Practice Problem Set #6
Solutions

Question #1:

a) Differentiating the particle's position vector, $\vec{r}(t) = ((1/2)t^2 + 3t)\hat{\imath} + 2t^2\hat{\jmath} - 7t\hat{k}$, with respect to time gives an expression for the particle's velocity

$$\vec{v}(t) = (t+3)\hat{\imath} + 4t\hat{\jmath} - 7\hat{k}. \tag{1}$$

Differentiating (1) with respect to time then gives an expression for the particle's acceration

$$\vec{a}(t) = \hat{\imath} + 4\hat{\jmath} + 0\hat{k}. \tag{2}$$

Now, using Newton II ($\vec{F} = m\vec{a}$) with $m = 1$, the force on the particle as a function of time is simply

$$\vec{F}(t) = \hat{\imath} + 4\hat{\jmath} + 0\hat{k}. \tag{3}$$

Using $P = dW/dt = \vec{F} \cdot \vec{v}$, we have

$$\begin{aligned} P(t) &= \vec{F} \cdot \vec{v}(t), \\ &= \left(\hat{\imath} + 4\hat{\jmath} + 0\hat{k}\right) \cdot \left((t+3)\hat{\imath} + 4t\hat{\jmath} - 7\hat{k}\right), \\ &= (t+3) + 16t, \\ &= 17t + 3. \end{aligned} \tag{4}$$

b) The total work, W, done on the particle by the external system from time $t = 0$ to $t = 1$ is then

$$\begin{aligned} W &= \int_0^1 P(t)dt = \int_0^1 [17t + 3]dt, \\ &= \left[\frac{17}{2}t^2 + 3t\right]_0^1 = \frac{23}{2}. \end{aligned} \tag{5}$$

Question #2:

Begin by considering the position of the object as a function of θ (see attached Figure 1) in the loop. The magnitude of the normal force on object by the loop is then simply $F_N(\theta) = mg\cos\theta$, while changes in angle $d\theta$ are related to displacements along the loop edge by $d\vec{s} = d\theta R\hat{\theta}$. Since the force of friction, $\vec{F}_f(\theta)$, always points anti-parallel to displacements $d\vec{s}$, we have

$$dW = -\vec{F}_f(\theta)\cdot d\vec{s} = F_f(\theta)Rd\theta. \tag{6}$$

Integrating over a dummy index, θ', from $\theta' = 0$ to $\theta' = \theta$ and using $F_f(\theta) = \mu F_N(\theta)$, we obtain an expression for the total work, W, done against friction

$$\begin{aligned} W &= \int_0^\theta F_f(\theta')Rd\theta' = \int_0^\theta \mu F_N(\theta')Rd\theta', \\ &= \int_0^\theta \mu Rmg\cos\theta' d\theta' = \mu Rmg\sin\theta. \end{aligned} \tag{7}$$

However, geometrically we have $\sin\theta = \frac{\sqrt{R^2-(R-h)^2}}{R}$, so the above expression can be rewritten

$$W = \mu mg\sqrt{R^2-(R-h)^2} = \mu mg\sqrt{2hR-h^2} \tag{8}$$

as desired.

Question #3:

Ascending the sloped road, the power required for the car to maintain a constant speed is opposite the rate of work done by the car against the air and gravity:

$$\begin{aligned} P_{req.} &= -\vec{F}_{net}\cdot\vec{v} = -\vec{F}_{drag}\cdot\vec{v} - \vec{F}_g\cdot\vec{v}, \\ &= F_{drag}v + mgv\sin\theta, \end{aligned} \tag{9}$$

where v is the speed of the car, F_{drag} is the magnitude of the drag force on the car, θ is the angle of the road above the horizontal, m is the mass of the car, and g is the acceleration due to gravity. Solving for θ we find

$$\theta = \arcsin\left(\frac{P_{req.} - vF_{drag}}{mgv}\right). \tag{10}$$

Evaluating (10) using $m = 1400\text{kg}$, $v = 60\text{km/h} = 16.7\text{m/s}$, $F_{drag} = 450\text{N}$, and $P_{tot} = 38\times10^3\text{W}$ we find $\theta = 7.7^0$.