Practical Problem Set 4: Solution

February 6, 2017

1. Two infinite slabs have thickness a and both are uniformly charged as shown in Fig. 1. The charge densities are ρ and $-\rho$ respectively. They have a distance between them of d. Calculate the electric field inside and outside the slabs.

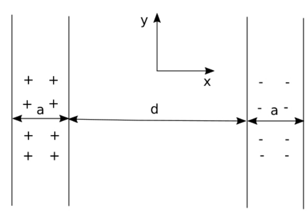

Figure 1: Two infinite charged slabs.

Solution:
Let the middle of d be the origin of x axis,
for $-d/2 < x < d/2$,

$$\boldsymbol{E} = \boldsymbol{E}_{left} + \boldsymbol{E}_{right} = \frac{\rho a}{2\varepsilon_0}\hat{\boldsymbol{e}_x} + \frac{\rho a}{2\varepsilon_0}\hat{\boldsymbol{e}_x} = \frac{\rho a}{\varepsilon_0}\hat{\boldsymbol{e}_x}; \tag{1}$$

where $\hat{\boldsymbol{e}_x}$ is the unit vector in $\hat{\boldsymbol{x}}$ direction;
for $x < -d/2 - a$ and $x > d/2 + a$, $\boldsymbol{E} = 0$;
for $-d/2 - a \leq x \leq -d/2$,

$$\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_{left1} + \boldsymbol{E}_{left2} + \boldsymbol{E}_{right} && (2)\\
&= \frac{\rho(x + a + d/2)}{2\varepsilon_0}\hat{\boldsymbol{e}_x} + \frac{\rho(-d/2 - x)}{2\varepsilon_0} \cdot -\hat{\boldsymbol{e}_x} + \frac{\rho a}{2\varepsilon_0}\hat{\boldsymbol{e}_x} && (3)\\
&= \frac{\rho(x + \frac{a+d}{2})}{\varepsilon_0}\hat{\boldsymbol{e}_x} + \frac{\rho a}{2\varepsilon_0}\hat{\boldsymbol{e}_x}; && (4)
\end{aligned}$$

for $d/2 \le x \le d/2 + a$,

$$\begin{aligned}
\boldsymbol{E} &= \boldsymbol{E}_{left} + \boldsymbol{E}_{right1} + \boldsymbol{E}_{right2} && (5)\\
&= \frac{\rho a}{2\varepsilon_0}\hat{\boldsymbol{e}_x} + \frac{\rho(x-d/2)}{2\varepsilon_0}\cdot -\hat{\boldsymbol{e}_x} + \frac{\rho(d/2+a-x)}{2\varepsilon_0}\cdot \hat{\boldsymbol{e}_x} && (6)\\
&= \frac{\rho a}{2\varepsilon_0}\hat{\boldsymbol{e}_x} + \frac{\rho(\frac{a+d}{2}-x)}{\varepsilon_0}\hat{\boldsymbol{e}_x}; && (7)
\end{aligned}$$

2. An infinitely long cylinder has charge uniformly distributed inside it with a positive volume charge density of ρ. Part of this cylinder has been removed to form another cylindrical shape. The vertical axis of this hollow shape is parallel to that of the original cylinder and these axes have a distance of a as shown in Fig. 2. Calculate the electric field in the hollow part.

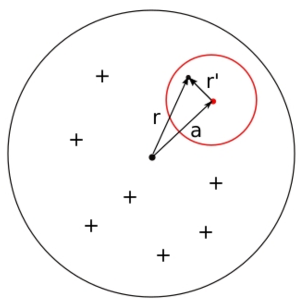

Figure 2: A infinitely long cylinder with a hollow cylindrically shaped part.

Solution: The system can be considered as a positive charged cylinder (the whole one) plus a negative charged cylinder (in the hollow shape).
For a infinitely charged solid cylinder, electric field at distance r from the centre is:

$$\boldsymbol{E} = \frac{\pi r^2 \cdot \rho}{2\pi r \varepsilon_0}\hat{\boldsymbol{e}_r} = \frac{\rho}{2\varepsilon_0}\boldsymbol{r} \tag{8}$$

By the superposition of the electric fields, inside the hollow cylindrically shaped part:

$$\boldsymbol{E}_{total} = \frac{\rho}{2\varepsilon_0}\boldsymbol{r} - \frac{\rho}{2\varepsilon_0}\boldsymbol{r}' = \frac{\rho}{2\varepsilon_0}\boldsymbol{a} \tag{9}$$

which means that the electric field in the hollow part is a constant electric field.

3. Imagine a point charge q located in the centre of a spherical conducting shell (which can be considered very thin, radius of R).

(1)Fig. 3 shows the diagram of electric field distribution inside and outside the shell. Determine the net charge on this conducting shell, and describe the charge distribution on the inner surface and outer surface;
(2)If the net charge on the whole spherical conducting shell is zero,

a. draw a diagram to indicate the electric field inside and outside the shell;

b. what if the point charge is shifted from the centre of the shell? Is the induced charge on the inner surface uniformly distributed? Is the induced charge on the outer surface uniformly distributed?

c. In Fig. 4, when the point charge is shifted d from the centre, calculate the electric field in the space.

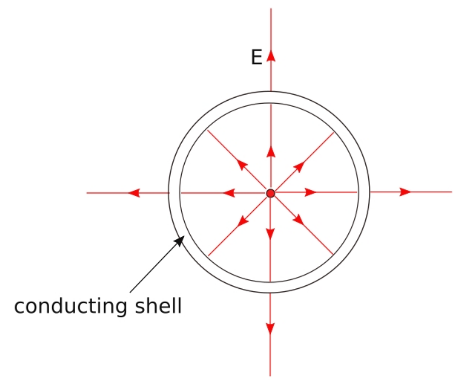

Figure 3: A point charge q located in the centre of a spherical conducting shell. The red lines with arrows indicate the electric field.

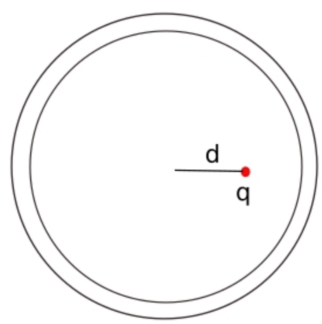

Figure 4: A point charge q shifted from the centre of a spherical conducting shell.

Solution:
(1)On the inner surface ($r \to R$), from Gauss's law,

$$\Phi_1 = q/\varepsilon_0 \tag{10}$$

Since the electric field inside the conducting shell is zero, the induced charge on the inner surface should be equal to the point charge q, and negative.
On the outer surface ($r \leftarrow R$),

$$\Phi_2 = q_{outer}/\varepsilon_0 \tag{11}$$

With $\Phi_1/\Phi_2 = 8/4 = 2$, $q_{outer} = q/2$, and positive. Hence the net charge on the shell is $-q/2$. And the charge on the inner and outer surface are both uniformly distributed because of symmetry.
(2) Fig. 5 shows the electric fields when the point charge located in the centre or shifted. In both cases, the charge on the outer surface is uniformly distributed. But the charge on the inner surface is not uniformly distributed any more when the point charge is shifted.
c. For this problem, we have the conditions: 1. the total charge on the shell must be zero;

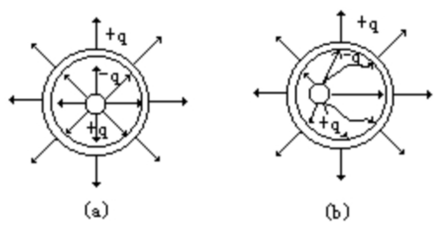

Figure 5: electric fields for question (a),(b)

2. the electric potential must be constant over the shell (but not necessarily equal to zero).
To satisfy these conditions,
step 1: we assume there is only an amount of charge $-q$ on the inner surface, and the potential of the sphere is zero;
step 2: we distribute an amount of charge $+q$ uniformly on the outer surface.
At step 1, we search a system of image charges whose potential, summed up with the point charge potential, would yield a zero potential on the internal surface of the shell. Inside the shell the electric field and the potential will be those produce by q and the image charge.
Fig. 6 illustrates such an image charge. In order to determine the position d' (i.e., distance from centre of the shell) and charge amount q' of the image charge, one solves the potential at point a and b as zero. Thus $d' = R^2/d$, $q' = -qR/d$.
Inside the shell, the electric field is:

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\boldsymbol{r}} - \frac{qR}{4\pi\varepsilon_0 d r^2}\hat{\boldsymbol{r'}} \tag{12}$$

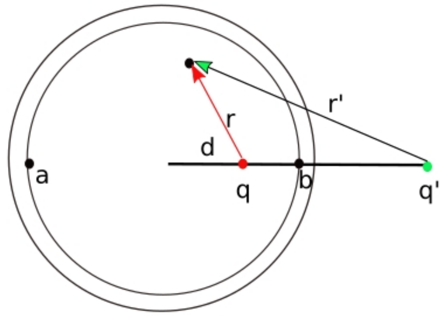

Figure 6: An image charge q' is applied to give a zero-potential shell with the point charge q.

Outside the shell, the electric field is produced by the charge q uniformly distributed on the outer surface (i.e., those of a point charge locate in the centre of the shell).