PHY489 Lecture 16

Quantum Electrodynamics

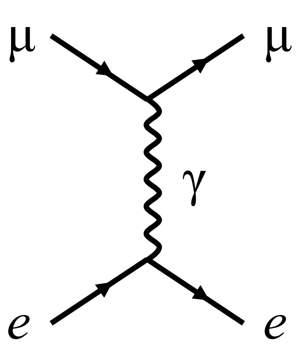
- Follow Griffiths sections 7.5-7.7 + example 7.7 from section 7.8, using the $e^- \mu^- \rightarrow e^- \mu^-$ scattering as an example.

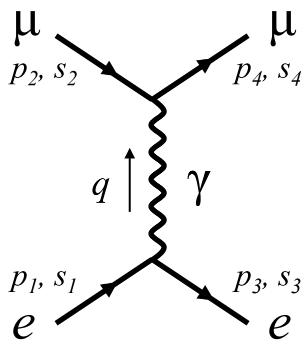
  - Feynman rules for Quantum Electrodynamics
  - $e^- \mu^- \rightarrow e^- \mu^-$ scattering
  - Spin-averaging of amplitudes
  - Scattering of an electron from a heavy spin-1/2 particle

Feynman Rules for Quantum Electrodynamics

1. Notation: label incoming and outgoing four-momenta & corresponding spins; label the internal four momenta; assign arrows to lines as follows:
   - Arrows on external lines indicate whether they represent particles or anti-particles (the latter flow backwards in time).
   - Arrows on external photon lines point forwards.
   - Arrows on internal lines are in the direction that preserves the “direction of flow”: each vertex must have at least one arrow entering and one arrow leaving.

EXAMPLE: $e^- \mu^- \rightarrow e^- \mu^-$ scattering (in QED ! No weak interaction contribution)





so at low energies, where the weak interaction contribution is negligible (suppressed by the high mass of the weak gauge bosons) e.g.

$$\frac{1}{q^2} \quad \text{vs.} \quad \frac{1}{q^2 - M^2c^2}$$

for the propagator.

Think of the μ here as a heavy charged particle with $M >> m_e$.

Feynman Rules for QED cont'd

2. External Lines: contribute factors to $\mathcal{M}$ as follows:

- electrons — incoming — u
  e.g. fermion — outgoing — $\bar{u} \equiv u^\dagger \gamma^0$
- positrons — incoming — v
  e.g. anti-fermion — outgoing — $\bar{v} \equiv v^\dagger \gamma^0$
- photons — incoming — ε^μ
  outgoing — $\varepsilon^{\mu *}$
  polarization vectors: see § 7.4

3. Vertices: each vertex contributes a factor of $ig\gamma^\mu$ (photon is spin-1).
   (here my g is Griffiths $g_e \equiv e\sqrt{4\pi / \hbar c} = \sqrt{4\pi\alpha}$).

3. Propagators: each internal line contributes a factor of:

- electrons and positrons $\dfrac{i\left(\gamma^\mu q_\mu + mc\right)}{q^2 - m^2c^2}$ [e.g. internal fermion line]

- photons $-i\dfrac{g_{\mu\nu}}{q^2}$

Feynman Rules for QED cont's

5. Conservation of energy and momentum: for each vertex write a factor

$$(2\pi)^4 \delta^4 (k_1 + k_2 + k_3)$$

enforcing overall energy and momentum conservation at that vertex.
(here each k represents a 4 momentum; incoming four momenta are positive and outgoing are negative).

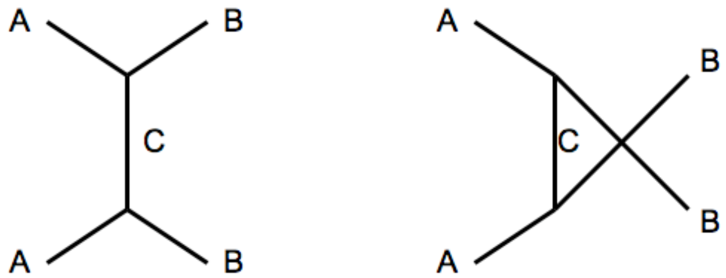
6. Integrate over all internal momenta: for each internal momentum q write a factor of $(2\pi)^{-4} d^4q$ and integrate.

7. Cancelling the remaining δ-function (expressing overall energy and momentum conservation) leaves you with -$i\mathcal{M}$.

8. Anti-symmetrization: include a relative minus sign between diagrams differing only by the exchange of two incoming (or outgoing) electrons (or positrons) or of and incoming electron with an outgoing positron (or vice versa) [see next slide].

Anti-symmetrization of QED diagrams

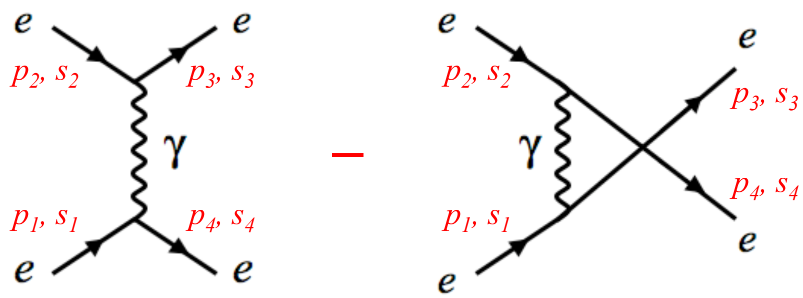
Recall the ABC model scattering process $AA \rightarrow BB$. There are two diagrams that contribute at lowest order:



We summed the amplitudes for these two diagrams to get the total amplitude (so, with a relative positive sign).

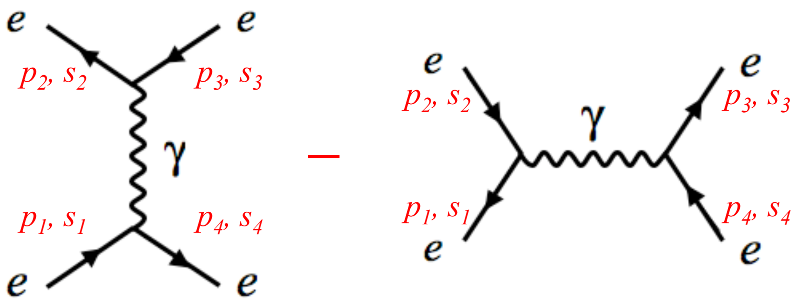
For fermions the relative sign between such diagrams is *negative*.

Here for $e^- e^- \rightarrow e^- e^-$

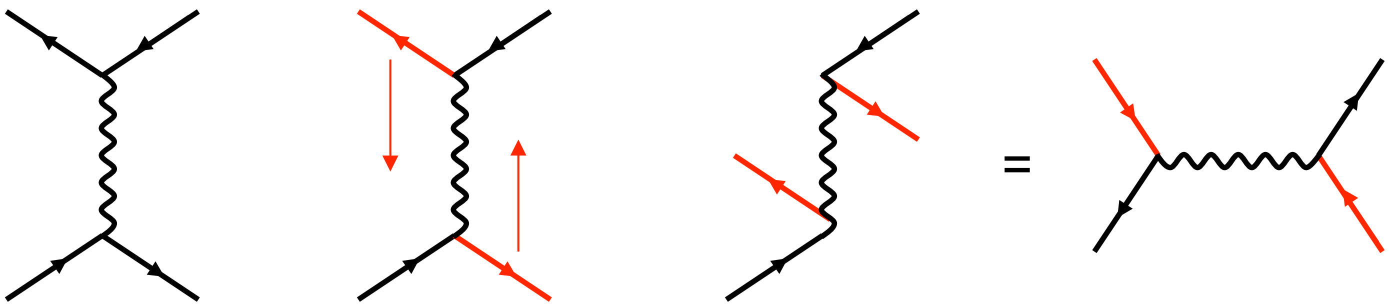


More on anti-symmetrization of QED diagrams

Consider electron positron scattering: $e^+e^- \rightarrow e^+e^-$ (Bhabha scattering)



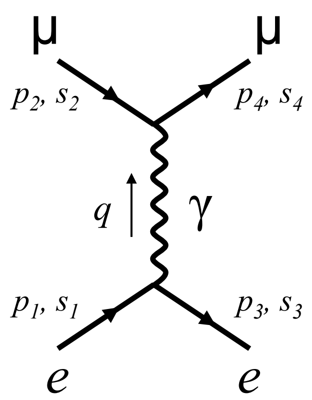
Relative negative sign since diagrams differ by exchange of incoming positron and outgoing electron:



Electron-Muon Scattering in QED

Now back to $e^- \mu^- \rightarrow e^- \mu^-$ scattering in QED:

We had:



Now apply Feynman rules to obtain $\mathcal{M}$.

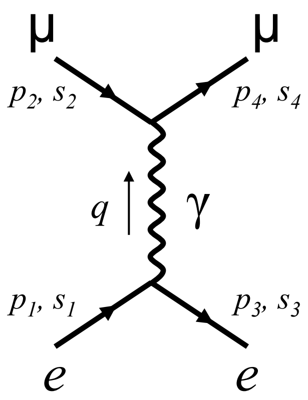
Procedure is to write down terms working backwards in time along each fermion line:

electron line: $\bar{u}^{(s_3)}(p_3)\; ig\gamma^{\mu}\; u^{(s_1)}(p_1)\;(2\pi)^4\,\delta^4(p_1 - p_3 - q)$

outgoing electron spinor	vertex coupling	incoming electron spinor	δ-function for conservation of energy and momentum at electron vertex.

muon line: $\bar{u}^{(s_4)}(p_4)\; ig\gamma^{\nu}\; u^{(s_2)}(p_2)(2\pi)^4\,\delta^4(p_2 + q - p_4)$

Electron-Muon Scattering cont'd



Propogator is $\frac{-ig_{\mu\nu}}{q^2}$. Applying the Feynman rules we obtain

[spin indices have been suppressed]

$$\int \left[\bar{u}(p_3)\ ig\gamma^{\mu}\ u(p_1)(2\pi)^4\delta^4(p_1-p_3-q)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}(p_4)\ ig\gamma^{\nu}\ u(p_2)(2\pi)^4\delta^4(p_2+q-p_4)\right]\frac{d^4q}{(2\pi)^4}$$

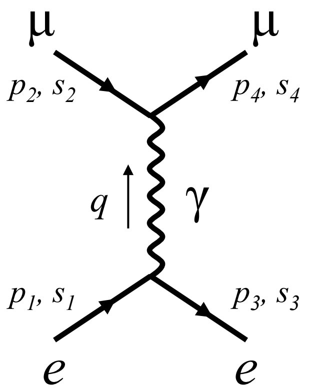
Use first δ-function for integration over d^4q. This leaves:

$$\frac{-ig^2}{(p_1-p_3)^2}\left[\bar{u}(p_3)\gamma^{\mu}u(p_1)\right]\left[\bar{u}(p_4)\gamma_{\mu}u(p_2)\right]\cancel{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)}$$

Canceling the overall δ-function leaves us with -i$\mathcal{M}$:

$$\mathcal{M}=\frac{g^2}{(p_1-p_3)^2}\left[\bar{u}(p_3)\gamma^{\mu}u(p_1)\right]\left[\bar{u}(p_4)\gamma_{\mu}u(p_2)\right]$$

Electron-Muon Scattering cont'd



$$\mathcal{M} = \frac{g^2}{\left(p_1 - p_3\right)^2}\underbrace{\left[\bar{u}\left(p_3\right)\gamma^{\mu}u\left(p_1\right)\right]}\underbrace{\left[\bar{u}\left(p_4\right)\gamma_{\mu}u\left(p_2\right)\right]}$$

Each of component of each of these factors has the form:

$$\begin{pmatrix} 1 \times 4 \end{pmatrix}\begin{pmatrix} 4 \times 4 \end{pmatrix}\begin{pmatrix} 4 \\ \times \\ 1 \end{pmatrix}$$

This is just a number, so the above expression is just a number (*e.g.* a scalar quantity), which we will learn to calculate.

But first, we need to learn how to deal with spin.

Dealing with Spin

If we know the spins of the incoming and outgoing particles, we can write down the appropriate spinors and do the matrix multiplication.

In practice, we often have (experimentally) beams of particles with random spins, and in doing an experiment (for example $e^-\mu^- \to e^-\mu^-$) we might measure only the number of particles scattered at a particular angle.

This means we effectively average over the initial spins and sum over the final spins. Can do this too when we calculate the amplitude. Could compute the amplitude $\mathcal{M}_i$ for each possible configuration, and then sum and average, to get the so-called spin-average amplitude $\left\langle |\mathcal{M}|^2 \right\rangle$. However, we can also calculate this directly.

We had $$\mathcal{M} = \frac{g^2}{(p_1 - p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]$$ [with, *e.g.* $\bar{u}(3) \equiv \bar{u}^{(s_3)}(p_3)$]

$$|\mathcal{M}|^2 = \frac{g^4}{(p_1 - p_3)^4}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(3)\gamma^\nu u(1)\right]^*\left[\bar{u}(4)\gamma_\nu u(2)\right]^*$$

We therefore need to compute quantities of the form $G \equiv \left[\bar{u}(a)\Gamma_1 u(b)\right]\left[\bar{u}(a)\Gamma_2 u(b)\right]^*$ where a, b label the appropriate spins and momenta and Γ_1, Γ_2 are two 4x4 matrices.

Spin-averaging cont'd

We need to evaluate factors like $G \equiv \left[\bar{u}(a)\Gamma_1 u(b)\right]\left[\bar{u}(a)\Gamma_2 u(b)\right]^*$

Note that $\left[\bar{u}(a)\Gamma_2 u(b)\right]$ is just a number (or a 1x1 matrix), so

$$\left[\bar{u}(a)\Gamma_2 u(b)\right]^* = \left[\bar{u}(a)\Gamma_2 u(b)\right]^\dagger = \left[u(a)^\dagger \gamma^0 \Gamma_2 u(b)\right]^\dagger = u(b)^\dagger \Gamma_2^\dagger \gamma^{0\dagger} u(a)$$

$$= u(b)^\dagger \left(\gamma^0\gamma^0\right)\Gamma_2^\dagger \gamma^0 u(a)$$ [since $\gamma^0\gamma^0 = 1,\ \gamma^{0^\dagger} = \gamma^0$]

$$= \bar{u}(b)\gamma^0\Gamma_2^\dagger\gamma^0 u(a) = \bar{u}(b)\bar{\Gamma}_2 u(a)$$ where $\bar{\Gamma}_2 \equiv \gamma^0\Gamma_2^\dagger\gamma^0$

So $G \equiv \left[\bar{u}(a)\Gamma_1 u(b)\right]\left[\bar{u}(a)\Gamma_2 u(b)\right]^* = \left[\bar{u}(a)\Gamma_1 \underbrace{u(b)\right]\left[\bar{u}(b)}\bar{\Gamma}_2 u(a)\right]$

Remember,we have to sum this quantity over all possible spins. Write it in this form to take advantage of the completeness relation for the spinors, when performing this spin sum:

$$\left.\begin{aligned} \sum_{s=1,2} u^{(s)}\bar{u}^{(s)} &= \gamma^\mu p_\mu + mc \\ \sum_{s=1,2} v^{(s)}\bar{v}^{(s)} &= \gamma^\mu p_\mu - mc \end{aligned}\right\}$$

See eqn 7.99 and problem 7.24

Summing G over the spin orientations of particle b we obtain

$$\sum_{b-spins} G = \bar{u}(a)\Gamma_1\left\{\sum_{s_b=1,2} u^{(s_b)}(p_b)\bar{u}^{(s_b)}(p_b)\right\}\bar{\Gamma}_2 u(a)$$

$$= \bar{u}(a)\Gamma_1\left(\gamma^\mu p_{b\mu} + m_b c\right)\bar{\Gamma}_2 u(a)$$

$$= \bar{u}(a)\Gamma_1\left(\not{p}_b + m_b c\right)\bar{\Gamma}_2 u(a) \quad [\text{ where } \quad \not{p}_b \equiv \gamma^\mu p_{b\mu} : \text{ in general } \not{a} \equiv \gamma^\mu a_\mu]$$

$$= \bar{u}(a) Q u(a) \quad [\text{with } Q = \Gamma_1\left(\not{p}_b + m_b c\right)\bar{\Gamma}_2]$$

[Note that Q is still just a 4x4 matrix]

Need to also sum over possible spin states of particle a:

$$\sum_{a-spins}\left(\sum_{b-spins} G\right) = \sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a) Q u^{(s_a)}(p_a)$$

Need to evaluate $\sum_{a-spins}\left(\sum_{b-spins} G\right) = \sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a) Q u^{(s_a)}(p_a)$

Write out the matrix multiplication explicitly:

$$\sum_{i,j=1,4}\sum_{s_a=1,2} \bar{u}^{(s_a)}(p_a)_i Q_{ij} u^{(s_a)}(p_a)_j = \sum_{i,j=1,4} Q_{ij}\left\{\sum_{s_a=1,2} u^{(s_a)}(p_a)\bar{u}^{(s_a)}(p_a)\right\}_{ji}$$

[see footnote on pg 251 of text]

$$\sum_i\sum_j\sum_{s=1,2} \overbrace{\bar{u}_i^{(s)} Q_{ij} u_j^{(s)}}^{\text{each of these 3 quantities is just a number}} \Rightarrow \sum_i\sum_j Q_{ij}\sum_{s=1,2}\bar{u}_i^{(s)} u_j^{(s)}$$

using $\bar{u}_i u_j = (u^\dagger\gamma^0)_i u_j = (u\bar{u})_{ji}$ $\quad \sum_{i,j} Q_{ij}\left\{\sum_{s=1,2}\bar{u}_i^{(s)} u_j^{(s)}\right\} = \sum_{i,j} Q_{ij}\left\{\sum_{s=1,2} u^{(s)}\bar{u}^{(s)}\right\}_{ji}$

so $\sum_{a-spins}\left(\sum_{b-spins} G\right) = \sum_{i,j=1,4} Q_{ij}\left\{\sum_{s_a=1,2} u^{(s_a)}(p_a)\bar{u}^{(s_a)}(p_a)\right\}_{ji} = \sum_{i,j=1,4} Q_{ij}(\not{p}_a + m_a c)_{ji} = Tr\left(Q(\not{p}_a + m_a c)\right)$

Show that $\bar{u}_i u_j = \left(u^\dagger \gamma^0\right)_i u_j = \left(u\bar{u}\right)_{ji}$ (which we used on the previous slide).

$$\bar{u}_i u_j = \left(u^\dagger \gamma^0\right)_i u_j = \left\{ \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\}_i u_j = \begin{matrix} u_i^* u_j & (i=1,2) \\ -u_i^* u_j & (i=3,4) \end{matrix}$$

$$u\bar{u} = uu^\dagger\gamma^0 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* & u_3^* & u_4^* \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{pmatrix} u_1^* & u_2^* & -u_3^* & -u_4^* \end{pmatrix} = \begin{pmatrix} u_1 u_1^* & u_1 u_2^* & \boxed{-u_1 u_3^*} & -u_1 u_4^* \\ u_2 u_1^* & u_2 u_2^* & -u_2 u_3^* & -u_2 u_4^* \\ \boxed{u_3 u_1^*} & u_3 u_2^* & -u_3 u_3^* & -u_3 u_4^* \\ u_4 u_1^* & u_4 u_2^* & -u_4 u_3^* & -u_4 u_4^* \end{pmatrix}$$

$$i=1, j=3 \quad \Rightarrow \quad \bar{u}_i u_j = u_1^* u_3 = \boxed{\left(u\bar{u}\right)_{3,1}}$$

$$i=3, j=1 \quad \Rightarrow \quad \bar{u}_i u_j = -u_3^* u_1 = \boxed{\left(u\bar{u}\right)_{1,3}}$$

In the last step of the calculation we stated $\sum_{i,j=1,4} Q_{ij}\left(\not{p}_a + m_a c\right)_{ji} = Tr\left(Q\left(\not{p}_a + m_a c\right)\right)$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} Q_{ij}B_{ji} = \sum_{j} Q_{1j}B_{j1} + Q_{2j}B_{j2} + Q_{3j}B_{j3} + Q_{4j}B_{j4}$$

$$\begin{aligned} &= \boxed{Q_{11}B_{11} + Q_{21}B_{12} + Q_{31}B_{13} + Q_{41}B_{14}} \\ &+ \boxed{Q_{12}B_{21} + Q_{22}B_{22} + Q_{32}B_{23} + Q_{42}B_{24}} \\ &+ \boxed{Q_{13}B_{31} + Q_{23}B_{32} + Q_{33}B_{33} + Q_{43}B_{34}} \\ &+ \boxed{Q_{14}B_{41} + Q_{24}B_{42} + Q_{34}B_{43} + Q_{44}B_{44}} \end{aligned}$$

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{pmatrix}$$

$$QB = \begin{pmatrix} \boxed{Q_{11}B_{11} + Q_{12}B_{21} + Q_{13}B_{31} + Q_{14}B_{41}} & Q_{11}B_{12} + Q_{12}B_{22} + Q_{13}B_{32} + Q_{14}B_{42} & Q_{11}B_{13} + Q_{12}B_{23} + Q_{13}B_{33} + Q_{14}B_{43} & Q_{11}B_{14} + Q_{12}B_{24} + Q_{13}B_{34} + Q_{14}B_{44} \\ Q_{21}B_{11} + Q_{22}B_{21} + Q_{23}B_{31} + Q_{24}B_{41} & \boxed{Q_{21}B_{12} + Q_{22}B_{22} + Q_{23}B_{32} + Q_{24}B_{42}} & Q_{21}B_{13} + Q_{22}B_{23} + Q_{23}B_{33} + Q_{24}B_{43} & Q_{21}B_{14} + Q_{22}B_{24} + Q_{23}B_{34} + Q_{24}B_{44} \\ Q_{31}B_{11} + Q_{32}B_{21} + Q_{33}B_{31} + Q_{34}B_{41} & Q_{31}B_{12} + Q_{32}B_{22} + Q_{33}B_{32} + Q_{34}B_{42} & \boxed{Q_{31}B_{13} + Q_{32}B_{23} + Q_{33}B_{33} + Q_{34}B_{43}} & Q_{31}B_{14} + Q_{32}B_{24} + Q_{33}B_{34} + Q_{34}B_{44} \\ Q_{41}B_{11} + Q_{42}B_{21} + Q_{43}B_{31} + Q_{44}B_{41} & Q_{41}B_{12} + Q_{42}B_{22} + Q_{43}B_{32} + Q_{44}B_{42} & Q_{41}B_{13} + Q_{42}B_{23} + Q_{43}B_{33} + Q_{44}B_{43} & \boxed{Q_{41}B_{14} + Q_{42}B_{24} + Q_{43}B_{34} + Q_{44}B_{44}} \end{pmatrix}$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} Q_{ij}B_{ji} = Tr\left(QB\right)$$

$$\sum_{all\ spins}\left[\bar{u}(a)\Gamma_1 u(b)\right]\left[\bar{u}(a)\Gamma_2 u(b)\right]^* = Tr\left(\Gamma_1\left(\not{p}_b + m_b c\right)\bar{\Gamma}_2\left(\not{p}_a + m_a c\right)\right)$$

Note that for antiparticles (v rather than u) the mass terms change sign

Note that there are NO spinors left in this expression, only matrix multiplication and the evaluation of the trace.

Return now to electron muon scattering: we had

$$|\mathcal{M}|^2 = \frac{g^4}{(p_1 - p_3)^4}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(3)\gamma^\nu u(1)\right]^*\left[\bar{u}(4)\gamma_\nu u(2)\right]^*$$

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{4}\frac{g^4}{(p_1 - p_3)^4}\sum_{spins}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]\left[\bar{u}(3)\gamma^\nu u(1)\right]^*\left[\bar{u}(4)\gamma_\nu u(2)\right]^*$$

$$\left\langle|\mathcal{M}|^2\right\rangle = \frac{1}{4}\frac{g^4}{(p_1 - p_3)^4}Tr\left(\gamma^\mu\left(\not{p}_1 + mc\right)\gamma^\nu\left(\not{p}_3 + mc\right)\right)Tr\left(\gamma_\mu\left(\not{p}_2 + Mc\right)\gamma_\nu\left(\not{p}_4 + Mc\right)\right)$$

where $m = m_e$, $M = m_\mu$ and the factor of 1/4 is from averaging over possible initial spin states.

Griffiths §7.7 lists many identities that are needed for the evaluation of the types of traces we will encounter. Here list only those that are relevant to the current calculation:

1. $Tr(A+B) = Tr(A) + Tr(B)$

2. $Tr(\alpha A) = \alpha Tr(A)$

10. The trace of the product of an odd number of γ matrices is 0

12. $Tr(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

13. $Tr(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda})$

$$Tr\left(\gamma^\mu (\not{p}_1 + mc)\gamma^\nu (\not{p}_3 + mc)\right) \xrightarrow{1,2} Tr\left(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\right) + mc\Big[\overbrace{\cancel{Tr\left(\gamma^\mu \not{p}_1 \gamma^\nu\right)} + \cancel{Tr\left(\gamma^\mu \gamma^\nu \not{p}_3\right)}}^{=\,0\text{ by rule 10.}}\Big] + (mc)^2\, Tr\left(\gamma^\mu \gamma^\nu\right)$$

$$= Tr\left(\gamma^\mu \not{p}_1 \gamma^\nu \not{p}_3\right) + (mc)^2\, Tr\left(\gamma^\mu \gamma^\nu\right)$$

$$Tr\left(\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3\right)+(mc)^2\,Tr\left(\gamma^{\mu}\gamma^{\nu}\right)$$

$$Tr\left(\gamma^{\mu}\not{p}_1\gamma^{\nu}\not{p}_3\right)=(p_1)_{\lambda}(p_3)_{\sigma}\,Tr\left(\gamma^{\mu}\gamma^{\lambda}\gamma^{\nu}\gamma^{\sigma}\right)$$ [since each element of p_1 or p_3 is just a number]

$$=(p_1)_{\lambda}(p_3)_{\sigma}\,4\left(g^{\mu\lambda}g^{\nu\sigma}-g^{\mu\nu}g^{\lambda\sigma}+g^{\mu\sigma}g^{\lambda\nu}\right)$$ [using rule 13]

$$=4\left(p_1^{\mu}p_3^{\nu}-g^{\mu\nu}\left(p_1\cdot p_3\right)+p_3^{\mu}p_1^{\nu}\right)$$ [simply contracting the indices]

$$(mc)^2\,Tr\left(\gamma^{\mu}\gamma^{\nu}\right)=4(mc)^2\,g^{\mu\nu}$$ [using rule 12]

$$Tr\left(\gamma^{\mu}\left(\not{p}_1+mc\right)\gamma^{\nu}\left(\not{p}_3+mc\right)\right)=4\left(p_1^{\mu}p_3^{\nu}+p_3^{\mu}p_1^{\nu}+g^{\mu\nu}\left((mc)^2-p_1\cdot p_3\right)\right)$$

The other trace in our calculation is the same, but with $m \rightarrow M$, $1 \rightarrow 2$, $3 \rightarrow 4$ and the greek indices lowered; that is,

$$Tr\left(\gamma_{\mu}\left(\not{p}_2+Mc\right)\gamma_{\nu}\left(\not{p}_4+Mc\right)\right)=4\left(p_{2\mu}p_{4\nu}+p_{4\mu}p_{2\nu}+g^{\mu\nu}\left((Mc)^2-p_2\cdot p_4\right)\right)$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4}\frac{g^4}{(p_1 - p_3)^4} Tr\left(\gamma^{\mu}\left(\not{p}_1 + mc\right)\gamma^{\nu}\left(\not{p}_3 + mc\right)\right) Tr\left(\gamma_{\mu}\left(\not{p}_2 + Mc\right)\gamma_{\nu}\left(\not{p}_4 + Mc\right)\right)$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4}\frac{g^4}{(p_1 - p_3)^4}\left\{4\left(p_1^{\mu}p_3^{\nu} + p_3^{\mu}p_1^{\nu} - g^{\mu\nu}\left((mc)^2 - p_1 \cdot p_3\right)\right)\right\}\left\{4\left(p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} - g_{\mu\nu}\left((Mc)^2 - p_2 \cdot p_4\right)\right)\right\}$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 4\frac{g^4}{(p_1 - p_3)^4}\left(p_1^{\mu}p_3^{\nu} + p_3^{\mu}p_1^{\nu} - g^{\mu\nu}\left((mc)^2 - p_1 \cdot p_3\right)\right)\left(p_{2\mu}p_{4\nu} + p_{4\mu}p_{2\nu} - g_{\mu\nu}\left((Mc)^2 - p_2 \cdot p_4\right)\right)$$

$$\begin{aligned}\left\langle |\mathcal{M}|^2 \right\rangle = 4\frac{g^4}{(p_1 - p_3)^4}\Big\{ & p_1^{\mu}p_3^{\nu}p_{2\mu}p_{4\nu} + p_1^{\mu}p_3^{\nu}p_{4\mu}p_{2\nu} + p_1^{\mu}p_3^{\nu}g_{\mu\nu}\left((Mc)^2 - p_2 \cdot p_4\right) \\ & + p_3^{\mu}p_1^{\nu}p_{2\mu}p_{4\nu} + p_3^{\mu}p_1^{\nu}p_{4\mu}p_{2\nu} + p_3^{\mu}p_1^{\nu}g_{\mu\nu}\left((Mc)^2 - p_2 \cdot p_4\right) \\ & + p_{2\mu}p_{4\nu}g^{\mu\nu}\left((mc)^2 - p_1 \cdot p_3\right) + p_{4\mu}p_{2\nu}g^{\mu\nu}\left((mc)^2 - p_1 \cdot p_3\right) \\ & + g^{\mu\nu}\left((mc)^2 - p_1 \cdot p_3\right)g_{\mu\nu}\left((Mc)^2 - p_2 \cdot p_4\right)\Big\}\end{aligned}$$

$$\left\langle \left|\mathcal{M}\right|^2 \right\rangle = 4\frac{g^4}{\left(p_1 - p_3\right)^4}\Big\{\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)+\left(p_3\cdot p_2\right)\left(p_1\cdot p_4\right)+\left(p_3\cdot p_4\right)\left(p_1\cdot p_2\right)$$

$$+\left(p_1\cdot p_3\right)\left(\left(Mc\right)^2 - p_2\cdot p_4\right)+\left(p_1\cdot p_3\right)\left(\left(Mc\right)^2 - p_2\cdot p_4\right)$$

$$+\left(p_2\cdot p_4\right)\left(\left(mc\right)^2 - p_1\cdot p_3\right)+\left(p_2\cdot p_4\right)\left(\left(mc\right)^2 - p_1\cdot p_3\right)$$

$$+4\left(\left(mc\right)^2 - p_1\cdot p_3\right)\left(\left(Mc\right)^2 - p_2\cdot p_4\right)\Big\}$$

$$\left\langle \left|\mathcal{M}\right|^2 \right\rangle = 4\frac{g^4}{\left(p_1 - p_3\right)^4}\Big\{2\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)+2\left(p_2\cdot p_3\right)\left(p_1\cdot p_4\right)$$

$$+2\left(p_1\cdot p_3\right)\left(Mc\right)^2 - 2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)$$

$$+2\left(p_2\cdot p_4\right)\left(mc\right)^2 - 2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)$$

$$+4\left(\left(mc\right)^2 - p_1\cdot p_3\right)\left(\left(Mc\right)^2 - p_2\cdot p_4\right)\Big\}$$

$$\left\langle \left|\mathcal{M}\right|^2 \right\rangle = 4\frac{g^4}{\left(p_1 - p_3\right)^4}\Big\{2\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)+2\left(p_2\cdot p_3\right)\left(p_1\cdot p_4\right)-2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)$$

$$+2\left(p_1\cdot p_3\right)\left(Mc\right)^2 + 2\left(p_2\cdot p_4\right)\left(mc\right)^2$$

$$+4\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+4\left(mc\right)^2\left(Mc\right)^2$$

$$-4\left(mc\right)^2\left(p_2\cdot p_4\right)-4\left(Mc\right)^2\left(p_1\cdot p_3\right)\Big\}$$

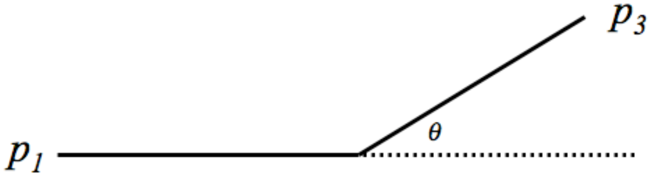
$$\left\langle |\mathcal{M}|^2 \right\rangle = 4\frac{g^4}{(p_1 - p_3)^4}\left\{2(p_1 \cdot p_2)(p_3 \cdot p_4) + 2(p_2 \cdot p_3)(p_1 \cdot p_4) - 2(p_1 \cdot p_3)(Mc)^2 - 2(p_2 \cdot p_4)(mc)^2 + 4(mMc^2)^2\right\}$$

$$= 8\frac{g^4}{(p_1 - p_3)^4}\left\{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_3)(Mc)^2 - (p_2 \cdot p_4)(mc)^2 + 2(mMc^2)^2\right\}$$

[Griffiths 7.129]

Note that this is independent of the reference frame. To consider the problem in a specific reference frame, write out the relevant four vectors in that reference frame and do the calculation.

Consider the case of electron scattering from a heavy muon at rest:



Assume $M >> m$ so that we can ignore the recoil of the heavy spin-1/2 particle

$$\frac{d\sigma}{d\Omega} = \frac{\hbar}{8\pi Mc}\left\langle |\mathcal{M}|^2 \right\rangle$$

[we did this in a recent lecture]

$$p_1 = \left(\frac{E}{c}, \vec{p}_1\right) \qquad p_2 = \left(Mc, \vec{0}\right)$$

$$p_3 = \left(\frac{E}{c}, \vec{p}_3\right) \qquad p_4 = \left(Mc, \vec{0}\right)$$

[ignoring the recoil]

$$|\vec{p}_1| = |\vec{p}_3| \equiv p$$

$$\vec{p}_1 \cdot \vec{p}_3 = p^2 \cos\theta$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\frac{g^4}{(p_1 - p_3)^4}\left\{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_3)(Mc)^2 - (p_2 \cdot p_4)(mc)^2 + 2(mMc^2)^2\right\}$$

$$(p_1 - p_3)^2 = -(\vec{p}_1 - \vec{p}_3)^2 = -|\vec{p}_1|^2 - |\vec{p}_3|^2 + 2\vec{p}_1 \cdot \vec{p}_3 = -2p^2(1 - \cos\theta) = -4p^2 \sin^2(\theta/2)$$

$$(p_1 \cdot p_3) = \frac{E^2}{c^2} - (\vec{p}_1 \cdot \vec{p}_3) = p^2 + m^2c^2 - p^2\cos\theta = m^2c^2 + 2p^2\sin^2(\theta/2)$$

$$(p_1 \cdot p_2)(p_3 \cdot p_4) = (p_1 \cdot p_4)(p_2 \cdot p_3) = (ME)^2 \qquad (p_2 \cdot p_4) = (Mc)^2$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\frac{g^4}{\left(-4p^2\sin^2(\theta/2)\right)^2}\left\{(ME)^2 + (ME)^2 - \left(m^2c^2 + 2p^2\sin^2(\theta/2)\right)(Mc)^2 - (Mc)^2(mc)^2 + 2(mMc^2)^2\right\}$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\left(\frac{g^2}{4p^2\sin^2(\theta/2)}\right)^2\left\{2(ME)^2 - \left(m^2c^2 + 2p^2\sin^2(\theta/2)\right)(Mc)^2 - (Mc)^2(mc)^2 + 2(mMc^2)^2\right\}$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\left(\frac{g^2}{4p^2\sin^2(\theta/2)}\right)^2\left\{2M^2\left(p^2c^2 + m^2c^4\right) - 2(Mc)^2 p^2\sin^2(\theta/2) - (m^2c^2)(Mc)^2 - (M^2c^2)(m^2c^2) + 2(mMc^2)^2\right\}$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 2\left(\frac{2g^2}{4p^2\sin^2(\theta/2)}\right)^2\left\{2M^2c^2\left(m^2c^2 + p^2 - p^2\sin^2(\theta/2)\right)\right\} = \left(\frac{g^2 Mc}{p^2\sin^2(\theta/2)}\right)^2\left\{m^2c^2 + p^2\cos^2(\theta/2)\right\}$$

$$\frac{d\sigma}{d\Omega} = \frac{\hbar}{8\pi Mc}\left\langle |\mathcal{M}|^2 \right\rangle \qquad g = \sqrt{4\pi\alpha} \qquad \frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{2p^2 \sin^2(\theta/2)}\right)^2 \left\{(mc)^2 + p^2\cos^2(\theta/2)\right\}$$

[Mott formula]

This is a good approximation to low energy electron-proton scattering (e.g. scattering of an electron off a heavy spin-1/2 particle).

For the case $p^2 << (mc)^2$ this expression reduces to

$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha\hbar}{2p^2 \sin^2(\theta/2)}\right)^2 (mc)^2 = \left(\frac{\alpha\hbar mc}{2p^2 \sin^2(\theta/2)}\right)^2 = \left(\frac{e^2 m}{2m^2v^2 \sin^2(\theta/2)}\right)^2 = \left(\frac{e^2 m}{2mv^2 \sin^2(\theta/2)}\right)^2$$

using $\alpha = \frac{e^2}{\hbar c}$ $p = mv$

$$\frac{d\sigma}{d\Omega} = \left(\frac{e^2 m}{2mv^2 \sin^2(\theta/2)}\right)^2$$

This is the formula for Rutherford scattering. Compare Griffiths Example 6.4, noting that here (above) we have

$$2mv^2 = 4\left(\frac{1}{2}mv^2\right) = 4E$$

Which holds in the non-relativistic limit we have used.