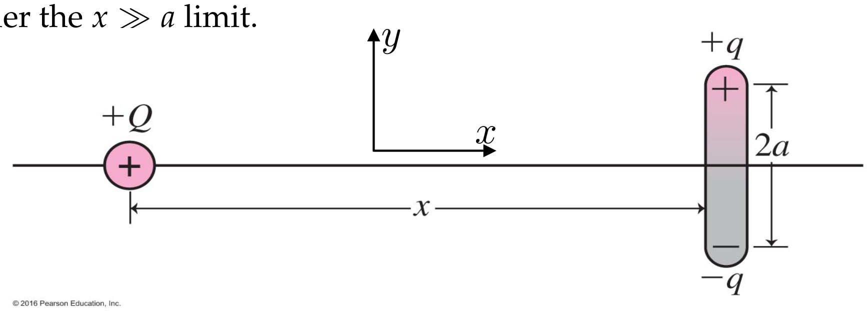
Practice Problem Set 3

#1. *A dipole and a charge*

A dipole with charges $\pm q$ and separation $2a$ is located a distance x from a point charge Q, oriented as shown in Figure 20.32 of the textbook (reproduced here for convenience). For both a) and b), consider the $x \gg a$ limit.



a) What is the net torque (magnitude and direction) on the dipole?

We use the formula derived in the textbook exressing the torque on a dipole in terms of its electric dipole moment and the electric field at its position:

$$\vec{\tau} = \vec{p} \times \vec{E}. \tag{1}$$

In terms of the coordinates shown on the figure below, the field generated by the point charge at the location of the dipole is

$$\vec{E} = \frac{kQ}{x^2}\,\vec{i}. \tag{2}$$

The dipole moment, on the other hand, is

$$\vec{p} = q(2a)\,\vec{j}, \tag{3}$$

so the torque is given by

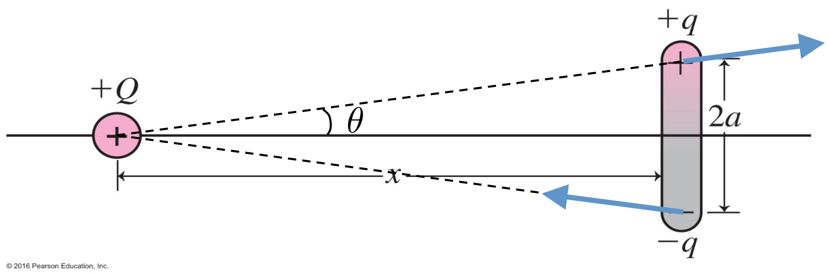
$$\begin{aligned}\vec{\tau} &= 2aq\,\vec{j} \times \frac{kQ}{x^2}\,\vec{i} \\ &= \frac{2kaqQ}{x^2}\,(-\vec{k}),\end{aligned} \tag{4}$$

where $\vec{k} = \vec{i} \times \vec{j}$ points out of the page. This indicates that the torque tends to rotate the dipole in the clockwise direction, in accord with our intuition.

b) What is the net force (magnitude and direction) on the dipole?

We begin by not yet assuming $x \gg a$, and treating each consituent of the dipole one at a time. Each constituent of the dipole is subjected to an electric force from the monopole's field, as shown in the adjacent figure. Though the net force on the dipole in the x direction cancels exactly, it does not in the y direction. Instead, it is given by

$$F_y = 2\,\frac{kqQ}{x^2 + a^2}\,\sin(\theta). \tag{5}$$



But since

$$\sin(\theta) = \frac{a}{\sqrt{x^2 + a^2}}, \tag{6}$$

we arrive at

$$F_y = \frac{kQ(2aq)}{(x^2 + a^2)^{3/2}}. \tag{7}$$

We can now invoke the $x \gg a$ limit. In the present case, the leading-order behaviour (i.e. the most significant contribution) is given by the first term in the small-u expansion $(1+u)^\alpha \approx 1 + \alpha u$, so we only retain that term:

$$\left(x^2 + a^2\right)^{3/2} = x^3 \left(1 + \frac{a^2}{x^2}\right)^{3/2} \approx x^3 \left(1 + \underbrace{\frac{3}{2}\frac{a^2}{x^2}}_{\text{higher-order term}}\right) \approx x^3. \tag{8}$$

This can be compared to #2 c), in which both terms in the expansion must be retained, because the first is cancelled by some other term.

Hence, we find

$$\vec{F} = \frac{kQ(2aq)}{x^3}\,\vec{j}. \tag{9}$$

c) How could you have you have predicted it based on other results from the chapter?

This result could have been forseen using Newton's third law (which electromagnetism obeys). Indeed, we know the field generated by a dipole ($\vec{E} = -k\vec{p}/r^3$ on the perpendicular bisector – covered in the textbook), and we know that the monopole experiences a force $\vec{F} = Q\vec{E}$. According to Newton's third law, the dipole must feel and equal and opposite force due to the monopole.

#2. *A Charged Cross, or Practice With Integrals*

a) Consider a thin rod of length $2a$ placed along the x axis and centered at the origin (see Figure 1). Considering the rod has a uniformly distributed charge q (where $q > 0$), find the electric field at the point P at a distance x from the origin on the positive side of the x axis (assume

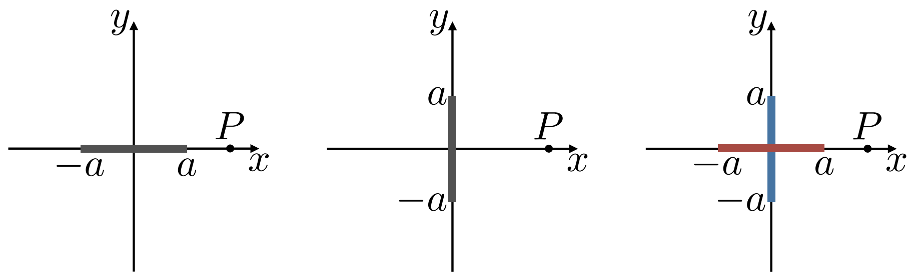


Figure 1 Figure 2 Figure 3

$x > a$). Does your result agree with your expectations in the $x \gg a$ limit?

By symmetry, the field at point P is directed along the x axis, and clearly points away from the charge distribution. Furthermore, in this case, there are no sine or cosine factors to include, because the entire charge distribution is along this axis.

We set up the integral:

$$E_x(P) = k \int \frac{dq}{r^2}, \tag{10}$$

where r is the distance bewteen P and the infinitesimal charge dq. We let $dq = \lambda dx'$ (where $\lambda = q/2a$ is the line charge density), in which case $r = x - x'$, and the integration ranges from $-a$ to a:

$$\begin{aligned} E_x(P) &= k\lambda \int_{-a}^{a} \frac{dx'}{(x'-x)^2} \\ &= k\lambda \left(\frac{-1}{x'-x} \right) \Bigg|_{-a}^{a} \\ &= k\lambda \left(\frac{1}{x-a} - \frac{1}{x+a} \right) \\ &= k\lambda \frac{2a}{x^2 - a^2} \\ &= \frac{kq}{x^2 - a^2}, \end{aligned} \tag{11}$$

where we have let $2a\lambda = q$. In the $x \gg a$ limit, we may let $x^2 - a^2 \approx x^2$ (which, as in #1 b), amounts to only keeping the leading-order term in the Taylor expansion), in which case the field reduces to that of a point charge q located at the origin, as expected.

b) Consider a thin rod, also of length $2a$, placed along the y axis and centered at the origin (see Figure 2). The rod has a uniformly distributed charge $-q$ (where $q > 0$). Find the electric field at the point P at a distance x from the origin on the positive side of the x axis. Does your result agree with your expectations in the $x \gg a$ limit?

Again, by symmetry, the field at point P is directed along the x axis, though now there is a trigonometric factor to pick out the x component of the field generated at point P.

The integral is as follows:

$$E_x(P) = k \int \frac{dq}{r^2} \cos(\theta). \tag{12}$$

This time, the integral is over $y\prime$ between $-a$ and a, the line charge density is $\lambda = -q/2a$, and r is given by Pythagoras' theorem. Furthermore, $\cos(\theta)$ must be expressed in terms of the integration variable:

$$\cos(\theta) = \frac{x}{\sqrt{y'^2 + x^2}}. \tag{13}$$

The integral hence becomes

$$\begin{aligned} E_x(P) &= k\lambda \int_{-a}^{a} \frac{dy'}{(y'^2 + x^2)} \frac{x}{\sqrt{y'^2 + x^2}} \\ &= k\lambda x \int_{-a}^{a} \frac{dy'}{(y'^2 + x^2)^{3/2}}. \end{aligned} \tag{14}$$

The integral is evaluated either via trigonometric substitution or consultation of an integral table:

$$\begin{aligned} E_x(P) &= \frac{k\lambda}{x} \left(\frac{y'}{\sqrt{y'^2 + x^2}} \right) \Bigg|_{-a}^{a} \\ &= \frac{k\lambda}{x} \left(\frac{a}{\sqrt{a^2 + x^2}} - \frac{-a}{\sqrt{a^2 + x^2}} \right) \\ &= \frac{k\lambda}{x} \frac{2a}{\sqrt{a^2 + x^2}} \\ &= \frac{k(-q)}{x\sqrt{a^2 + x^2}}, \end{aligned} \tag{15}$$

where in the last step we let $2a\lambda = -q$. In the $x \gg a$ limit, once again, we only retain the leading-order term in the Taylor expansion:

$$\sqrt{x^2 + a^2} = x\sqrt{1 + \frac{a^2}{x^2}} \approx x \left(1 + \underbrace{\frac{1}{2}\frac{a^2}{x^2}}_{\text{higher-order term}} \right) \approx x, \tag{16}$$

in which case the field reduces to that of a point charge $-q$ located at the origin, as expected.

c) Now, suppose these two metal rods are positioned as shown in Figure 3 (assume no charge is exchanged), with the negatively charged rod placed horizontally. What is the field at the point P at a distance x from the origin, on the positive side of the x axis?

Try taking the $x \gg a$ limit. In this limit, with what power of x does the electric field decay? Is this behaviour reminiscent of a specific charge distribution?

We can use the principle of superposition to combine the answers from parts a) and b) and find the electric field at P in this situation:

$$\begin{aligned} E_x(P) &= \frac{kq}{x^2-a^2} + \frac{k(-q)}{x\sqrt{a^2+x^2}} \\ &= kq\left(\frac{1}{x^2-a^2} - \frac{1}{x\sqrt{a^2+x^2}}\right) \\ &= kq\left(\frac{x\sqrt{a^2+x^2}-(x^2-a^2)}{(x^2-a^2)x\sqrt{a^2+x^2}}\right) \\ &= \frac{kq}{(x^2-a^2)x\sqrt{a^2+x^2}}\left(x^2\sqrt{\frac{a^2}{x^2}+1}-x^2+a^2\right). \end{aligned} \tag{17}$$

Now we consider the $x \gg a$ limit. *In the denominator*, as for the cases above, keeping leading-order behaviour amounts to simply letting $x^2 - a^2 \approx x^2$, and $\sqrt{a^2+x^2} \approx x$.

However, care must be taken *in the parentheses to the right*, as this is an $\infty - \infty$ situation. We use the approximation $(1+u)^\alpha \approx 1 + \alpha u$, valid for small u:

$$\begin{aligned} E_x(P) &= \frac{kq}{x^2x\sqrt{x^2}}\left(x^2\left(1+\frac{1}{2}\frac{a^2}{x^2}\right)-x^2+a^2\right) \\ &= \frac{kq}{x^4}\left(x^2+\frac{a^2}{2}-x^2+a^2\right) \\ &= \frac{kq}{x^4}\frac{3}{2}a^2. \end{aligned} \tag{18}$$

Note how the larger term from the expansion, x^2, disappears in a cancellation, so now the *leading-order term* comes (in part) from the second term in the expansion.

We see that in this limit, the field decays as x^{-4} – this is reminiscent of a quadrupole, as you will discover in this week's assignment, and indeed, squinting a little makes the cross in Figure 3 resemble a quadrupole[1].

[1]Disclaimer: squinting is generally not a good way to do physics.