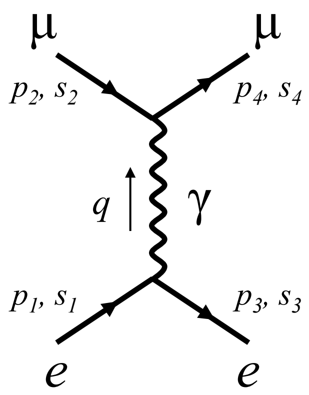
PHY489 Lecture 17

Last Time: electron-muon scattering



$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4} \frac{g^4}{(p_1 - p_3)^4} Tr\left(\gamma^\mu (\not{p}_1 + mc)\gamma^\nu (\not{p}_3 + mc)\right) Tr\left(\gamma_\mu (\not{p}_2 + Mc)\gamma_\nu (\not{p}_4 + Mc)\right)$$

$$Tr\left(\gamma^\mu (\not{p}_1 + mc)\gamma^\nu (\not{p}_3 + mc)\right) = 4\left(p_1^\mu p_3^\nu + p_3^\mu p_1^\nu + g^{\mu\nu}\left((mc)^2 - p_1 \cdot p_3\right)\right)$$

$$Tr\left(\gamma_\mu (\not{p}_2 + Mc)\gamma_\nu (\not{p}_4 + Mc)\right) = 4\left(p_{2\mu} p_{4\nu} + p_{4\mu} p_{2\nu} + g^{\mu\nu}\left((Mc)^2 - p_2 \cdot p_4\right)\right)$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8 \frac{g^4}{(p_1 - p_3)^4} \left\{ (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - (p_1 \cdot p_3)(Mc)^2 - (p_2 \cdot p_4)(mc)^2 + 2\left(mMc^2\right)^2 \right\}$$

Last time we looked at low-energy scattering of an electron off a muon at rest (in the limit where we ignore the recoil of the muon).

Today, start by looking at high-energy electron-muon scattering.

High Energy Electron-Muon Scattering

We have derived Griffiths 7.129 for electron-muon scattering (in QED)

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\frac{g^4}{(p_1 - p_3)^4}\left\{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4) - \cancel{(p_1 \cdot p_3)(Mc)^2} - \cancel{(p_2 \cdot p_4)(mc)^2} + \cancel{2(mMc^2)^2}\right\}$$

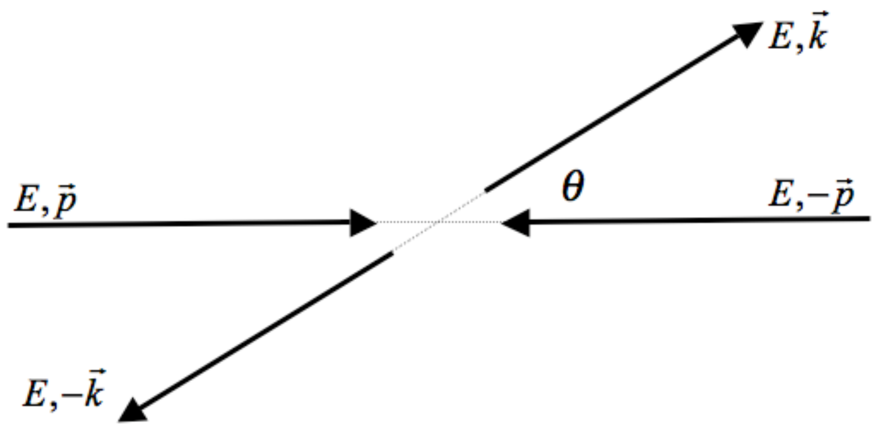
Let's do problem 7.38 which asks us to start with the above expression and derive the differential cross-section for high-energy electron-muon scattering in the CM frame. In this limit, we can ignore the mass terms:

Griffiths 6.47 tells us:
[see lecture 12]

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|} \Rightarrow \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{(2E)^2}\left\langle |\mathcal{M}|^2 \right\rangle$$ [in this case]

[since $|\vec{p}_f| = |\vec{p}_i|$ and $E_1 = E_2$ (in this high energy limit where we can ignore the masses)]

$$\left\langle \left|\mathcal{M}\right|^2 \right\rangle = 8\frac{g^4}{\left(p_1 - p_3\right)^4}\left\{\left(p_1 \cdot p_2\right)\left(p_3 \cdot p_4\right) + \left(p_2 \cdot p_3\right)\left(p_1 \cdot p_4\right)\ \right\}$$



$$p_1 = \left(\frac{E}{c}, \vec{p}\right) \qquad p_2 = \left(\frac{E}{c}, -\vec{p}\right)$$

$$p_3 = \left(\frac{E}{c}, \vec{k}\right) \qquad p_4 = \left(\frac{E}{c}, -\vec{k}\right)$$

$$\left|\vec{p}\right|^2 = \left|\vec{k}\right|^2 \equiv p^2$$

$E_1^2 = m_1^2 c^4 + \left|\vec{p}_1\right|^2 c^2 \approx \left|\vec{p}_1\right|^2 c^2 \Rightarrow \dfrac{E_1}{c} \approx \left|\vec{p}\right| \equiv p$ Similarly, $\dfrac{E_2}{c} \approx \left|\vec{p}\right| \equiv p$ and $\dfrac{E_3}{c} = \dfrac{E_4}{c} \approx \left|\vec{k}\right| \equiv p$

$$\left(p_1 - p_3\right)^2 = \left(\vec{p} - \vec{k}\right)^2 = p^2 + k^2 - 2\vec{p} \cdot \vec{k} = 2p^2(1 - \cos\theta) = 4p^2 \sin^2\left(\theta / 2\right)$$

$$\left(p_1 \cdot p_2\right) = \frac{E^2}{c^2} + p^2 \approx 2p^2 \qquad \left(p_3 \cdot p_4\right) = \frac{E^2}{c^2} + k^2 = \frac{E^2}{c^2} + p^2 \approx 2p^2$$

$$\left(p_1 \cdot p_4\right) = \frac{E^2}{c^2} + \vec{p} \cdot \vec{k} = p^2\left(1 + \cos\theta\right) = 2p^2 \cos^2\left(\theta / 2\right) \qquad \left(p_2 \cdot p_3\right) = \frac{E^2}{c^2} + \vec{p} \cdot \vec{k} = p^2\left(1 + \cos\theta\right) = 2p^2 \cos^2\left(\theta / 2\right)$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\frac{g^4}{(p_1 - p_3)^4}\left\{(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_2 \cdot p_3)(p_1 \cdot p_4)\right\}$$

$$= \frac{8g^4}{\left(4p^2 \sin^2(\theta/2)\right)^2}\left\{(2p^2)(2p^2) + \left(2p^2\cos^2(\theta/2)\right)^2\right\}$$

$$= \frac{8g^4}{16p^4 \sin^4(\theta/2)}(4p^4)\left\{1 + \cos^4(\theta/2)\right\} = 2g^4\left\{\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}\right\}$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{1}{(2E)^2} 2g^4 \left\{\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}\right\} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{g^4}{2E^2}\left\{\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}\right\}$$

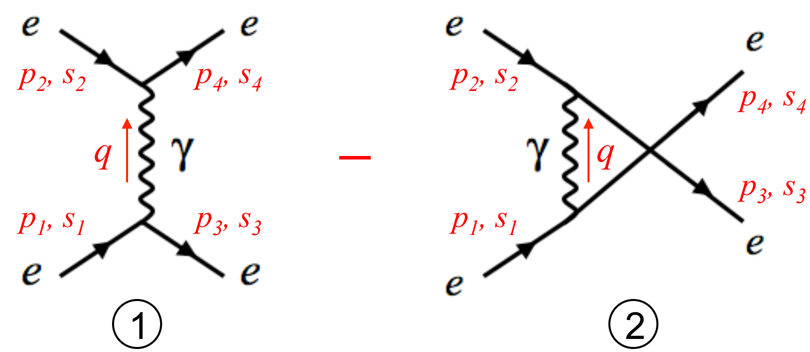
Note that this blows up for $\theta = 0$ (which makes sense since this case is indistinguishable from the case of no scattering - or scattering via an a photon of negligible energy (momentum transfer).

Can nevertheless calculate the total cross-section for this process within some angular region -- e.g. that covered by a detector if you are trying to compare with experimental results.

Detector cannot cover all the way to the beam-line ($\theta = 0,\pi$)

Other QED examples from Griffiths

electron-electron scattering:



For 1 $$\int \left[\bar{u}(p_3)\; ig\gamma^\mu\; u(p_1)(2\pi)^4\delta^4(p_1-p_3-q)\right]\frac{-ig_{\mu\nu}}{q^2}\left[\bar{u}(p_4)\; ig\gamma^\nu\; u(p_2)(2\pi)^4\delta^4(p_2+q-p_4)\right]\frac{d^4q}{(2\pi)^4}$$

$$=\frac{ig^2}{(p_1-p_3)^2}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_4)\gamma_\mu u(p_2)\right]\cancel{(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)}$$

$$\mathcal{M}_1=-\frac{g^2}{(p_1-p_3)^2}\left[\bar{u}(p_3)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_4)\gamma_\mu u(p_2)\right]$$

For $\mathcal{M}_2$ $q=p_1-p_4$ and p_1 couples to p_4, p_2 to p_3, so by inspection

$$\mathcal{M}_2=\frac{g^2}{(p_1-p_4)^2}\left[\bar{u}(p_4)\gamma^\mu u(p_1)\right]\left[\bar{u}(p_3)\gamma_\mu u(p_2)\right]$$

where the sign change comes because the diagrams differ by the exchange of two outgoing electron lines (as discussed in the last lecture)

Compton Scattering: External Photons

First need to know how to deal with external photons (see § 7.4) for details:

Feynman rules: incoming photon: $\varepsilon^{\mu}(p)$

outgoing photon: $\varepsilon^{\mu *}(p)$

$$\left\{ \varepsilon^{\mu}_{(s)} \ \text{ for } \ s = 1,2 \right\}$$

Here ε^{μ} is a polarization vector which satisfies the following properties:

$$\varepsilon^{\mu} p_{\mu} = 0 \qquad \varepsilon^{\mu *}_{(1)} \varepsilon_{(2)\mu} = 0 \ \text{(orthogonal)} \qquad \varepsilon^{\mu *} \varepsilon_{\mu} = 1 \ \text{(normalized)}$$

In the Coloumb gauge: $\varepsilon^{0} = 0, \ \vec{\varepsilon} \cdot \vec{p} = 0 \ \ (\text{since } \ \varepsilon^{\mu} p_{\mu} = 0)$

If p is chosen to be along the z direction, one choice is: $\varepsilon_{1} = (1,0,0) \ \ \varepsilon_{2} = (0,1,0)$

Completeness relation: $\sum_{s=1,2} \left(\varepsilon_{(s)}\right)_{i} \left(\varepsilon_{(s)}\right)_{j} = \delta_{ij} - \hat{p}_{i} \hat{p}_{j}$ [some details follow on next slides]

Photons in QED

Maxwell's equations:
[Gaussian units]

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

In relativistic notation the fields $\vec{E}$ and $\vec{B}$ together form an antisymmetric second-rank tensor (Field strength tensor) while the sources ρ and $\vec{J}$ form a four-vector:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad J^{\mu} = \left(c\rho, \vec{J}\right)$$

Using this notation, the inhomogeneous Maxwell's equations (e.g. those involving sources) can be written as:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \quad \text{e.g. (for } \nu = 0) \quad \partial_\mu F^{\mu 0} = \frac{4\pi}{c} J^0 = \frac{4\pi}{c} c\rho = 4\pi\rho$$

$$\frac{1}{c}\frac{\partial}{\partial t}(0) + \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z \quad = \vec{\nabla}\cdot\vec{E}$$

For $\nu = 1$: $$\partial_\mu F^{\mu 1} = \frac{4\pi}{c} J^1 \quad \Rightarrow \quad \frac{1}{c}\frac{\partial}{\partial t}(-E_x) + \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}B_z + \frac{\partial}{\partial z}(-B_y) \quad = \frac{4\pi}{c} J_x$$

$$-\frac{1}{c}\frac{\partial}{\partial t}(E_x) + \left(\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y\right) \quad = \frac{4\pi}{c} J_x \qquad \left[-\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \vec{\nabla}\times\vec{B}\right]_x \quad = \frac{4\pi}{c} J_x$$

This is just the x-component of $$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial}{\partial t}\vec{E} = \frac{4\pi}{c}\vec{J}$$

Doing the same for $\nu = 2,3$ gives the corresponding y and z components.

So we have: $\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$

From the anti-symmetric property $F^{\mu\nu} = -F^{\nu\mu}$ it follows that: $\partial_\nu \partial_\mu F^{\mu\nu} = 0$

[since $\partial_\nu \partial_\mu$ is symmetric under $\mu \leftrightarrow \nu$ while $F^{\mu\nu}$ is anti-symmetric].

$$\partial_\nu \partial_\mu F^{\mu\nu} = \frac{4\pi}{c} \partial_\nu J^\nu = 0 \quad \Rightarrow \quad \partial_\nu J^\nu = 0$$

$$\partial_\mu J^\mu = 0 \quad \to \quad \frac{1}{c} \frac{\partial}{\partial t}(c\rho) + \vec{\nabla} \cdot \vec{J} = 0 \quad \to \quad \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

which is the continuity equation for electric charge.

For the homogeneous Maxwell equations (i.e. no sources) we have:

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad \vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$$

so we can write $\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t}$ as the gradient of a scalar.

Using $\vec{B} = \vec{\nabla} \times \vec{A}$ one can write $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ where $A^\mu = \left(V, \vec{A}\right)$.

In term of A^μ the inhomogeneous Maxwell's equations becomes:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \left(\partial_\mu A^\mu \right) = \frac{4\pi}{c} J^\nu$$

The potential formulation automatically takes care of the homogeneous Maxwell equations:

$$\text{recall } \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{F} \right) = 0 \text{ for all } \vec{F}, \text{ so } \vec{\nabla} \cdot \vec{B} = 0 \text{ is guarateed}$$

The “defect” with the potential formulation is that the potentials are not uniquely defined:

$\vec{E} = -\vec{\nabla} V$ so adding a constant to the potential does not change anything.

Here (current case) $A'_\mu \rightarrow A_\mu + \partial_\mu \lambda(\vec{x}, t)$ works just as well in the expression (above) for $F^{\mu\nu}$.

e.g. $$\partial_\mu A'^\nu - \partial^\nu\left(\partial_\mu A'^\mu\right) = \partial_\mu A^\nu - \partial^\nu\left(\partial_\mu A^\mu\right)$$

A change of potential that does not affect the fields is called a gauge transformation (this turns out to be a fundamental concept in the theoretical description of all interactions – this is discussed in chapter 11 of the text).

Can exploit this freedom to define $\partial_\mu A^\mu = 0$. This additional constraint on A^μ is referred to as the Lorentz condition.

$$\partial_\mu \partial^\mu A^\nu - \cancel{\partial^\nu\left(\partial_\mu A^\mu\right)} = \frac{4\pi}{c} J^\nu \qquad \rightarrow \qquad \partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} J^\nu$$

The Lorentz condition does not uniquely specify A^{μ} since we can write a gauge transformation that leaves $\partial_{\mu}A^{\mu}=0$ unaffected.

One can choose to live with this indeterminacy or impose additional constraints: this spoils Lorentz invariance, but is nevertheless a common approach.

In empty space $\left(J^{\mu}=0\right)$ we define $A^{0}=0$ (known as the Coulomb gauge).

The Lorentz condition $\partial_{\mu}A^{\mu}=0$ then becomes $\vec{\nabla}\cdot\vec{A}=0$.

In QED, A^{μ} represents the photon wavefunction. A free photon satisfes:

$$\partial_{\mu}\partial^{\mu}A^{\nu}=0$$

Klein-Gordon Eqn. for massless particles (e.g. in the absence of sources).

Plane-wave solutions with four-momentum $p = \left(\frac{E}{c}, \vec{p} \right)$ are of the form:

$$A^{\mu}(x) = a e^{-\frac{i}{\hbar} p \cdot x} \varepsilon^{\mu}(p)$$

Normalization factor
(don't care here)

Polarization vector
characterizing the
photon spin-state.

$$\partial_{\mu}\partial^{\mu} A^{\nu} = 0 \quad \rightarrow \quad p_{\mu} p^{\mu} = 0 \;\text{ or }\; E = |\vec{p}|c \;\; \text{(correct for massless particle).}$$

The polarization vector ε^{μ} nominally has four components, but the Lorentz condition:

$$\partial_{\mu} A^{\mu} = 0 \quad \rightarrow \quad p_{\mu} \varepsilon^{\mu} = 0$$

removes one degree of freedom.

The Coulomb gauge corresponds to the choice: $\varepsilon^0 = 0 \quad \Rightarrow \quad \vec{\varepsilon} \cdot \vec{p} = 0$

so the polarization vector is perpendicular to the photon three-momentum $\vec{p}$.

For solutions $A^\mu(x) = ae^{-\frac{i}{\hbar}p\cdot x}\varepsilon^\mu(p)$ we then have $A^0(x) = ae^{-\frac{i}{\hbar}p\cdot x}\varepsilon^0(p) = 0$

A free photon is said to be transversely polarized. There are two linearly independent three vectors perpendicular to $\hat{p}$ (here along $\hat{z}$)

$$\vec{\varepsilon}^{(1)} = (1,0,0) \qquad \vec{\varepsilon}^{(2)} = (0,1,0)$$

instead of the four one might have expected. There is no longitudinal polarization state for a massless particle.

Polarization vectors: completeness relation

Completeness relation: $\sum_{s=1,2} \left(\varepsilon_{(s)}\right)_i \left(\varepsilon_{(s)}\right)_j = \delta_{ij} - \hat{p}_i \hat{p}_j$

For example, in the case where p is along z, $\vec{p} = \left|\vec{p}\right| \hat{z}$

$$\hat{p}_i \hat{p}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{so} \quad \delta_{ij} - \hat{p}_i \hat{p}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Feynman Rules for External Photons

Feynman rules: incoming photon: $\varepsilon^{\mu}(p)$

outgoing photon: $\varepsilon^{\mu *}(p)$

$\left\{\varepsilon^{\mu}_{(s)} \text{ for } s = 1,2\right\}$

Here ε^{μ} is the polarization vector which satisfies the following properties:

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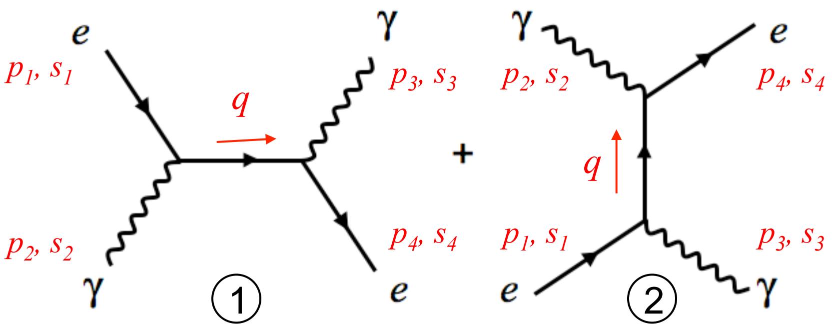
The completeness relation is: $\sum_{s=1,2} \left(\varepsilon_{(s)}\right)_{i} \left(\varepsilon_{(s)}\right)_{j} = \delta_{ij} - \hat{p}_{i} \hat{p}_{j}.$

Compton Scattering



For 1

write down contribution from fermion line:

Write down factors for two external photons (coupled to the correct vertex - e.g. with the correct μ or ν index)

Integrate over all internal momenta

$$\int \varepsilon_\mu^*(3)\left[\bar{u}(4)\; ig\gamma^\mu\; \frac{i(\not{q}+mc)}{q^2-m^2c^2} ig\gamma^\nu u(1)\right]\varepsilon_\nu(2)(2\pi)^4\delta^4(p_1+p_2-q)(2\pi)^4\delta^4(q-p_3-p_4)\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M}_1=-\frac{g^2}{(p_1+p_2)^2-m^2c^2}\left[\bar{u}(4)\not{\varepsilon}^*(3)(\not{p}_1+\not{p}_2+mc)\not{\varepsilon}(2)u(1)\right]$$

By inspection $\mathcal{M}_2=\mathcal{M}_1$ with $q=p_1-p_3$, ε(2) couples to p_4 and ε(3)* couples to p_1

$$\mathcal{M}_2=-\frac{g^2}{(p_1-p_3)^2-m^2c^2}\left[\bar{u}(4)\not{\varepsilon}(2)(\not{p}_1-\not{p}_3+mc)\not{\varepsilon}^*(3)u(1)\right]$$

and $\mathcal{M}=\mathcal{M}_2+\mathcal{M}_1$.