PHY489 Lecture 13

Feynman Rules for Fundamental Processes

So far we have learned about:

- ✓ particle content
- ✓ interactions (allowed vertices)
- ✓ conservation laws
- ✓ relativistic kinematics
- ✓ Fermi's golden rule for scattering and decays
- ✓ Two-body decay rates, differential scattering cross-sections
  - ➢ Lorentz invariant form
  - ➢ CM, lab (fixed target) reference frames.

A comment on reference frames…..

Note that these are just the two most typical experimental scenarios, but there are also others. HERA, for instance, was an ep collider which collided 30 GeV electrons (or positrons) with 820 GeV protons. This clearly produces collisions which are NOT in the CM frame or in a frame in which either of the initial state particles is at rest. Other examples include the B-factories which produced collisions of the form $e^{+}e^{-} \to \Upsilon(4s) \to B\bar{B}$ but did so using unequal beam energies such that $\sqrt{s} = M_{\Upsilon(4s)}$ but with a final state that is not at rest (e.g. the final state particles are Lorentz boosted in the direction of the more energetic beam). These experiments are designed to make precision measurements of CP violation in the b-quark system, which rely on the ability to resolve the decay vertices of the B mesons. This is easier if they are boosted (and thus have longer decay lengths in the lab frame).

HERA experiments: ZEUS, H1

B-factory experiments: Belle, Babar

Note that $p\bar{p}$ or pp collisions are also not typically in the CM frame, or rather, the fundamental parton-parton collision is not typically in it’s CM frame since the two interacting partons in general carry different fractions of the total proton momentum.

Calculating Amplitudes

So far, we have done only the *kinematics* associated with two body scattering or decays.

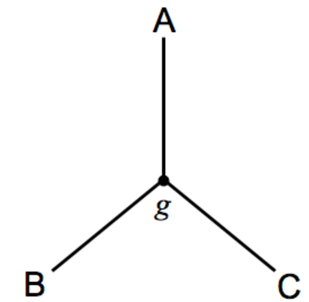
The *amplitude* $\mathcal{M}$ is calculated using the "Feynman rules" of the theory, which assign mathematical factors to elements of the Feynman diagrams, such as external lines, internal lines, vertices, etc.

These rules come from the Lagrangian of the theory (which displays the full particle content of the theory and the allowed interactions).

$$L_{QED} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \not{D} \equiv \gamma^{\mu} D_{\mu} \qquad D_{\mu} = \partial_{\mu} + iA_{\mu}$$

We will take the Feynman rules as given, and learn how to apply them.

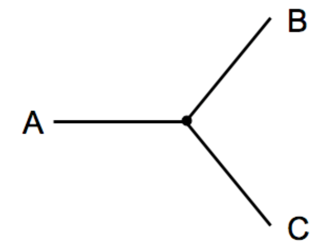
Following the text, to avoid the complications due to spin, look first at a toy model with three scalar (e.g. spin 0) particles A, B and C which are their own antiparticles and which interact via a single fundamental vertex coupling them all together (strength g):



The ABC Model

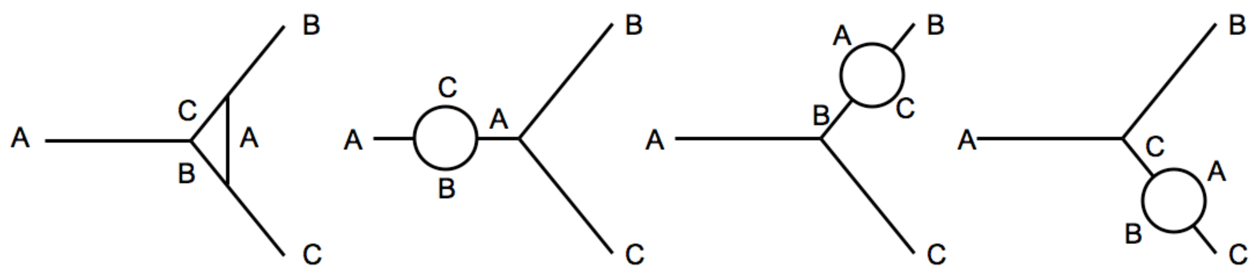
The vertex we just wrote down assumes no particular direction of time.

If we assume that $M_A > M_B + M_C$ then the lowest order process in the theory is the decay of A:



[Don't need arrows on lines here since $A = \bar{A}$ $B = \bar{B}$ $C = \bar{C}$]

There are also higher-order contributions:



Note that if you try to draw other versions of the vertex correction diagram, you end up with the same one

You should be able to draw the diagrams for higher-order corrections, but we will not calculate them in this course. They are, however, very important, so please read §6.3.3 (we will not cover this in class).

The Feynman Rules for the ABC Model

1. Notation: Label the incoming and outgoing four-momenta $p_1, p_2, ...p_n$. Label the internal momenta $q_1, q_2, \ldots$.. Put an arrow on each line to keep track of the direction of "positive" momentum flow (this is arbitrary for internal line, but you need to keep track of it).
2. Coupling constant (vertex factor): For each vertex write a factor of $-ig$, where g is the coupling constant specifying the strength of the interaction.
3. Propagator: For each internal line write a factor of

$$\frac{i}{q_j^2 - m_j^2 c^2}$$

   where q_j is the four-momentum of the (j^{th}) internal line and m_j is the mass of the particle that this line represents (note that $q_j^2 \neq m_j^2 c^2$ for a virtual particle)

4. Conservation of energy and momentum: for each vertex write a factor of $(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$ where the k's are the four-momenta coming *into* the vertex (so use a minus sign for any that flow *out* of the vertex).
5. Integration over internal momenta: for each internal line write a factor of $\frac{d^4 q_j}{(2\pi)^4}$ and integrate over all internal momenta.
6. Cancel the remaining delta function: step 5 will leave an expression that includes a delta function expressing overall conservation of energy and momentum. Canceling this factor leaves you with $-i\mathcal{M}$.

What is the lifetime of A ?

The first (and simplest) question we can address is: what is the lifetime of A?

We have done the kinematics already: $\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m_A^2 c}|\mathcal{M}|^2$

Here $S = 1$ (since B and C are distinct) and $\vec{p}$ is determined by the masses of the three particles. (We have also done the kinematics for this).

Applying the Feynman rules to the calculation of $\mathcal{M}$ in this case, trivially yields the results $\mathcal{M} = g$.

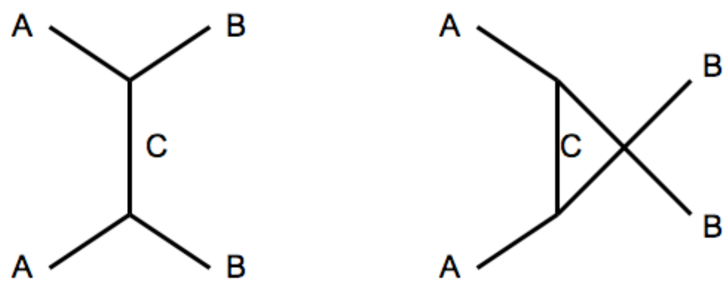
We know that $\tau_A = \frac{1}{\Gamma_A}$ $\tau_A = \frac{8\pi\hbar m_A^2 c}{g^2|\vec{p}|}$. Units are $\frac{[MeV \cdot s][MeV^2/c^4]c}{g^2[MeV/c]} = \frac{MeV^2/c^2}{g^2}s$

Which is OK since (for this toy model) g has units of MeV/c.

Lifetime larger for lower mass difference M_A-M_B-M_C (less phase space leading to smaller $|\vec{p}|$) and longer for weaker coupling g (as $g \rightarrow 0$, $\tau \rightarrow \infty$).

Two-body scattering

Consider the case of two-body scattering (we've also done the kinematics for this already). Look at $AA \rightarrow BB$. Diagrams at lowest-order are:



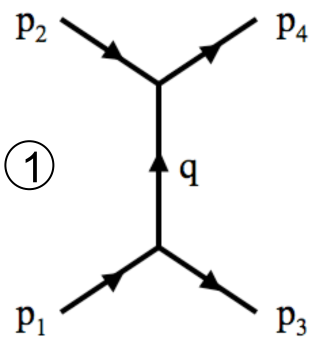
Here the second diagram is necessary since one cannot tell which outgoing particle connects to which vertex (so we need to consider both possibilities). There will be a factor of 1/2 in the expression for the differential cross-section coming from the statistical factor S, to account for this, *e.g.* in the CM frame:

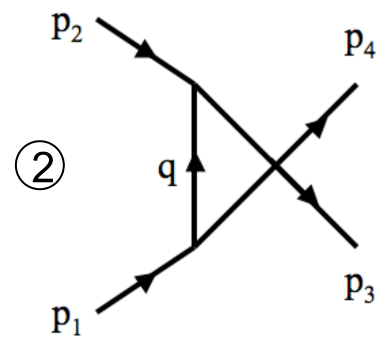
$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\left|\mathcal{M}\right|^2}{(E_1+E_2)^2} \frac{\left|\vec{p}_f\right|}{\left|\vec{p}_i\right|}$$

Let's calculate the amplitude for this…..

Amplitude for AA➔BB

Step 1: draw and label the contributing Feynman diagrams:





Note that the arrows are for momentum flow only.

Look first at $\mathcal{M}_1$ (go through the remaining steps)

$$\int \underset{\uparrow}{\left(-ig\right)^2} \underbrace{\left(\frac{i}{q^2 - m_C^2 c^2}\right)}_{\text{propagator}} \underbrace{\left(2\pi\right)^4 \delta^4\left(p_1 - q - p_3\right)}_{\delta\text{-function for 1st vertex}} \underbrace{\left(2\pi\right)^4 \delta^4\left(p_2 + q - p_4\right)}_{\delta\text{-function for 2nd vertex}} \frac{d^4 q}{\left(2\pi\right)^4}$$

two vertex factors

integrate over internal momenta

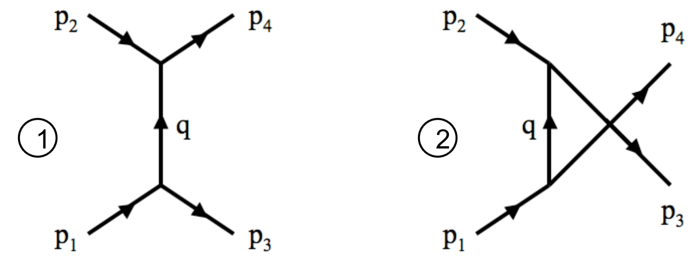
step: 2 3 4 4 5

$$= -\frac{ig^2}{\left(p_1 - p_3\right)^2 - m_C^2 c^2} \cancel{\left(2\pi\right)^4 \delta^4\left(p_1 + p_2 - p_3 - p_4\right)} \qquad \Rightarrow -i\mathcal{M}_1$$

$$\boxed{\mathcal{M}_1 = \frac{g^2}{\left(p_1 - p_3\right)^2 - m_C^2 c^2}}$$

Amplitude for AA➔BB

Now look at $\mathcal{M}_2$



Calculation of this amplitude is exactly the same, except for the p_3 , p_4 swap which means the integration picks out $q = p_1 - p_4$. This means that

$$\mathcal{M}_2 = \frac{g^2}{(p_1 - p_4)^2 - m_C^2 c^2}$$

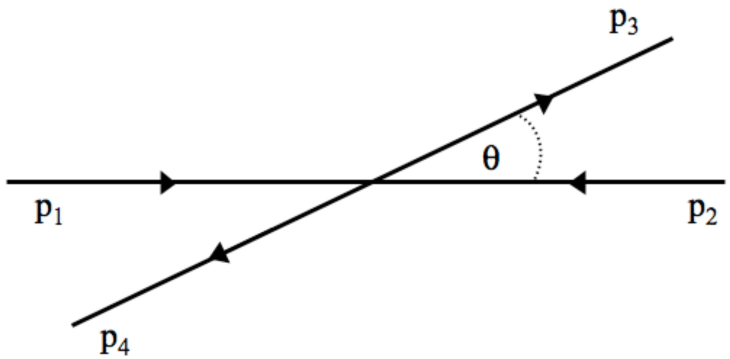
and thus

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \left\{ \frac{g^2}{(p_1 - p_3)^2 - m_C^2 c^2} + \frac{g^2}{(p_1 - p_4)^2 - m_C^2 c^2} \right\}$$

Note that this amplitude is frame-independent. Lorentz invariance of $\mathcal{M}$ is enforced by the Feynman rules.

If we want (for example) $d\sigma/d\Omega$ in the CM frame, need to specify the four vectors in that reference frame and work through the resulting amplitude.

dσ/dΩ for AA➔BB in CM frame



$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \left\{ \frac{g^2}{(p_1 - p_3)^2 - m_C^2 c^2} + \frac{g^2}{(p_1 - p_4)^2 - m_C^2 c^2} \right\}$$

Consider the case for which $m_A = m_B = m,\ m_C = 0$:

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = \left\{ \frac{g^2}{(p_1 - p_3)^2} + \frac{g^2}{(p_1 - p_4)^2} \right\}$$

$$(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3$$

$$= \frac{E^2}{c^2} - |\vec{p}|^2 + \frac{E^2}{c^2} - |\vec{p}|^2 - 2\left(\frac{E}{c}, \vec{p}_1\right) \cdot \left(\frac{E}{c}, \vec{p}_3\right)$$

$$= 2\cancel{\frac{E^2}{c^2}} - 2|\vec{p}|^2 - 2\cancel{\left(\frac{E}{c}\right)^2} - 2\vec{p}_1 \cdot \vec{p}_3 = -2|\vec{p}|^2 - 2|\vec{p}|^2 \cos\theta = -2|\vec{p}|^2 (1 + \cos\theta)$$

Similarly, we have:

$$(p_1 - p_4)^2 = p_1^2 + p_4^2 - 2p_1 \cdot p_4 = -2|\vec{p}|^2 (1 - \cos\theta)$$

dσ/dΩ AA➔BB in CM frame

$$\mathcal{M} = \frac{g^2}{-2|\vec{p}|^2(1-\cos\theta)} + \frac{g^2}{-2|\vec{p}|^2(1+\cos\theta)} = \frac{g^2(1+\cos\theta)+g^2(1-\cos\theta)}{-2|\vec{p}|^2(1-\cos\theta)(1+\cos\theta)} = \frac{-2g^2}{2|\vec{p}|^2(1-\cos^2\theta)}$$

$$\boxed{\mathcal{M} = \frac{-g^2}{|\vec{p}|^2\sin^2\theta}}$$

two identical particles in final state: S = 1/2! = 1/2

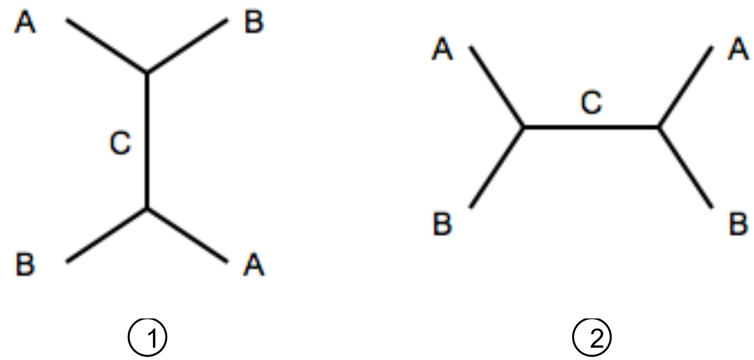
$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1+E_2)^2} \boxed{\frac{|\vec{p}_f|}{|\vec{p}_i|}} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\left(\frac{1}{2}\right)|\mathcal{M}|^2}{4E^2} = \left(\frac{\hbar c}{8\pi}\right)^2\left(\frac{1}{2}\right)\frac{1}{4E^2}\frac{g^4}{|\vec{p}|^4\sin^4\theta} = \frac{1}{2}\left(\frac{\hbar c g^2}{16E|\vec{p}|^2\sin^2\theta}\right)^2$$

initial and final state particle have same masses so momenta are the same and this factor is 1.

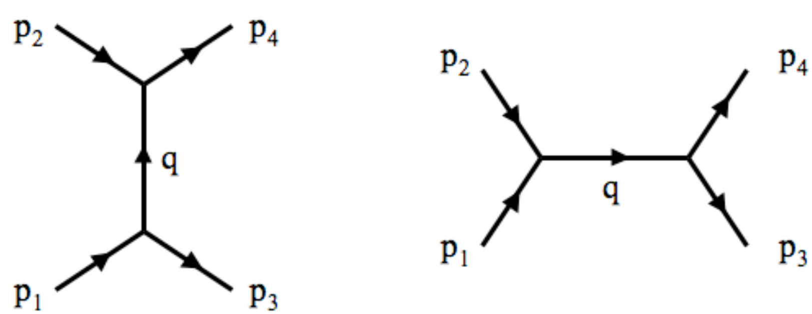
Initial state particles have same mass and momentum, so same energy E.

AB ➔ AB Scattering

Now consider AB➔AB scattering: there are again two diagrams at lowest order:



Labeling these for the amplitude calculation we have



For $\mathcal{M}_1$ we write down $\displaystyle\int (-ig)^2 \left(\frac{i}{q^2 - m_C^2 c^2}\right)(2\pi)^4 \delta^4(p_1 - q - p_3)(2\pi)^4 \delta^4(p_2 + q - p_4)\frac{d^4 q}{(2\pi)^4}$

$$\int (-ig)^2 \left(\frac{i}{q^2 - m_C^2 c^2} \right) (2\pi)^4 \delta^4(p_1 - q - p_3)(2\pi)^4 \delta^4(p_2 + q - p_4) \frac{d^4 q}{(2\pi)^4}$$

$$\Rightarrow \quad \frac{-ig^2}{(p_1 - p_3)^2 - m_C^2 c^2} \cancel{(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)} = -i\mathcal{M}_1 \quad \Rightarrow \quad \mathcal{M}_1 = \frac{g^2}{(p_1 - p_3)^2 - m_C^2 c^2}$$

As in the previous exercise, we can get $\mathcal{M}_2$ from $\mathcal{M}_1$ simply by substituting for the proper form of q (= $p_1 + p_2$ for diagram 2).

$$\mathcal{M}_2 = \frac{g^2}{(p_1 + p_2)^2 - m_C^2 c^2} \qquad \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \quad \Rightarrow \quad \mathcal{M} = g^2 \left\{ \frac{1}{(p_1 - p_3)^2 - m_C^2 c^2} + \frac{1}{(p_1 + p_2)^2 - m_C^2 c^2} \right\}$$

Again, this is a Lorentz invariant form for the AB➔ AB scattering amplitude.

AB ➔ AB Scattering in the Lab Frame

Find the differential cross-section AB ➔ AB in the lab frame, assuming $m_B >> m_A$ and B remains stationary (*e.g.* we ignore any recoil momentum).

We've done the kinematics for this already: $$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_B c}\right)^2 |\mathcal{M}|^2$$

Assume $m_B >> m_A, m_C, E/c^2$, *e.g.* for incoming particle A of energy E.

$$(p_1 - p_3)^2 = \underset{\cancel{m_A^2 c^2}}{p_1^2} + \underset{m_B^2 c^2}{p_3^2} - 2\underset{\cancel{\frac{E^2}{c^2} - \vec{p}_1 \cdot \vec{p}_3}}{p_1 \cdot p_3} \approx m_B^2 c^2$$

$$(p_1 + p_2)^2 = \underset{\cancel{m_A^2 c^2}}{p_1^2} + \underset{m_B^2 c^2}{p_2^2} + 2\underset{\cancel{\frac{E^2}{c^2} - \vec{p}_1 \cdot \vec{p}_2}}{p_1 \cdot p_2} \approx m_B^2 c^2$$

$$\mathcal{M} = g^2 \left\{ \frac{1}{(p_1 - p_3)^2 - m_C^2 c^2} + \frac{1}{(p_1 + p_2)^2 - m_C^2 c^2} \right\} \Rightarrow \mathcal{M} \approx g^2 \left\{ \frac{1}{m_B^2 c^2} + \frac{1}{m_B^2 c^2} \right\} = \frac{2g^2}{m_B^2 c^2}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi m_B c}\right)^2 \left(\frac{2g^2}{m_B^2 c^2}\right)^2 = \left(\frac{\hbar g^2}{4\pi m_B^3 c^3}\right)^2$$

$$\sigma = 4\pi \left(\frac{\hbar g^2}{4\pi m_B^3 c^3}\right)^2 = \frac{1}{4\pi}\left(\frac{\hbar g^2}{m_B^3 c^3}\right)^2$$