material
Key Facts about Granular Materials

- collections of macroscopic (\(\sim\) mm) particles interacting through contact forces (e.g. friction)

- Thermal energy irrelevant (\(k_BT \sim 10^{-21} \text{ J} \ll PE \sim 10^{-5} \text{ J}\))
  * systems “frozen” in metastable state, unable to move to lower energy state without external help

- forces propagate in linear chains

\[
2F \sin \theta = W \\
\Rightarrow F = \frac{W}{2 \sin \theta}
\]
Ordinary granular materials are easily manipulated.
Geometrically cohesive particles are different

- Why are rodpiles so rigid?
- Is the rigid rodpile qualitatively different from the sandpile?
- What governs how granular materials rearrange?
Rods: Static Packings and Solid Plugs

- **Philipse** *(Langmuir, 1996)*
  - above aspect ratio $\sim 35$ pile emerges as solid plug
  - mean-field *Random Contact Model* (packing fraction $\propto v_{\text{excl}}^{-1}$)
  - assumes no orientational correlation, constant coordination number

- **Blouwolff and Fraden** *(Europhysics Letters, 2006)*
  - small variation in coordination number $(6 \leq \langle z \rangle \leq 10)$, essentially validates RCM

![Graph showing packing fraction vs. aspect ratio](image-url)
GCGM display solid & granular behavior  Desmond & SVF (2006)
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Stick-slip

Transition

Solid

Smaller Container
Column Collapse of Granular Rods

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Experimental Setup

- $D = 11.43, 15.24$ cm cylinders
- Sand, Acrylic & Teflon Rods
  - $L : 2.5 - 7.5$ cm
  - $w : 0.16 - 0.6$ cm
  - aspect ratio 4-48
- independent of cylinder velocity
- average runoff in 4 directions
Geometric Transition

Low Piles

Tall Piles
\[ \tilde{r} = \begin{cases} 
\tilde{H}^{1.2 \pm 0.1} & \tilde{H} < 1.1 \pm 0.3 \\
\tilde{H}^{0.6 \pm 0.1} & \tilde{H} > 1.1 \pm 0.3 
\end{cases} \]
Conservation of Volume I: Mesas

\[
\pi r_0^2 H_0 \quad = \quad \pi \left( \frac{H_f}{r_0} \right)^3 \left[ \frac{1}{3 \tan^2 \theta_c} - \frac{1}{\tan \theta_c H_f} + \left( \frac{r_f}{H_f} \right)^2 \right]
\]

- \( H_f = H_0, \tan \theta_c = \frac{H_f}{r_f} \)

\[
\frac{r_f}{r_0} \quad = \quad \frac{1}{2 \tan \theta_c} \left( \frac{H_f}{r_0} + \sqrt{4 \tan^2 \theta_c - \frac{H_f^2}{3 r_0^2}} \right) \quad \approx \quad \frac{1}{2 \tan \theta_c r_0} H_f
\]
Conservation of Volume II: Cones

"Classic" angle of repose theory

\[ \pi r_0^2 H_0 = \frac{1}{3} \pi H_f r_f^2 \implies \frac{r_f}{r_0} \sim (H_0)^{1/3} \]

\[ \frac{H_f}{r_f} = \tan \theta_c \]
Conservation of Volume II: Cones

"Classic" angle of repose theory

\[
\pi r_0^2 H_0 = \frac{1}{3} \pi H_f r_f^2 \implies \frac{r_f}{r_0} \sim (H_0)^{1/3}
\]

Internal stable cone

\[
\frac{H_f}{r_0} = \tan \theta_c
\]

\[
\pi r_0^2 H_0 = \frac{1}{3} \pi H_f r_f^2 \implies \pi r_0^2 H_0 = \frac{1}{3} \pi r_0 \tan \theta_c r_f^2
\]

\[
\implies r_f \sim H_0^{1/2}
\]
Long thin rods form stable piles

AR 24: $h = 3r$
Moderately long rods also don’t collapse!

AR 16: $h = 0.35r$

AR 16: $h = 0.35r$

AR 16: $h = r$
Transition from Solid to Flow

- **Granular Behavior**
- **Transition Region**
- **Solid Behavior**

The graph shows the relationship between pile height/particle length ($\tilde{H}_p, \tilde{H}_d$) and particle aspect ratio ($\alpha$) with data points indicating the transition from solid to flow behavior.
Linear Increase in Collapse Probability

\[ H = (\tilde{H} - \tilde{H}_l) / (\tilde{H}_u - \tilde{H}_l) \]

Collapse Probability

0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

0
0.2
0.4
0.6
0.8
1

H=(\tilde{H} - \tilde{H}_l) / (\tilde{H}_u - \tilde{H}_l)
Staples
Nick Gravish, SVF, David Hu, Dan Goldman (2012)
Biological geometric cohesion: ant rafts

- Rafts contain \( \sim 10^5 \) ants, essential for colony survival
- Cohere through interlocked limbs (stable even when dead)
Vibration induced melting
Height relaxes as stretched exponential $h(t) \propto \exp[-(t/\tau)^\beta]$

- Large $\tau$: rigid pile not shaking hard
Timescale $\tau$ vs. vibration acceleration $\Gamma$: $\tau \propto \exp[\Gamma_0/\Gamma]$

- Large $\Gamma_0$: pile much more resistant to less vigorous shaking
Optimally Rigid Barb Ratio

\[ \Gamma_0 \]

Barb ratio \( l/w \)

\[ \max[\Delta] \]
Entanglement Number
Nonlinear decay/growth of packing fraction, entanglement
Entanglement density peaks with $\Gamma_0$
Entanglement density of arcs

- no jamming at $\theta_{\text{subtended}} = 0, 2\pi$
- can “easily” find distance between two arcs

$$(x^A_c, y^A_c, \theta_A), (x^B_c, y^B_c, \theta_B)$$

$$D_{AB}^2 \equiv \left( (x^A_c + r \cos \theta_A) - (x^B_c + r \cos \theta_B) \right)^2$$
$$+ \left( (y^A_c + r \sin \theta_A) - (y^B_c + r \sin \theta_B) \right)^2$$
Entanglement density of arcs

- no jamming at $\theta_{\text{subtended}} = 0, 2\pi$
- can “easily” find distance between two arcs

\[ D_{AB}^2 \equiv \left( (x_c^A + r \cos \theta_A) - (x_c^B + r \cos \theta_B) \right)^2 \]
\[ + \left( (y_c^A + r \sin \theta_A) - (y_c^B + r \sin \theta_B) \right)^2 \]

Lagrangian formulation actually easier. Minimize

\[ D^2 = (x_A - x_B)^2 + (y_A - y_B)^2 \]

subject to constraints: $(x_{A/B}, y_{A/B})$ lie on circle $A/B$
Rheology of Geometrically Cohesive Granular Materials

- \( l = 1.3\text{cm} \)
- \( w = 0.64\text{cm} \)
- \( l/w = 0.5 \)
- \( d = 0.1\text{cm} \)
- \( \phi \approx 0.24 \)
- \( D = 3.1\text{cm} \)
- \( L = 2 - 11\text{cm} \)
Canonical stick-slip behavior
Event details

![Graph showing force vs. elongation with labels for $F_{\text{yield}}$ and $F_{\text{drop}}$.]
• Treat all 7543 slip events as independent failures
Longer piles significantly weaker
Weibullian “weakest link” statistics (Ken Kamrin)

- **Basic assumptions**
  - multiple small units $\delta L$, if any one fails, sample fails
  - probability of a unit failing is $P_f \propto F^m \delta L$

- **Success probability of $N$ elements:**
  \[ S = \prod [1 - \alpha F^m \delta L] \]
  \[
  \ln S = \sum \ln [1 - \alpha F^m \delta L] \approx \sum -\alpha F^m dL \\
  \implies S \approx \exp [-\alpha LF^m]
  \]

- **Probability of failure if force $F$ and length $L$:**
  \[ P_f(F, L) = 1 - e^{-\alpha LF^m} \]
Cumulative Failure distribution function

\[ P_f(F, L) = 1 - e^{-\alpha LF^m} \]
Prediction #1 (fixed length): \( P_f(F') = 1 - e^{-\beta F^m} \)
Fixed length failure probability distribution: $m = 1$
Prediction #2: Mean yield force \( \sim L^{-1} \)

- \(< F_Y > = \int_0^\infty \alpha L F e^{-\alpha L F} dF \sim L^{-1}\)
Mean yield force $\sim L^{-1}$
Prediction #3: Collapse all data onto single curve

\[ P_f(F, L) = 1 - e^{-\alpha L F^m} \implies \frac{-1}{L} \log \left[ 1 - \frac{P_f(F, L)}{P_f(L)} \right] \propto F \]
Prediction #3: Collapse all data onto single curve

\[ P_f(F, L) = 1 - e^{-\alpha L F^m} \Rightarrow \frac{-1}{L} \log \left[ 1 - \frac{P_f(F, L)}{P_f(L)} \right] \propto F \]
Conclusions

- Geometrically Cohesive Granular Materials (GCGM) exhibit solid-like and stick-slip *extensional* rheology
- Statistical/thermodynamic models show some success in explaining behavior
- Extensional rheology well-modeled by weakest-link theory that assumes yield probability proportional to applied force
Flow of Particles Through Wedge Hoppers

aperture: 0-50cm
rods: \( d = 0.08 - 0.6 \text{cm} \)
\( L = 0.6 - 8 \text{cm} \)
aspect ratio: 1 - 50

Summer Saraf
Ordinary (round) Hoppers: Exit Mass Distribution Decays Exponentially

- Zuriguel (2005), distributions scale as $\langle s \rangle$ which may diverge as hopper aperture $R \rightarrow R_c \approx 5D$ ($D$=particle diameter)
- no memory effects
3+ Decades of Power Law

Large events *more* common than expected
Independent of hit number (Aperture = 10\textit{d})

\begin{itemize}
\item \text{\textit{P}}(n)
\item \text{\textit{n}} \quad \text{number of particles to exit before a jam occurs}
\end{itemize}
Random Walk Models

- Exponential decay implies no correlations:
  - 1 particle exits w/probability $p$
  - $n$ particles: $P(n) = p \cdot p \ldots p \cdot (1 - p) = p^n \cdot (1 - p)$

\[
\log P(n) = \log p^n (1 - p) = \log(1 - p) + n \log p
\]

\[
\exp[\log P(n)] = e^{\log(1-p) + n \log p}
\]

\[
P(n) = e^{\log(1-p)} e^{n \log p} = (1 - p) e^{n \log p}
\]

$p < 1$ and so $\log p < 0$ and $P(N)$ exponentially decays

- generate random numbers until get one $> p$
- number of random numbers is “event size”
- do this many times, make histogram of event size
Simulation: March 2009

Wedge-shaped hopper w/uniform probability

\[
\begin{array}{cccccccc}
\text{p} & \text{p} & \text{p} & \text{p} & \text{p} & \text{p} & \text{p} & \text{p} \\
\end{array}
\]
Wedge-shaped hopper w/ uniform probability

nonuniform probability: cell $j$ has probability $p_j$

Simulation: March 2009
Wedge-shaped hopper w/uniform probability

nonuniform probability: cell $j$ has probability $p_j$
“String” orientation

\[ p_c \]

\[ p_d \]

\[ \perp : \text{low exit probability} \]
“String” orientation

\( p_c \): probability of entering the string orientation

\( p_d \): probability of exiting the string orientation

\( \perp \): low exit probability

\( > \): larger exit probability
“String” orientation

\[ p_c, p_d \]

\( \perp \): low exit probability
\( > \): larger exit probability
\( \parallel \): largest exit probability

Distribution of exit probabilities \( p(\theta) \). Need to average over string orientations to find \( \langle P(n) \rangle \).
Final Picture

\[ P(N) = \int_{p_c}^{p_d} p^N (1 - p) \, dp \]

\[ = \left[ \frac{p^{N+1}}{N+1} - \frac{p^{N+2}}{N+2} \right]_{p_c}^{p_d} \]

\[ \approx \frac{p_d^{N+1}}{N+1} \approx \frac{\exp[N \ln p_d]}{1+N} \]

As long as \( p_d < 1 \), exponential decay.

As soon as \( p_d = 1 \):

\[ \frac{1}{N+1} - \frac{1}{N+2} \approx \frac{1}{N^2} \]
Transition to Power Law

Probability Distribution $P(N)$

- $p_+ = 0.9$
- $p_+ = 0.975$
- $p_+ = 0.995$
- $p_+ = 1.0$

Exit mass $N$
Success!
Conclusions II

- Exit-mass probability distribution in wedge hoppers shows broad power-law tail.
- Model that assumes characteristic length-scale (strings) with orientation dependent exit probability.
  * Exponential or power-law tail depending on aperture geometry.
- Model and experiment agree over many decades.