Repeated Measurements

- When you make repeated measurements of the same thing, you often do *not* get the same number over and over.
- Instead, you get a **distribution** of numbers, which, when considered together, give you and idea of the true number you are seeking.

Mean

- Suppose you make N measurements of a quantity x, and you expect these measurements to be normally distributed
- Each measurement, or trial, you label with a number i, where $i = 1, 2, 3$, etc
- You can estimate the **mean** by adding up all the individual measurements and dividing by N:

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Standard Deviation

- Suppose you make N measurements of a quantity x, and you expect these measurements to be normally distributed
- The best estimate of the **standard deviation** is:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$$

- The quantity $N - 1$ is called the **number of degrees of freedom**
- In this case, it is the number of measurements minus one because you used one number from a previous calculation (mean) in order to find the standard deviation.

Example on Excel

	A	B	C
1	t1	5.334	
2	t2	5.462	
3	t3	5.459	
4	t4	5.328	
5	t5	5.331	
6	t6	5.592	
7	t7	5.480	
8	=AVERAGE(B1:B7)	5.427	**Mean**
9	=STDEV(B1:B7)	0.10014	**Standard Deviation**

Statistical Uncertainty

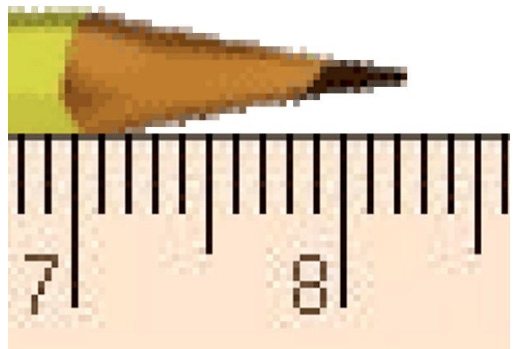
- There is roughly a 68% chance that any measurement of a sample taken at random will be within one standard deviation of the mean
- Usually the mean is what we wish to know and each individual measurement almost certainly differs from the true value of the mean by some uncertainty
- There is a 68% chance that any single measurement lies with one standard deviation of this true value of the mean
- Thus it is reasonable to say that:

$$u_x = \sigma$$

- This uncertainty is often called *statistical*

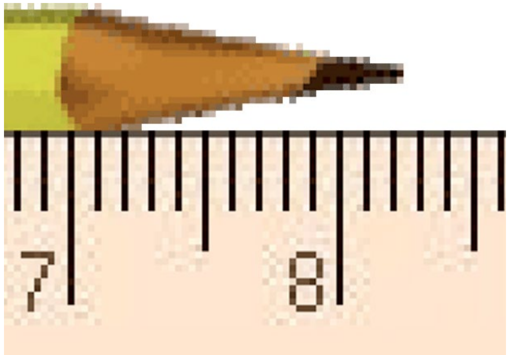
Reading Uncertainty (Analog)

- Imagine you use a ruler to measure the length of a pencil
- You line up the tip of the eraser with 0, and the image below shows what you see over near 8 cm



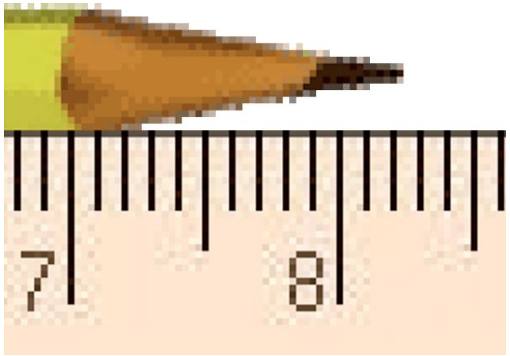
- The pencil appears to be about 8.25 cm long, but what is the reading uncertainty?
- There is no fixed rule that will allow us to answer this question
- We must use our *intuition* and *common sense*!

Reading Uncertainty (Analog)



- Could the pencil actually be as long as 8.3 cm? …no, I don't think so
- Could it be 8.28 cm? …maybe
- And it could be as short as 8.23 cm, but, in my opinion, no shorter
- So the range is about 8.23 to 8.28 cm

Reading Uncertainty (Analog)



- The range is about 8.23 to 8.28 cm
- A reasonable estimate of the reading uncertainty of this measurement is half the range: ± 0.025 cm
- To be cautious, we might round up to 0.03 cm
- We say "The length of the pencil is 8.25 ± 0.03 cm."
- Meaning: if we get a collection of objective observers together to look at the pencil above, we expect most (ie more than 68%) of all observers will report a value between 8.22 and 8.28 cm

Reading Uncertainty (Digital)

- For a measurement with an instrument with a digital readout, the reading uncertainty is usually "± one-half of the last digit."
- This means one-half of the power of ten represented in the last digit.
- With the digital thermometer shown, the last digit represents values of a tenth of a degree, so the reading uncertainty is $\frac{1}{2} \times 0.1 = 0.05$°C

- You should write the temperature as 12.80 ± 0.05 °C.

Choosing between Statistical and Reading Uncertainty

- In most cases, when you have both a standard deviation and a reading uncertainty, one is much larger than the other, and then you should choose the **larger** to be the uncertainty
- For example, if every time you measure something you always get the same numerical answer, this indicates that the reading uncertainty is dominant
- However, if every time you measure something you get different answers which differ more than the reading uncertainty you might estimate, then the standard deviation is dominant

Significant Figures

- Imagine you have a set of 7 timing measurements for which the statistical uncertainty is clearly dominant
- You use an equation to estimate that the standard deviation is 0.10014 seconds
- Consider one of these measurements, the 5th one, for which we measured 5.331 seconds
- Using the standard deviation as the uncertainty, this measurement should be written as 5.331 ± 0.10014 s
- What this means is that there is about a 68% chance that the true value is somewhere between 5.23086 and 5.43114 seconds….

????

Significant Figures

$$5.331 \pm 0.10014 \text{ s } ????$$

- Clearly we are using too many significant figures here!
- It would be just as instructive to say that there is about a 68% chance that the true value is somewhere between 5.2 and 5.4 seconds
- Or, you could say the measurement is: **5.3 ± 0.1 s**
- In fact it is not only more concise to report this, but it is more honest

Significant Figures

- There are two general rules for significant figures used in experimental sciences:

1. Uncertainties should be specified to one, or at most two, significant figures.
2. The most precise column in the number for the uncertainty should also be the most precise column in the number for the value.

- So if the uncertainty is specified to the 1/100th column, the quantity itself should also be specified to the 1/100th column.

Propagation of Uncertainties of Precision

- When you have two or more quantities with known uncertainties you may sometimes want to combine them to compute a derived number
- You can use the rules of Uncertainty Propagation to infer the uncertainty in the derived quantity
- We assume that the two directly measured quantities are x and y, with uncertainties u_x and u_y, respectively
- The measurements x and y must be independent of each other.
- The fractional uncertainty is the value of the uncertainty divided by the value of the quantity: u_x / x
- To use these rules for quantities which cannot be negative, the fractional uncertainty should be much less than one

Propagation of Uncertainties

- Rule #1 (sum or difference rule):
- If $z = x + y$
- or $z = x - y$
- then

$$u_z = \sqrt{{u_x}^2 + {u_y}^2}$$

- Rule #2 (product or division rule):
- If $z = xy$
- or $z = x/y$
- then $\frac{u_z}{z} = \sqrt{\left(\frac{u_x}{x}\right)^2 + \left(\frac{u_y}{y}\right)^2}$

Propagation of Uncertainties

- Rule #2.1 (multiply by exact constant rule):
- If $z = xy$ or $z = x/y$
- and x is an exact number, so that $u_x = 0$
- then

$$u_z = |x| u_y$$

- Rule #3 (exponent rule):
- If $z = x^n$
- then

$$\frac{u_z}{z} = n \frac{u_x}{x}$$

The Uncertainty in the Mean

- Suppose many individual, independent measurements are repeated N times
- Each individual measurement has the same uncertainty u_x
- Using uncertainty propagation, you can show that the uncertainty in the estimated mean is:

$$u_{\bar{x}} = \frac{u_x}{\sqrt{N}}$$

Example on Excel

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- The 7 individual measurements should be reported as $t_1 = 5.3 \pm 0.1$ s, $t_2 = 5.5 \pm 0.1$ s, $t_3 = 5.5 \pm 0.1$ s, $t_4 = 5.3 \pm 0.1$ s, $t_5 = 5.3 \pm 0.1$ s, $t_6 = 5.6 \pm 0.1$ s, and $t_7 = 5.5 \pm 0.1$ s.
- The uncertainty in the mean is $u_t/\sqrt{N} = 0.10014\sqrt{7} = 0.038$
- The mean should be reported as $\bar{t} = 5.43 \pm 0.04$ s.