PHY489 Lecture 12

Fermi's Golden Rule for Transitions

transition probability $W = \frac{2\pi}{\hbar}\left|\mathcal{M}_{if}\right|^2 \rho_f\left(E\right)$

Density of final states available for energy *E.*

matrix element $\left|\mathcal{M}_{if}\right|^2 = \left|\left\langle \psi_f \middle| V_{if} \middle| \psi_i \right\rangle\right|^2$

Matrix element contains the fundamental physics (e.g. the dynamics).

There are cases where $\mathcal{M}_{if}$ = constant, in which case the reaction is entirely governed by the kinematics, described by $\rho_f(E)$. Need this to be Lorentz invariant (we will see that $\mathcal{M}$ is always a Lorentz scalar).

Reminder: $\left\langle \psi_f \middle| H_{\text{int}} \middle| \psi_i \right\rangle$ for $H_{TOT} = H_0 + H_{int}$ with $H_{int} \ll H_0$ treated with perturbation theory, starting with free particle wave-functions (plane-wave solutions). Conventional wave-function normalization is such that

$$\psi = \frac{1}{\sqrt{V}} e^{-i(\vec{p}\cdot\vec{r} - Et)} \qquad \int \psi^* \psi \, dV = 1$$

e.g. normalize to a particle density of *1/V.*

Lorentz Invariant Wave-function Normalization

Consider a particle in an interaction to be at rest w.r.t. a box of volume *V* containing it and only it. Now consider this system viewed in a reference frame $S^{/}$ in which the box is moving with relative velocity *v*. In this frame (in which we want to calculate *W*) the volume *V* is Lorentz contracted by a factor of $\gamma = E/mc^2$ (assuming the particle has mass).

$$V' = \frac{mc^2}{E} V \qquad \int \psi^* \psi \, dV' = \frac{E}{mc^2} \quad \text{(in S}'\text{)}$$

e.g. the particle density has increased by a factor of γ. To ensure a reference-frame independent particle density normalization need to incorporate a factor of $\sqrt{2E}$ in the wave-function normalization:

$$\psi = \sqrt{\frac{2E}{V}} e^{-i(\vec{p}\cdot\vec{r} - Et)}$$

This wave-function normalization is Lorentz invariant.

Density of States

The state of a single-particle with momentum between 0 and p, confined to a volume *V*, is specified by a point in *6D* phase space (x,y,z,p_x,p_y,p_z)

The extent to which spatial and momentum coordinates along each axis can be simultaneously specified is limited by the uncertainty principle, to *h*, so the "elemental volume" of phase space is h^3.

The number of states available to a single particle is thus

$$N = \frac{\text{total phase space available}}{\text{elemental volume}} = \frac{1}{(2\pi\hbar)^3}\int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = \frac{V}{(2\pi\hbar)^3}\int d^3\vec{p}$$

For a system of *n* particles, the number of available final states is the product of the individual factors for each particle:

$$N_n = \left[\frac{V}{(2\pi\hbar)^3}\right]^n \int \prod_{i=1}^{n} d^3\vec{p}_i$$

Phase space factor defined as the number of states / unit energy / unit volume

$$\rho(E) = \frac{dN_n}{dE} = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\vec{p}_i$$

[ignore factors of *V* for now]

Note that not all momenta are independent (conservation of momentum). Could integrate over *n-1* momenta, but simpler to use Dirac δ-function to impose the constraint:

Total three momentum of final state

$$\vec{P} - \sum_{i=1}^{n} \vec{p}_i = 0 \qquad \Rightarrow \qquad \int d^3\vec{p}\,\delta\left(\vec{P} - \sum_{i=1}^{n} \vec{p}_i\right) = 1$$

$$\rho(E) = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\vec{p}_i \delta\left(\vec{P} - \sum \vec{p}_i\right)$$

Energy conservation gives us a further constraint $\left(\sum_{i=1}^{n} E_i\right) - E = 0$

and thus $\int dE\,\delta\left(\sum_{i=1}^{n} E_i - E\right) = 1$ so that we can write

$$\rho(E) = \frac{1}{(2\pi\hbar)^{3n}} \frac{d}{dE} \int \prod_{i=1}^{n} d^3\vec{p}_i \, dE\,\delta\left(\vec{P} - \sum_{i=1}^{n} \vec{p}_i\right) \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

Using the relation $\frac{d}{dE}\int f(E)\,dE = f(E)$ this becomes:

$$\rho(E) = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3\vec{p}_i \, \delta\left(\vec{P} - \sum_{i=1}^{n} \vec{p}_i\right) \delta\left(\sum_{i=1}^{n} E_i - E\right)$$

so each particle has a phase space factor (before integration) of $\frac{d^3\vec{p}}{(2\pi\hbar)^3}$

Unfortunately, this is NOT Lorentz invariant. Note that the factors of *V* (which we've been ignoring) will cancel with factors from the wave-function normalization in the matrix element.

Lorentz Invariance of Phase Space Factor

For Lorentz invariance now need additional factors of *2E* from the Lorentz invariant wave-function normalization. The phase space factor of the form $\dfrac{d^3\vec{p}_i}{E}$ is Lorentz invariant. Show this:

$$p'_x = \gamma(p_x - \beta \frac{E}{c}) \qquad \frac{E'}{c} = \gamma(\frac{E}{c} - \beta p_x) \quad p'_y = p_y \quad p'_z = p_z$$

$$\frac{dp'_x}{dp_x} = \frac{d}{dp_x}\left(\gamma(p_x - \beta\frac{E}{c})\right) = \gamma\left(1 - \frac{\beta}{c}\frac{dE}{dp_x}\right)$$

$$\frac{dE}{dp_x} = \frac{d}{dp_x}\left(p^2c^2 + m^2c^4\right)^{1/2} = \left(2p_xc^2\right)\left(\frac{1}{2}\right)\left(p^2c^2 + m^2c^4\right)^{-1/2} = \frac{p_x}{E}c^2$$

$$\frac{dp'_x}{dp_x} = \gamma\left(1 - \frac{\beta}{c}\frac{p_x}{E}c^2\right) = \gamma\left(1 - \beta\frac{p_x}{E}c\right) = \gamma\left(1 - \beta\frac{p_x}{\left(E/c\right)}\right) = \frac{\gamma\left(\left(E/c\right) - \beta p_x\right)}{\left(E/c\right)} = \frac{\left(E'/c\right)}{\left(E/c\right)} = \frac{E'}{E}$$

$$\frac{dp'_x}{dp_x} = \frac{E'}{E} \quad \Rightarrow \quad \frac{dp'_x}{E'} = \frac{dp_x}{E} \quad \text{or} \quad \boxed{\frac{d^3\vec{p}'}{E'} = \frac{d^3\vec{p}}{E}}$$

Golden Rule for Decays

Consider decays of the form 1➔ 2+3+ ……..+ N

$$d\Gamma = \left|\mathcal{M}\right|^2 \frac{S}{2\hbar m_1}\left[\left(\frac{cd^3\vec{p}_2}{(2\pi)^3 2E_2}\right)\left(\frac{cd^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\cdots\cdots\left(\frac{cd^3\vec{p}_n}{(2\pi)^3 2E_n}\right)\right]\times(2\pi)^4\delta^4\left(p_1 - p_2 - \ldots\ldots - p_n\right)$$

(note that some the factors of $\hbar$ have vanished…..)

δ-function ensures overall energy and momentum conservation. This is the differential decay rate for the case in which the momentum of final state particle *i* lies in the range $d^3\vec{p}_i$ about $\vec{p}_i$ for all *i*.

Need to integrate over all outgoing momenta. Decaying particle is at rest. For the two-body decay 1➔ 2+3

$$\Gamma = \frac{S}{\hbar m_1}\left(\frac{c}{4\pi}\right)^2 \frac{1}{2}\int \frac{\left|\mathcal{M}\right|^2}{E_2 E_3}\delta^4\left(p_1 - p_2 - p_3\right) d^3\vec{p}_2 d^3\vec{p}_3$$

In general the amplitude $\mathcal{M}$ can depend on both $\vec{p}_2$ and $\vec{p}_3$, so it cannot be pulled out of the integration. For two body decays, the integration can be done.

Two Body Decay to Massless Particles

Consider the decay of a massive ^ spinless particle (mass m_1=m) into a final state consisting of two massless particles (m_2=m_3=0) so $E_2 = |\vec{p}_2|c$ and $E_3 = |\vec{p}_3|c$.

Start with assumption that $\mathcal{M} = \mathcal{M}(\vec{p}_2, \vec{p}_3)$ and find Γ:

It should be clear from the start that $\vec{p}_2 = -\vec{p}_3$ so that what we really have here is $\mathcal{M} = \mathcal{M}(\vec{p}_2)$. One can also note that in the absence of spin, there is no direction against which to measure components of $\vec{p}_2$, so expect that $\mathcal{M} = \mathcal{M}(|\vec{p}_2|)$.

Rewrite the δ-function in terms of $E_1 = mc^2,\ \vec{p}_1 = 0,\ E_2 = |\vec{p}_2|c,\ E_3 = |\vec{p}_3|c$

$$\delta^4\left(p_1 - p_2 - p_3\right) = \delta\left(mc - \frac{E_2}{c} - \frac{E_3}{c}\right)\delta^3\left(-\vec{p}_2 - \vec{p}_3\right)$$

$$\Gamma = \frac{S}{\hbar m}\left(\frac{1}{4\pi}\right)^2 \frac{1}{2}\int \frac{|\mathcal{M}|^2}{|\vec{p}_2||\vec{p}_3|}\delta\left(mc - |\vec{p}_2| - |\vec{p}_3|\right)\delta^3\left(-\vec{p}_2 - \vec{p}_3\right) d^3\vec{p}_2 d^3\vec{p}_3$$

Now integrate over $\vec{p}_3$ using $\delta^3\left(-\vec{p}_2 - \vec{p}_3\right)$

$$\Gamma = \frac{S}{2(4\pi)^2 \hbar m} \int \frac{|\mathcal{M}|^2}{|\vec{p}_2|^2} \delta\left(mc - 2|\vec{p}_2|\right) d^3\vec{p}_2 \qquad \mathcal{M} = \mathcal{M}(|\vec{p}_2|).$$

N.B. $\quad d^3\vec{p}_2 = |\vec{p}_2|^2 \, d|\vec{p}_2| \sin\theta d\theta d\phi \qquad \int \sin\theta d\theta d\phi = 4\pi \qquad \delta(kx) = \frac{1}{|k|}\delta(x)$

$$\Gamma = \frac{S}{8\pi\hbar m} \int_0^\infty |\mathcal{M}|^2 \frac{1}{2} \delta\left(|\vec{p}_2| - \frac{mc}{2}\right) d|\vec{p}_2| \qquad \Rightarrow \qquad \Gamma = \frac{S}{16\pi\hbar m} |\mathcal{M}|^2$$

where $\mathcal{M}$ is evaluated at $|\vec{p}_2| = \dfrac{mc}{2}$ $\quad \left[E_2 = E_3 = \dfrac{mc^2}{2} \Rightarrow |\vec{p}_2| = |\vec{p}_3| = \dfrac{mc}{2} \right]$

We have not discussed *S* yet, but it is a statistical factor related to the identity of the final state particles. If they are different, then *S*=1; if they are identical, then *S*=1/2 (we will see why later).

See §6.2.1.1 for the general case with arbitrary m_2, m_3.

$$\Gamma = \frac{S\left|\vec{p}\right|}{8\pi\hbar m_1^2 c}\left|\mathcal{M}\right|^2$$

Where $\mathcal{M}$ is again evaluated at the momentum dictated by conservation of energy and momentum (here $\vec{p}$).

N.B. for $m_2 = m_3 = 0$ (as before)

$$\left|\vec{p}\right| = \frac{m_1 c}{2} \quad \Rightarrow \quad \Gamma = \frac{S m_1 c}{16\pi\hbar m_1^2 c}\left|\mathcal{M}\right|^2 \quad \Rightarrow \quad \Gamma = \frac{S}{16\pi\hbar m_1}\left|\mathcal{M}\right|^2$$

as we just saw.

So for 1 ➔ 2 + 3, we have for the general case that $\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m_1^2 c}|\mathcal{M}|^2$

Where $\vec{p}$ is the momentum dictated by the conservation of energy and momentum and the matrix element $\mathcal{M}$ is evaluated at that momentum.

Note the "phase space factor" $|\vec{p}|$. This is (for the case of two body decays) the origin of the "rule of thumb" stating that, for a given system, the decay with the largest energy release will proceed more rapidly (*e.g.* has a larger width Γ).

But again, this does not typically trump dynamical effects. For instance, we have seen that $\Gamma(D^0 \to K^-\pi^+) > \Gamma(D^0 \to \pi^-\pi^+)$ because the effect of the CKM matrix elements dominates:

$$\frac{\Gamma(D^0 \to \pi^-\pi^+)}{\Gamma(D^0 \to K^-\pi^+)} = \frac{|V_{cd}|^2}{|V_{cs}|^2}\left(\frac{p_{\pi^-}}{p_{K^-}}\right) \approx \lambda^2\left(\frac{922}{861}\right)$$

e.g. kinematic enhancement factor is 1.07 but dynamical suppression factor is λ^2 ~ .05.

Golden Rule for Scattering

For the process 1 + 2 ➔ 3 + 4 + …. + N

$$d\sigma = \left|\mathcal{M}\right|^2 \frac{\hbar^2 S}{4\sqrt{\left(p_1\cdot p_2\right)^2-\left(m_1 m_2 c^2\right)^2}}\left[\left(\frac{c d^3\vec{p}_3}{(2\pi)^3 2E_3}\right)\left(\frac{c d^3\vec{p}_4}{(2\pi)^3 2E_4}\right)\cdots\cdots\left(\frac{c d^3\vec{p}_n}{(2\pi)^3 2E_n}\right)\right]\times(2\pi)^4\delta^4\left(p_1+p_2-\ldots\ldots-p_n\right)$$

Kinematic term in front of phase space factor now contains p_1 and p_2 . [Was the mass of the decaying particle in the case of the GR for decays.] Phase space factors look the same.

This describes the differential cross-section for the process in which the momentum of the *i*th final state particle is in the range $d^3\vec{p}_i$ about $\vec{p}_i$ for all *i*.

Typically, one might measure only a single quantity, such as the angle at which particle 3 emerges from the scattering. Need to integrate over everything else.

We will only deal with the case of two body scattering 1 + 2 ➔ 3 + 4. Consider this process in the CM frame where

$$\vec{p}_1 = -\vec{p}_2 \qquad p_1\cdot p_2 = \frac{E_1 E_2}{c^2} + \left|\vec{p}_1\right|^2$$

Two Body Scattering in the CM frame

One part of problem 6.7 (may be on your next assignment) asks you to show that, for 2-body scattering in the CM frame,

$$\sqrt{\left(p_1 \cdot p_2\right)^2 - \left(m_1 m_2 c^2\right)^2} = \left(E_1 + E_2\right)\left|\vec{p}_1\right| / c$$

In this case we have

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S\left|\mathcal{M}\right|^2 c}{(E_1 + E_2)\left|\vec{p}_1\right|} \frac{d^3\vec{p}_3}{E_3} \frac{d^3\vec{p}_4}{E_4} \times \delta^4\left(p_1 + p_2 - p_3 - p_4\right)$$

$$\frac{1}{4}\left(\frac{1}{\left(2\pi\right)^3} \cdot \frac{1}{2}\right)\left(\frac{1}{\left(2\pi\right)^3} \cdot \frac{1}{2}\right)\left(2\pi\right)^4 = \frac{1}{16}\frac{\left(2\pi\right)^4}{\left(2\pi\right)^6} = \frac{1}{64\pi^2} = \frac{1}{\left(8\pi\right)^2}$$

Rewrite the δ-function using $\vec{p}_1 = -\vec{p}_2$

This will enforce $\vec{p}_3 = -\vec{p}_4$ in the final state.

$$\delta^4\left(p_1 + p_2 - p_3 - p_4\right) = \delta\left(\frac{E_1 + E_2 - E_3 - E_4}{c}\right)\delta^3\left(-\vec{p}_3 - \vec{p}_4\right)$$

Use $E_3 = c\sqrt{m_3^2c^2 + |\vec{p}_3|^2}$ and $E_4 = c\sqrt{m_4^2c^2 + |\vec{p}_4|^2}$ and integrate over $d^3\vec{p}_4$.

$$d\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2 c}{(E_1+E_2)|\vec{p}_1|} \times \frac{\delta\left(\frac{E_1+E_2}{c} - \sqrt{m_3^2c^2 + |\vec{p}_3|^2} - \sqrt{m_4^2c^2 + |\vec{p}_3|^2}\right)}{\sqrt{m_3^2c^2 + |\vec{p}_3|^2}\sqrt{m_4^2c^2 + |\vec{p}_3|^2}} d^3\vec{p}_3$$

At this stage when we discussed decays we concluded that $\mathcal{M} = \mathcal{M}(|\vec{p}_3|)$ but in this case we can have dependence also on the direction of $\vec{p}_3$ since the trajectory of the incoming particles defines an axis against which we can measure this (*i.e.* instead of having only one scalar $|\vec{p}_3|$ we also have the scalar $\vec{p}_1 \cdot \vec{p}_3$). So we cannot perform the angular integration (at least not over θ), so defining $d^3\vec{p}_3 = \rho^2 d\rho d\Omega \quad \rho \equiv |\vec{p}_3|$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1+E_2)|\vec{p}_1|} \int_0^\infty |\mathcal{M}|^2 \frac{\delta\left(\frac{E_1+E_2}{c} - \sqrt{m_3^2c^2 + \rho^2} - \sqrt{m_4^2c^2 + \rho^2}\right)}{\boxed{\sqrt{m_3^2c^2 + \rho^2}\sqrt{m_4^2c^2 + \rho^2}}} \boxed{\rho^2 d\rho}$$

To do the integration, use $E = c\left[\left(m_3^2c^2+\rho^2\right)^{1/2}+\left(m_4^2c^2+\rho^2\right)^{1/2}\right]$

(this is the total energy of the final state or the initial state)

$$dE = c\left\{\frac{1}{2}\left(m_3^2c^2+\rho^2\right)^{-1/2}(2\rho)+\frac{1}{2}\left(m_4^2c^2+\rho^2\right)^{-1/2}(2\rho)+\right\}d\rho$$

$$dE = c\rho\left\{\frac{1}{\left(m_3^2c^2+\rho^2\right)^{1/2}}+\frac{1}{\left(m_4^2c^2+\rho^2\right)^{1/2}}\right\}d\rho = c\rho\left\{\frac{\left(m_3^2c^2+\rho^2\right)^{1/2}+\left(m_4^2c^2+\rho^2\right)^{1/2}}{\left(m_3^2c^2+\rho^2\right)^{1/2}\left(m_4^2c^2+\rho^2\right)^{1/2}}\right\}d\rho$$

$$dE = \frac{E\rho}{\left(m_3^2c^2+\rho^2\right)^{1/2}\left(m_4^2c^2+\rho^2\right)^{1/2}}d\rho \quad\Rightarrow\quad \frac{dE}{E} = \boxed{\frac{\rho\, d\rho}{\left(m_3^2c^2+\rho^2\right)^{1/2}\left(m_4^2c^2+\rho^2\right)^{1/2}}}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2\frac{Sc}{(E_1+E_2)\left|\vec{p}_1\right|}\underbrace{\int_{(m_3+m_4)c^2}^{\infty}}_{\substack{\text{p from 0 to }\infty\\ \text{implies E from}\\ (m_3+m_4)c^2\text{ to }\infty}}\left|\mathcal{M}\right|^2\frac{\rho}{E}\underbrace{\delta\left(\frac{E_1+E_2}{c}-\frac{E}{c}\right)}_{\substack{\text{Overall conservation of energy}\\ =c\delta\left(E_1+E_2-E\right)}}dE$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc^2}{(E_1+E_2)|\vec{p}_1|} |\mathcal{M}|^2 \left(\frac{\rho_0}{E_1+E_2}\right) \qquad \rho_0 \equiv \left\{ \begin{array}{l} \text{Magnitude of final state momentum} \\ \text{dictated by conservation of energy and} \\ \text{momentum} \end{array} \right.$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$$

where $\vec{p}_i$ and $\vec{p}_f$ are the initial and final state momenta (*e.g.* $\vec{p}_1$ and $\vec{p}_3$).

Now look at Griffiths Problem 6.8: Elastic scattering of the form A+B➔A+B in the lab frame. This is like 1+2➔3+4 with $m_1=m_3=m_A$, $m_2=m_4=m_B$ *(with particle 2 initially at rest)*

Consider the case where $m_Bc^2 >> E_A$ so that we can neglect the recoil momentum of particle B.

The general case is problem 6.10 which may be on your next assignment.

$$d\sigma = \left|\mathcal{M}\right|^2 \frac{\hbar^2 S}{4\sqrt{\left(p_1 \cdot p_2\right)^2 - \left(m_1 m_2 c^2\right)^2}} \left[\left(\frac{c d^3\vec{p}_3}{\left(2\pi\right)^3 2E_3}\right)\left(\frac{c d^3\vec{p}_4}{\left(2\pi\right)^3 2E_4}\right)\right] \times \left(2\pi\right)^4 \delta^4\left(p_1 - p_2 - p_3 - p_4\right)$$

Note that this is Lorentz invariant. Now choose a particular reference frame:

Evaluate $\sqrt{\left(p_1 \cdot p_2\right)^2 - \left(m_1 m_2 c^2\right)^2}$ in the lab frame:

This may be on your next assignment……..

$$\Rightarrow \boxed{\sqrt{\left(p_1 \cdot p_2\right)^2 - \left(m_1 m_2 c\right)^2} = m_2 c\left|\vec{p}_1\right|}$$

$$d\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\left|\mathcal{M}\right|^2}{m_B c\left|\vec{p}_1\right|}\left(\frac{d^3\vec{p}_3}{E_3}\right)\left(\frac{d^3\vec{p}_4}{E_4}\right) \times \delta^4\left(p_1 + p_2 - p_3 - p_4\right)$$

Rewrite the δ-function ………

$$\delta^4\left(p_1+p_2-p_3-p_4\right)=\delta\left(\frac{E_1}{c}+m_Bc-\frac{E_3}{c}-m_Bc\right)\delta^3\left(\vec{p}_1+\vec{0}-\vec{p}_3-\vec{p}_4\right)=c\delta\left(E_1-E_3\right)\delta^3\left(\vec{p}_1-\vec{p}_3-\vec{p}_4\right)$$

Integrate over $d^3\vec{p}_4$. In this case the integration doesn't do anything since we are ignoring $\vec{p}_4$ but the $\delta^3\left(\vec{p}_1-p_3-p_4\right)$ disappears with the integration.

$$d\sigma=\left(\frac{\hbar c}{8\pi}\right)^2\frac{|\mathcal{M}|^2}{m_Bc\left|\vec{p}_1\right|}\left(\frac{d^3\vec{p}_3}{E_3}\right)\left(\frac{1}{E_4}\right)\times c\delta\left(E_1-E_3\right)$$

$= m_Bc^2$ in the limit in which we are working

Now integrate over $d^3\vec{p}_3$: $\quad d^3\vec{p}_3=\left|\vec{p}_3\right|^2 d\left|\vec{p}_3\right|d\Omega$ [again, can't do the angular part]

$$\frac{d\sigma}{d\Omega}=\left(\frac{\hbar c}{8\pi}\right)^2\frac{1}{m_B^2c^2\left|\vec{p}_1\right|}\int_0^\infty|\mathcal{M}|^2\frac{\left|\vec{p}_3\right|^2 d\left|\vec{p}_3\right|}{E_3}\times\delta\left(E_1-E_3\right)$$

Write this in terms of E_3 so that we can integrate using the δ-function:

$$E_3^2=\left|\vec{p}_3\right|^2c^2+m_3^2c^4 \qquad 2E_3dE_3=2\left|\vec{p}_3\right|c^2d\left|\vec{p}_3\right| \qquad \Rightarrow \qquad d\left|\vec{p}_3\right|=\frac{E_3dE_3}{\left|\vec{p}_3\right|c^2} \quad \left|\vec{p}_3\right|^2=\frac{E_3^2}{c^2}-m_3^2c^2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{m_B^2 c^2 \left|\vec{p}_1\right|} \int_{m_A c^2}^{\infty} \left|\mathcal{M}\right|^2 \frac{\left(E_3^2 - m_3^2 c^4\right)}{c^2 E_3} \frac{E_3 dE_3}{c\left(E_3^2 - m_3^2 c^4\right)^{1/2}} \times \delta\left(E_1 - E_3\right)$$

$$= \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{m_B^2 c^2} \frac{\left(E_1^2 - m_A^2 c^4\right)^{1/2}}{\left|\vec{p}_1\right| c^3} \left|\mathcal{M}\right|^2$$

$$= \left(\frac{\hbar}{8\pi m_B c}\right)^2 \left|\mathcal{M}\right|^2$$

since $\dfrac{\left(E_1^2 - m_A^2 c^4\right)^{1/2}}{\left|\vec{p}_1\right| c} = \dfrac{\sqrt{\left|\vec{p}_1\right|^2 c^2}}{\left|\vec{p}_1\right| c} = 1$

As before $|\mathcal{M}|^2$ must be evaluated at the value of the final state momentum that is dictated by the conservation of energy and momentum,