PHY489 Lecture 15

Solutions to the Dirac Equation

We had $\left(i\hbar\gamma^{\mu}\partial_{\mu} - mc\right)\psi = 0$:

here the γ matrices are 4x4 and ψ is a 4-component Dirac spinor $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

Still need to discuss the form of the spinor ψ:

Consider first the case in which ψ is stationary (independent of position)

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial z} = 0$$

e.g. a state with $\vec{p} = 0$ since $p_{\mu} \to i\hbar\partial_{\mu} \equiv i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

In this case, the Dirac equation reduces to $\frac{i\hbar}{c}\gamma^{0}\frac{\partial\psi}{\partial t} - mc\psi = 0$ which we can write as:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \partial\psi_A/\partial t \\ \partial\psi_B/\partial t \end{pmatrix} = -i\frac{mc^2}{\hbar}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad \text{with} \quad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \partial\psi_A / \partial t \\ \partial\psi_B / \partial t \end{pmatrix} = -i\frac{mc^2}{\hbar} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \quad \text{with} \quad \psi_A = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \psi_B = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$$

four components

two components each

$$\frac{\partial\psi_A}{\partial t} = -i\frac{mc^2}{\hbar}\psi_A \qquad -\frac{\partial\psi_B}{\partial t} = -i\frac{mc^2}{\hbar}\psi_B$$

These equations have straightforward solutions:

$$\psi_A(t) = e^{-i\left(mc^2/\hbar\right)t}\psi_A(0) \qquad \psi_B(t) = e^{+i\left(mc^2/\hbar\right)t}\psi_B(0)$$

Basic quantum mechanics tells us that characteristic time evolution of a quantum state of energy E is given by $e^{-iEt/\hbar}$: Here $E=mc^2$ for the time dependence of $\psi_{A.}$ However, for ψ_B we have

$$\psi_B(t) = e^{+i\left(mc^2/\hbar\right)t}\psi_B(0)$$

Interpret these “negative energy” solutions as being positive energy solutions for anti-particles.

So, ψ_A describes electrons and ψ_B describes positrons: each has two spin states:

$$\psi_1 = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$ electron with spin up

$$\psi_2 = e^{-i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$ electron with spin down

$$\psi_3 = e^{+i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$ positron with spin up

$$\psi_4 = e^{+i\frac{mc^2}{\hbar}t}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$ positron with spin down

Now, consider general state (non-stationary): plane wave solutions of form

$$\psi(\vec{r},t) = \underset{\downarrow}{a}\, e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}u(E,\vec{p})$$

normalization factor: ignore for now

$$\psi(\vec{r},t) = ae^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}u(E,\vec{p}) \quad \Rightarrow \quad \psi(x) = ae^{-\frac{i}{\hbar}(x\cdot p)}u(p)$$ [in 4v notation]

$\partial_\mu \psi = -\frac{i}{\hbar} p_\mu a e^{-\frac{i}{\hbar}(x\cdot p)} u(p)$ so the Dirac equation $(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0$ becomes

$\gamma^\mu p_\mu a e^{-\frac{i}{\hbar}(x\cdot p)} u - mcae^{-\frac{i}{\hbar}(x\cdot p)} u = 0$ or, more compactly, $\boxed{(\gamma^\mu p_\mu - mc)u = 0}$.

This is referred to as the Dirac equation in momentum space. Note that this is purely algebraic (i.e. it contains no derivatives). If u satisfies this equation then ψ satisfies the Dirac equation.

Expand this and then write in 2x2 matrix form: $\gamma^\mu p_\mu = \gamma^0 p_0 - \vec{\gamma}\cdot\vec{p}$

$$= \frac{E}{c}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \vec{p}\cdot\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} E/c & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E/c \end{pmatrix} \qquad \left[\gamma^i = \begin{pmatrix} 0 & \vec{\sigma}_i \\ -\vec{\sigma}_i & 0 \end{pmatrix} \quad i = 1,2,3\right]$$

$$\left(\gamma^{\mu} p_{\mu} - mc\right) u = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{E}{c} - mc & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -\frac{E}{c} - mc \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

$$\Rightarrow \quad \begin{pmatrix} \left(\frac{E}{c} - mc\right) u_A - \left(\vec{p}\cdot\vec{\sigma}\right) u_B \\ \left(\vec{p}\cdot\vec{\sigma}\right) u_A - \left(\frac{E}{c} + mc\right) u_B \end{pmatrix} = 0$$

e.g. two equations in u_A and u_B.

$$\left(\frac{E}{c} - mc\right) u_A = \left(\vec{p}\cdot\vec{\sigma}\right) u_B \quad \Rightarrow \quad u_A = \frac{c}{E - mc^2}\left(\vec{p}\cdot\vec{\sigma}\right) u_B$$

$$\left(\vec{p}\cdot\vec{\sigma}\right) u_A = \left(\frac{E}{c} + mc\right) u_B \quad \Rightarrow \quad u_B = \frac{c}{E + mc^2}\left(\vec{p}\cdot\vec{\sigma}\right) u_A$$

$$u_A = \frac{c}{E - mc^2}\left(\vec{p}\cdot\vec{\sigma}\right)\frac{c}{E + mc^2}\left(\vec{p}\cdot\vec{\sigma}\right) u_A = \frac{c^2}{E^2 - m^2c^4}\left(\vec{p}\cdot\vec{\sigma}\right)^2 u_A$$

$$u_A = \frac{c}{E - mc^2}\left(\vec{p}\cdot\vec{\sigma}\right)\frac{c}{E + mc^2}\left(\vec{p}\cdot\vec{\sigma}\right)u_A = \frac{c^2}{E^2 - m^2c^4}\left(\vec{p}\cdot\vec{\sigma}\right)^2 u_A$$

Look at factor $\left(\vec{p}\cdot\vec{\sigma}\right)$

$$\vec{p}\cdot\vec{\sigma} = p_x\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_y\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + p_z\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$\left(\vec{p}\cdot\vec{\sigma}\right)^2 = \begin{pmatrix} p_z^2 + \left(p_x - ip_y\right)\left(p_x + ip_y\right) & p_z\left(p_x - ip_y\right) - p_z\left(p_x + ip_y\right) \\ p_z\left(p_x + ip_y\right) - p_z\left(p_x - ip_y\right) & \left(p_x + ip_y\right)\left(p_x - ip_y\right) + p_z^2 \end{pmatrix}$$

$$= \begin{pmatrix} p_x^2 + p_y^2 + p_z^2 & 0 \\ 0 & p_x^2 + p_y^2 + p_z^2 \end{pmatrix} = \left|\vec{p}\right|^2 \quad \left(e.g.\ \left|\vec{p}\right|^2\ \mathrm{I}_{2x2}\right)$$

$$u_A = \frac{c^2}{E^2 - m^2c^4}\left|\vec{p}\right|^2 u_A \quad \Rightarrow \quad \text{requires } E^2 = m^2c^4 + \left|\vec{p}\right|^2 c^2 \qquad E^2 = \pm\sqrt{m^2c^4 + \left|\vec{p}\right|^2 c^2}$$

We had $\quad u_A = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})u_B \qquad u_B = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})u_A \qquad \vec{p}\cdot\vec{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$

$$\left.\begin{aligned} u_A &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow u_B = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{E+mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \\ u_A &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow u_B = \frac{c}{E+mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E+mc^2}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \end{aligned}\right\} \quad E^2 = +\sqrt{m^2c^4 + |\vec{p}|^2 c^2}$$

$$\left.\begin{aligned} u_B &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow u_A = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{c}{E-mc^2}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix} \\ u_B &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow u_A = \frac{c}{E-mc^2}(\vec{p}\cdot\vec{\sigma})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{c}{E-mc^2}\begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix} \end{aligned}\right\} \quad E^2 = -\sqrt{m^2c^4 + |\vec{p}|^2 c^2}$$

Need energy as stated to avoid blow-up of denominator at $E=mc^2$.

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{c\left(p_z\right)}{E+mc^2} \\ \dfrac{c\left(p_x+ip_y\right)}{E+mc^2} \end{pmatrix} \qquad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \dfrac{c\left(p_x-ip_y\right)}{E+mc^2} \\ \dfrac{c\left(-p_z\right)}{E+mc^2} \end{pmatrix} \qquad E^2 = +\sqrt{m^2c^4+\left|\vec{p}\right|^2c^2}$$

$$u_3 = N\begin{pmatrix} \dfrac{c\left(p_z\right)}{E-mc^2} \\ \dfrac{c\left(p_x+ip_y\right)}{E-mc^2} \\ 1 \\ 0 \end{pmatrix} \qquad u_4 = N\begin{pmatrix} \dfrac{c\left(p_x-ip_y\right)}{E-mc^2} \\ \dfrac{c\left(-p_z\right)}{E-mc^2} \\ 0 \\ 1 \end{pmatrix} \qquad E^2 = -\sqrt{m^2c^4+\left|\vec{p}\right|^2c^2}$$

Normalization is $u^\dagger u = 2|E|/c$ (recall that we discussed Lorentz invariant wavefunction normalizations a few lectures ago).

$$u_1^\dagger u_1 = N^2 \begin{pmatrix} 1 & 0 & \frac{c(p_z)}{E+mc^2} & \frac{c(p_x - ip_y)}{E+mc^2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x + ip_y)}{E+mc^2} \end{pmatrix} = N^2 \left(1 + \frac{c^2 p_z^2}{(E+mc^2)^2} + \frac{c^2 (p_x^2 + p_y^2)}{(E+mc^2)^2} \right)$$

$$N^2 \left(\frac{(E+mc^2)^2 + c^2 |\vec{p}|^2}{(E+mc^2)^2} \right) = N^2 \left(\frac{E^2 + 2Emc^2 + \overbrace{m^2c^4 + c^2|\vec{p}|^2}^{=E^2}}{(E+mc^2)^2} \right) = N^2 \left(\frac{2E(E+mc^2)}{(E+mc^2)^2} \right) = N^2 \left(\frac{2E}{E+mc^2} \right)$$

$u_1^\dagger u_1 = 2|E|/c$ requires $N = \sqrt{(|E| + mc^2)/c}$ (need $|E|$ for u_3, u_4)

Spin Matrices for Dirac Spinors

$$\vec{S} = \frac{\hbar}{2}\vec{\Sigma} \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

u_1 looks like it might represent a spin-up electron, but it does not. u_1 is NOT an eigenvector of S_z, unless we make a specific choice for the direction of the z axis by choosing it along $\hat{p}$.

In this case, $p_x = p_y = 0$ and all four spinors represent states of definite helicity.

For example:

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{c(p_z)}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix} \Rightarrow N\begin{pmatrix} 1 \\ 0 \\ \frac{c|\vec{p}|}{E+mc^2} \\ 0 \end{pmatrix} \quad \Rightarrow \quad \Sigma_z u_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} N\begin{pmatrix} 1 \\ 0 \\ \frac{c|\vec{p}|}{E+mc^2} \\ 0 \end{pmatrix} = N\begin{pmatrix} 1 \\ 0 \\ \frac{c|\vec{p}|}{E+mc^2} \\ 0 \end{pmatrix}$$

$$\Sigma_z u_1 = N\begin{pmatrix} 1 \\ 0 \\ \dfrac{c|\vec{p}|}{E+mc^2} \\ 0 \end{pmatrix} = \sqrt{\left(|E|+mc^2\right)/c}\begin{pmatrix} 1 \\ 0 \\ \dfrac{c|\vec{p}|}{E+mc^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\left(|E|+mc^2\right)/c} \\ 0 \\ \dfrac{c|\vec{p}|}{E+mc^2}\sqrt{\left(|E|+mc^2\right)/c} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\left(|E|+mc^2\right)/c} \\ 0 \\ \sqrt{\left(|E|-mc^2\right)/c} \\ 0 \end{pmatrix}$$

since

$$\frac{c|\vec{p}|}{E+mc^2}\sqrt{\left(|E|+mc^2\right)/c} = \frac{\sqrt{c^2|\vec{p}|^2}\sqrt{\left(|E|+mc^2\right)/c}}{E+mc^2} = \frac{\sqrt{\left(E^2-m^2c^4\right)}\sqrt{\left(|E|+mc^2\right)/c}}{E+mc^2}$$

$$= \frac{\sqrt{\left(E-mc^2\right)\cancel{\left(E+mc^2\right)}}\sqrt{\cancel{\left(|E|+mc^2\right)}/c}}{\cancel{E+mc^2}} = \sqrt{\left(E-mc^2\right)/c}$$

$\boxed{u_1 \text{ and } u_3 \text{ are spin-up}}$ $\boxed{u_2 \text{ and } u_4 \text{ are spin-down}}$

To represent positron solutions in terms of the physical energy and momentum of the positron, we flip the signs of E and $\vec{p}$ for u_3 and u_4:

$$\psi\left(\vec{r},t\right) = ae^{-\frac{i}{\hbar}\left(Et-\vec{p}\cdot\vec{r}\right)}u\left(-E,-\vec{p}\right)$$

$$u_4 = N\begin{pmatrix} \dfrac{c\left(p_x - ip_y\right)}{E - mc^2} \\ \dfrac{c\left(-p_z\right)}{E - mc^2} \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad u_4\left(-E, -\vec{p}\right) = N\begin{pmatrix} \dfrac{c\left(-p_x + ip_y\right)}{-E - mc^2} \\ \dfrac{c\left(+p_z\right)}{-E - mc^2} \\ 0 \\ 1 \end{pmatrix} = N\begin{pmatrix} \dfrac{c\left(p_x - ip_y\right)}{E + mc^2} \\ \dfrac{c\left(-p_z\right)}{E + mc^2} \\ 0 \\ 1 \end{pmatrix}$$

The convention is to write positron states (spinors) as v and to forget about u_3 and u_4;

$$v_1\left(E,\vec{p}\right)=u_4\left(-E,-\vec{p}\right)=N\begin{pmatrix}\frac{c\left(p_x-ip_y\right)}{E+mc^2}\\ \frac{c\left(-p_z\right)}{E+mc^2}\\ 0\\ 1\end{pmatrix}\qquad v_2\left(E,\vec{p}\right)=u_3\left(-E,-\vec{p}\right)=-N\begin{pmatrix}\frac{c\left(p_z\right)}{E+mc^2}\\ \frac{c\left(p_x+ip_y\right)}{E+mc^2}\\ 1\\ 0\end{pmatrix}$$

where in each case the energy is $E=+\sqrt{m^2c^4+\left|\vec{p}\right|^2c^2}$

u solutions (representing electrons) satisfy $\left(\gamma^\mu p_\mu-mc\right)u=0$

v solutions (representing positrons) satisfy $\left(\gamma^\mu p_\mu+mc\right)v=0$

We will talk more about the properties of these spinors. Most important is that we know what they are, because the Feynman rules we will learn for QED associate a spinor with each external fermion line. Generally we average over spin states, so whether something is spin-up or spin-down is often not so critical.

We will mostly use the spinor in abstract rather than component form, but it is useful to have explicit forms for derivation of their properties.

Quantum Electrodynamics

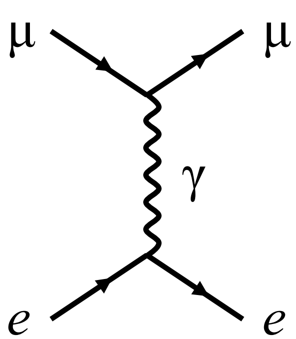
- Follow Griffiths sections 7.5-7.7 + example 7.7 from section 7.8, using the $e^-\mu^- \rightarrow e^-\mu^-$ scattering as an example.

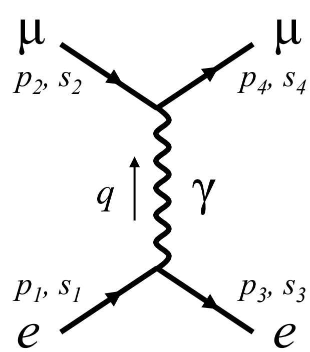
   - Feynman rules for Quantum Electrodynamics
   - $e^-\mu^- \rightarrow e^-\mu^-$ scattering
   - Spin-averaging of amplitudes (next time)
   - Scattering of an electron from a heavy spin-1/2 particle (next time).

Feynman Rules for Quantum Electrodynamics

1. Notation: label incoming and outgoing four-momenta & corresponding spins; label the internal four momenta; assign arrows to lines as follows:
   - Arrows on external lines indicate whether they represent particles or anti-particles (the latter flow backwards in time).
   - Arrows on external photon lines point forwards.
   - Arrows on internal lines are in the direction that preserves the “direction of flow”: each vertex must have at least one arrow entering and one arrow leaving.

EXAMPLE: $e^- \mu^- \rightarrow e^- \mu^-$ scattering (in QED ! No weak interaction contribution)





e.g. at low energies, where the weak interaction contribution is negligible (suppressed by the high mass of the weak gauge bosons) e.g.

$$\frac{1}{q^2} \quad \text{vs.} \quad \frac{1}{q^2 - M^2c^2}$$

for the propagator.

Feynman Rules for QED cont'd

2. External Lines: contribute factors to $\mathcal{M}$ as follows:

- electrons incoming u
  e.g. fermions outgoing $\bar{u} \equiv u^\dagger \gamma^0$
- positrons incoming v
  e.g. anti-fermions outgoing $\bar{v} \equiv v^\dagger \gamma^0$
- photons incoming ε^μ
  outgoing $\varepsilon^{\mu *}$

  polarization vectors: see § 7.4. We will discuss this in an upcoming lecture.

3. Vertices: each vertex contributes a factor of $ig\gamma^\mu$ (photon is spin-1).
   (here my g is Griffiths $g_e \equiv e\sqrt{4\pi / \hbar c} = \sqrt{4\pi\alpha}$).

4. Propagators: each internal line contributes a factor of:

- electrons and positrons $\dfrac{i\left(\gamma^\mu q_\mu + mc\right)}{q^2 - m^2c^2}$ [e.g. internal fermion line]
- photons $-i\dfrac{g_{\mu\nu}}{q^2}$

Feynman Rules for QED cont's

5. Conservation of energy and momentum: for each vertex write a factor

$$(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$$

   enforcing overall energy and momentum conservation at that vertex. (here each k represents a 4 momentum; incoming four momenta are positive and outgoing are negative).

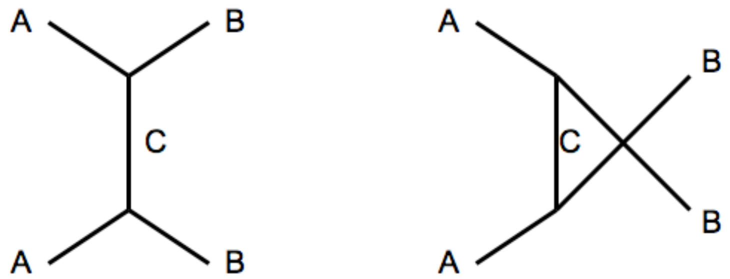
6. Integrate over all internal momenta: for each internal momentum q write a factor of $(2\pi)^{-4} d^4 q$ and integrate.

7. Cancelling the remaining δ-function (expressing overall energy and momentum conservation) leaves you with -$i\mathcal{M}$.

8. Anti-symmetrization: include a relative minus sign between diagrams differing only by the exchange of two incoming (or outgoing) electrons (or positrons) or of and incoming electron with an outgoing positron (or vice versa) [see next slide].

Anti-symmetrization of QED diagrams

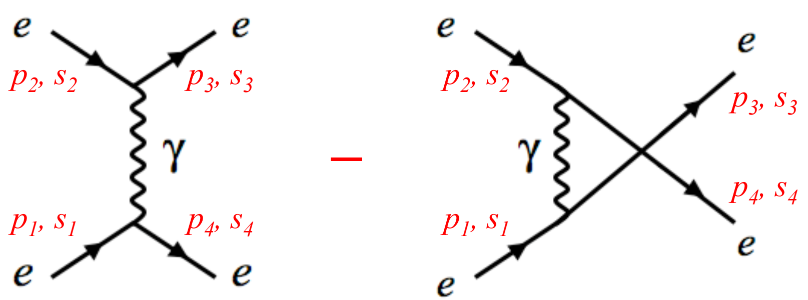
Recall the ABC model scattering process $AA \rightarrow BB$. There are two diagrams that contribute at lowest order:



We summed the amplitudes for these two diagrams to get the total amplitude (so, with a relative positive sign).

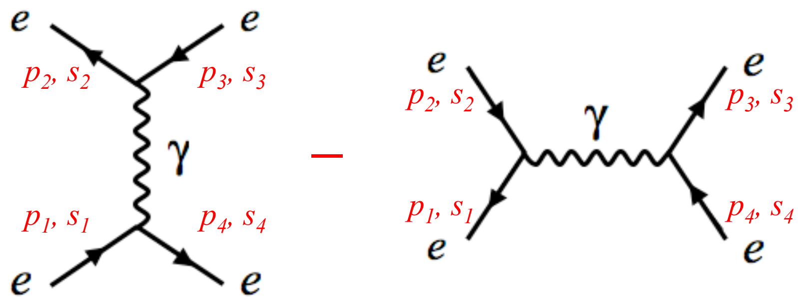
For fermions the relative sign between such diagrams is *negative*.

Here for $e^- e^- \rightarrow e^- e^-$

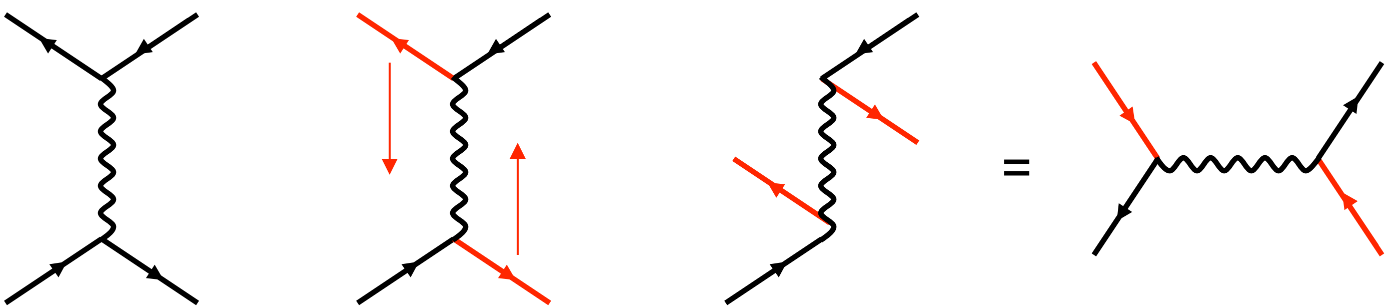


More on anti-symmetrization of QED diagrams

Consider electron positron scattering: $e^+e^- \to e^+e^-$ (Bhabha scattering)



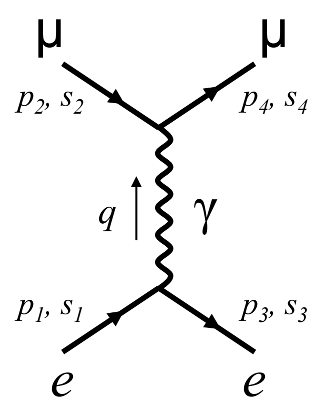
Relative negative sign since diagrams differ by exchange of incoming positron and outgoing electron:



Electron-Muon Scattering in QED

Now back to $e^-\mu^- \rightarrow e^-\mu^-$ scattering in QED:

We had:



Now apply Feynman rules to obtain $\mathcal{M}$.

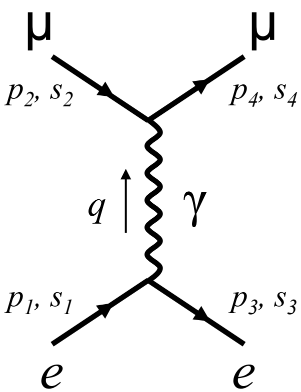
Procedure is to write down terms working backwards in time along each fermion line:

electron line: $$\bar{u}^{(s_3)}(p_3)\ ig\gamma^{\mu}\ u^{(s_1)}(p_1)\ (2\pi)^4\,\delta^4(p_1 - p_3 - q)$$

outgoing electron spinor	vertex coupling	incoming electron spinor	δ-function for conservation of energy and momentum at electron vertex.

muon line: $$\bar{u}^{(s_4)}(p_4)\ ig\gamma^{\nu}\ u^{(s_2)}(p_2)(2\pi)^4\,\delta^4(p_2 + q - p_4)$$

Electron-Muon Scattering cont'd



Propogator is $\frac{-ig_{\mu\nu}}{q^2}$. Applying the Feynman rules we obtain

[spin indices have been suppressed]

$$\int \left[\bar{u}(p_3)\; ig\gamma^{\mu}\; u(p_1)(2\pi)^4\,\delta^4(p_1-p_3-q)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}(p_4)\; ig\gamma^{\nu}\; u(p_2)(2\pi)^4\,\delta^4(p_2+q-p_4)\right]\frac{d^4q}{(2\pi)^4}$$

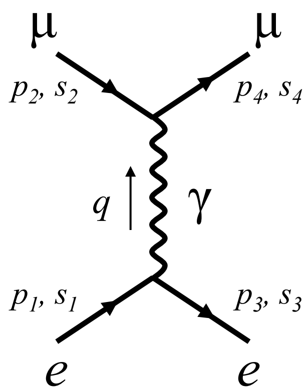
Use first δ-function for integration over d^4q. This leaves:

$$\frac{-ig^2}{(p_1-p_3)^2}\left[\bar{u}(p_3)\gamma^{\mu}u(p_1)\right]\left[\bar{u}(p_4)\gamma_{\mu}u(p_2)\right]\cancel{(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)}$$

Canceling the overall δ-function leaves us with -i$\mathcal{M}$:

$$\mathcal{M}=\frac{g^2}{(p_1-p_3)^2}\left[\bar{u}(p_3)\gamma^{\mu}u(p_1)\right]\left[\bar{u}(p_4)\gamma_{\mu}u(p_2)\right]$$

Electron-Muon Scattering cont'd



$$\mathcal{M} = \frac{g^2}{\left(p_1 - p_3\right)^2}\underbrace{\left[\bar{u}\left(p_3\right)\gamma^{\mu}u\left(p_1\right)\right]}\underbrace{\left[\bar{u}\left(p_4\right)\gamma_{\mu}u\left(p_2\right)\right]}$$

Each of component of each of these factors has the form:

$$\left(\quad 1 \times 4 \quad\right)\left(\quad 4 \times 4 \quad\right)\left(\begin{matrix}4\\ \text{x}\\ 1\end{matrix}\right)$$

This is just a number, so the above expression is just a number (*e.g.* a scalar quantity), which we will learn to calculate.

But first, we need to learn how to deal with spin.