Solutions to practice problem set 4

1 Chapter 3, Problem 61

You throw a baseball at a 45° angle to the horizontal, aiming at a friend who's sitting in a tree a distance h above level ground. At the instant you throw your ball, your friend drops another ball. ***(a)*** *Show that the two balls will collide, no matter what your ball's initial speed, provided it's greater than some minimum value.* ***(b)*** *Find an expression for that minimum speed.*

(a) Since the friend at a height h is at a 45° angle, they are also at a horizontal distance h. We choose coordinates such that you are located at the origin, and your friend is located at the point $(x, y) = (h, h)$.

Given an initial speed v_0, the equations of motion for your ball (x_1 and y_1) and your friend's ball (x_2 and y_2) are

$$x_1(t) = \frac{v_0}{\sqrt{2}}t, \qquad x_2(t) = h,$$
$$y_1(t) = \frac{v_0}{\sqrt{2}}t - \frac{1}{2}gt^2, \qquad y_2(t) = h - \frac{1}{2}gt^2.$$

A collision occurs if the two balls are *at the same place at the same time*: $x_1(t_c) = x_2(t_c)$ *and* $y_1(t_c) = y_2(t_c)$. First, we find the time t_c at which $x_1(t_c) = x_2(t_c)$,

$$\begin{aligned} x_1(t_c) &= x_2(t_c) \\ \frac{v_0}{\sqrt{2}}t_c &= h \\ t_c &= \frac{\sqrt{2}h}{v_0}, \end{aligned} \tag{1}$$

and then check that $y_1(t_c) = y_2(t_c)$:

$$\begin{aligned} y_1(t_c) - y_2(t_c) &= \left(\frac{v_0}{\sqrt{2}}t_c - \frac{1}{2}g{t_c}^2\right) - \left(h - \frac{1}{2}g{t_c}^2\right) \\ &= \frac{v_0}{\sqrt{2}}t_c - h \\ &= \frac{v_0}{\sqrt{2}}\frac{\sqrt{2}h}{v_0} - h \\ &= 0. \end{aligned} \tag{2}$$

Since this holds for any value of v_0, the question is answered.

(b) Though in part (a) we found a collision for any value of v_0, in reality, if the initial velocity is insufficient, the trajectories can be interrupted by the ground – the line $y = 0$ – before the collision can happen. Hence, we require that the collision happen with $y \geq 0$, i.e. $y_2(t_c) \geq 0$ (or, equivalently, $y_1(t_c) \geq 0$):

$$\begin{aligned} h &\geq \frac{1}{2}g{t_c}^2 \\ h &\geq \frac{1}{2}g\frac{2h^2}{{v_0}^2} \\ {v_0}^2 &\geq gh \\ v_0 &\geq \sqrt{gh}. \end{aligned} \tag{3}$$

The minimum speed is $\sqrt{gh}$.

2 Chapter 3, Problem 75

A jet is diving vertically downward at 1200 km/h. If the pilot can withstand a maximum acceleration of 5g (i.e., 5 times Earth's graviational acceleration) before losing consciousness, at what height must the plane start a 90° circular turn, from vertical to horizontal, in order to pull out of the dive? See Fig. 3.25, assume the speed remains constant, and neglect gravity.

As the sketch in Fig. 3.25 illustrates, the radius of the circular turn is equal to the height at which the plane must start the turn. We employ the relation between centripetal acceleration, speed, and radius:

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ r &= \frac{v^2}{a_c} \end{aligned} \tag{4}$$

Letting $a_c = 5g = 5(9.81\,\mathrm{m/s^2})$ and

$$\begin{aligned} v &= 1200\,\frac{\mathrm{km}}{\mathrm{h}}\,\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\,\frac{1000\,\mathrm{m}}{1\,\mathrm{km}} \\ &= 333.3\,\frac{\mathrm{m}}{\mathrm{s}}, \end{aligned} \tag{5}$$

we find

$$\begin{aligned} r &= \frac{(333.3\,\mathrm{m/s})^2}{5(9.81\,\mathrm{m/s^2})} \\ &= 2265\,\mathrm{m}. \end{aligned} \tag{6}$$

3 Chapter 3, Problem 79

A soccer player can kick the ball 28 m on level ground, with its initial velocity at 40° to the horizontal. At the same initial speed and angle to the horizontal, what horizontal distance can the player kick the ball on a 15° upward slope?

We denote the lauch angle by $\theta = 40°$, the slope angle by $\alpha = 15°$, and the level-ground rage by $x_0 = 28\,\mathrm{m}$. We choose the x axis to be horizontal and the y axis to be vertical, such that the initial velocity is in the direction $\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ and the initial position is at the origin.

We only need to know about the ball's trajectory, so we may use the equations of motion

$$x(t) = v_0 t\cos\theta \tag{7a}$$

$$y(t) = v_0 t\sin\theta - \frac{1}{2}gt^2 \tag{7b}$$

to eliminate time and obtain the equation for the trajectory:

$$y(x) = \tan\theta x - \frac{\sec^2\theta}{2l}x^2, \tag{8}$$

where we wrote $l = {v_0}^2/g$ as a shorthand. Factoring this equation,

$$\begin{aligned} y(x) &= -\frac{\sec^2\theta}{2l}x\left(x - 2l\cos^2\theta\tan\theta\right) \\ &= -\frac{\sec^2\theta}{2l}x\left(x - 2l\cos\theta\sin\theta\right) \\ &= -\frac{\sec^2\theta}{2l}x\left(x - l\sin(2\theta)\right), \end{aligned} \tag{9}$$

we recognize that the flat-ground range, given by the nonzero root of $y(x)$, is $x_0 = l\sin(2\theta)$; hence, we conclude that $l = x_0/\sin(2\theta)$.

To find the horizontal range on the slope, we must intersect the trajectory equation with the line $y = \tan\alpha x$. Calling the intersection x_0', we have

$$\begin{aligned} \tan\alpha x_0' &= \tan\theta x_0' - \frac{\sec^2\theta}{2l}{x_0'}^2 \\ 0 &= \tan\theta x_0' - \tan\alpha x_0' - \frac{\sec^2\theta}{2l}{x_0'}^2 \\ 0 &= -\frac{\sec^2\theta}{2l}x_0'\left(x_0' - 2l\cos^2\theta(\tan\theta - \tan\alpha)\right). \end{aligned} \tag{10}$$

The horizontal range is given by the nonzero root. Since $l = x_0/\sin(2\theta)$, we have

$$\begin{aligned} x_0' &= 2l\cos^2\theta(\tan\theta - \tan\alpha) \\ &= 2\frac{x_0}{\sin(2\theta)}\cos^2\theta(\tan\theta - \tan\alpha) \\ &= 2\frac{x_0}{2\sin\theta\cos\theta}\cos^2\theta(\tan\theta - \tan\alpha) \\ &= x_0\frac{1}{\tan\theta}(\tan\theta - \tan\alpha) \\ &= x_0\left(1 - \frac{\tan\alpha}{\tan\theta}\right). \end{aligned} \tag{11}$$

It is a good habit to do some simple consistency checks on an answer such as this one. For starters, when the slope is flat ($\alpha = 0$), then $x_0' = x_0$, as expected. When $\alpha > 0$, the range is decreased, and it goes to zero as $\alpha \to \theta$, as expected; when $\alpha < 0$, the range is increased, as it should be.

Substituting numbers, we find

$$\begin{aligned} x_0' &= (28\,\mathrm{m})\left(1 - \frac{\tan 15°}{\tan 40°}\right) \\ &= 19.1\,\mathrm{m}. \end{aligned} \tag{12}$$