Approximations Module Student Guide

Approximations are often used in physics, and they can almost always be explained in some way by the **Taylor Series**. In today's activities you will explore some of the most common applications of the Taylor Series in physics.

NOTE: Today's activities involve a lot of math, which we recommend that you do **on paper** with a pen or pencil – these are the tools of a mathematician! This means you should be ready with a camera to take many photos of your work and place them in the google slides so you can share and discuss with your partners. To research the easiest way to do this, you may want to do an internet search on 'add photos to google slides with iphone' or 'add photos to google slides with android' and try to find a good way. You may also want to use screen capture on your computer. With a PC you can use the app "Snip & Sketch" to capture part of a screen, then select Ctrl-C to copy, and Ctrl-V to paste it into your Google Slide. With a Mac you can use the app "Screenshot" to capture part of a screen, then select Cmnd-C to copy, and Cmnd-V to paste it into your Google Slide.

Activity 1: What is a Taylor Series? [15 minutes]

According to Wikipedia, the Taylor Series of a function is "an infinite sum of terms that are expressed in terms of the function's derivatives at a single point." If you have a function $f(x)$, you might be able to easily compute its value and its derivatives at a certain point $x = a$. The first derivative is denoted $f'(x)$, the second derivative is $f''(x)$, the third derivative is $f'''(x)$, etc. The Taylor Series is:

$$f(x) = f(a) + f'(a)[x-a] + \frac{f''(a)}{2}[x-a]^2 + \frac{f'''(a)}{3!}[x-a]^3 + \frac{f''''(a)}{4!}[x-a]^4 + \cdots \quad (1)$$

$x - a$ is sometimes called the "expansion parameter". If $x - a$ is small, then you can neglect higher order terms in the series.

If you attend the weekly physics colloquium, you will often hear physicists say "to the first order…" What they mean is that you only keep the first two terms in the Taylor Series, and neglect the squares, cubes, etc of $x - a$. In practice we often set $a = 0$ and then apply the Taylor Series when $x \ll 1$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots \quad (2)$$

A. Consider the function $f(x) = \frac{1}{1-x}$. Write down the first, second and third derivatives of this function. Evaluate $f(x)$ and its first, second and third derivatives when $x = 0$. Use Equation (2) to write down the Taylor Series for $\frac{1}{1-x}$

to the third order. [To make it exact, you can add "+ $\mathcal{O}(x^4)$" to the end, where this curly-O means "order of". This means you are adding assorted terms you don't care about that are proportional to x^n, where $n = 4, 5, 6, 7$, etc.]

B. Consider the function $f(x) = e^x$, where e is Euler's number $e = 2.71828$. As you may know, this function has the property that all of its derivatives are equal to itself! Write town the Taylor series for e^x to the third order.

C. Consider the function $f(x) = \sin(x)$, where the angle x is measured in radians. Write town the Taylor series for $\sin(x)$ to the third order. If you wrote it down to the fourth order, would it be any different?

Activity 2: The Lorentz Factor in Special Relativity [10 minutes]

According to special relativity, when an object is moving, time runs more slowly for that object. In the object's rest-frame, the time interval between two events is called the proper time, and is denoted as τ. In the lab-frame in which the object is moving at a constant velocity, the time interval between the same two events is $t = \gamma\tau$, where γ is a number greater than 1 called the Lorentz Factor. The Lorentz Factor is determined by the speed of the object, v, as measured in the lab frame:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where c is the speed of light $c = 3.00 \times 10^8$ m/s.

A. Define the ratio $\beta = v/c$, and consider a function $f(\beta) = \gamma = (1 - \beta^2)^{-1/2}$. Use a Taylor Series to the second order to write down the Lorentz Factor as a function of terms up to β^2.

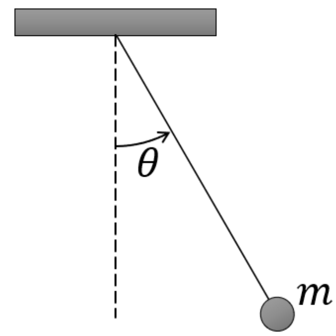
B. You can quantify the relative error you make with this first-order approximation as:

$$\mathcal{E} = \frac{|f_{true} - f_{approx}|}{|f_{true}|} \tag{3}$$

What is the relative error made by using this equation when $\beta = 0.01, 0.1, 0.5$?

C. The maximum speed that a human-built spacecraft has ever attained, relative to the Sun, is 70,220 m/s (Helios-B, in the 1970s). For this spacecraft, $\beta = 2.34 \times 10^{-4}$. Imagine we launch a probe at this speed, and it travels for exactly 1 year in its own frame. So the proper time is $\tau = 3.16 \times 10^7$ s. The time that elapses back on Earth will be $t = \gamma\tau \approx \left(1 + \frac{\beta^2}{2}\right)\tau = \tau + \frac{\beta^2\tau}{2}$, a bit longer than a year. Approximately how much longer than a year will have passed on Earth?

Activity 3: Simple Pendulum [10 minutes]

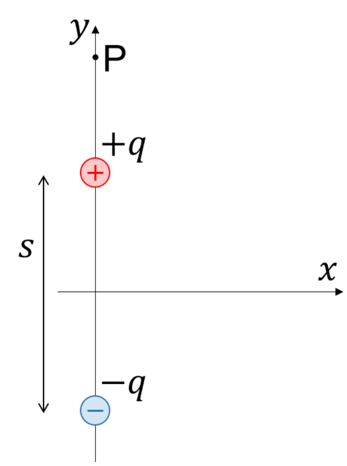


A point mass m is suspended on the bottom end of a light string. The string has a length L and is fixed at its top end. The mass is pulled to the right so that the string makes an angle θ with the downward vertical. For this question please always measure θ in **radians**.

A. What is the torque due to gravity on the mass? Consider the pivot point to be the point where the string is attached at its top end. Define positive torques to be counterclockwise.`
B. Use a Taylor Series to the first order to write down the torque as a linear function of θ.
C. For the gravitational torque on the pendulum, what is the relative error when $\theta = 0.1, 0.5, 1$ radians?

Activity 4: Electric Dipole On-Axis Field [20 minutes]

Recall that the magnitude of the electric field at some point, **P**, due to a single charge q is $E = \frac{Kq}{r^2}$, where r is the distance between **P** and q. The direction of the electric field is away from q if q is positive, or toward q if q is negative.



For this activity, consider the field of an electric dipole: two charges of equal magnitude, q, but opposite sign, are fixed in position on the y-axis, and separated by a distance s. The origin is exactly half-way between them, as shown.

A. What is the exact y-component of the electric field, E_y, of a point at a point **P**, at a position y on the y-axis, where $y > s/2$.
B. Pull out a factor of $1/_{y^2}$ from this equation, and show that:

$$E_y = \frac{Kq}{y^2}\left[\left(1 - \frac{s}{2y}\right)^{-2} - \left(1 + \frac{s}{2y}\right)^{-2}\right]$$

where K is the Coulomb's constant. Define a new variable, $z = s/2y$. If $y \gg s/2$, then $z \ll 1$ is small. Temporarily treat y as a constant, and define the function $f(z) = E_y$. Write down a first order Taylor Series for $f(z)$. Show your work. Neglect terms of second or higher order, and find a simplified approximation for E_y. Compare this with Eq.23.8 in your text. Note that $K = \frac{1}{4\pi\epsilon_0}$.

C. [If you have time] Consider a real dipole for which $q = 10^{-9}$ C, and $s = 0.02$ m. Compare the true and approximate values of E_y in N/C when $y = 0.05$ m, 0.5 m, 5 m. What is the relative error (Eq.3) made by using your approximation equation for these three points?