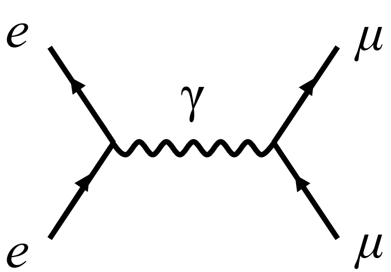
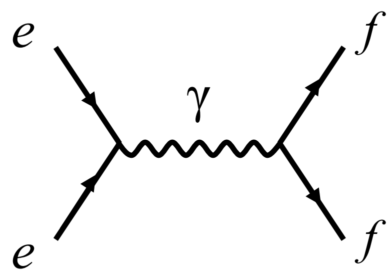
PHY489 Lecture 18

Electrodynamics of Quarks and Hadrons

Here we won't discuss contribution from the weak interaction (yet). Note that QED describes the interaction of the photon with charged spin-1/2 fermions. We just discussed electron-muon scattering. Can also discuss the process $e^+e^- \rightarrow \mu^+\mu^-$ (which we will do here). Note however, that this process (when mediated by photon exchange) has the same structure regardless of what fermion anti-fermion pair is in the final state (excluding electrons, since in that case there are additional diagrams)

N.B. Arrows need fixing on some diagrams

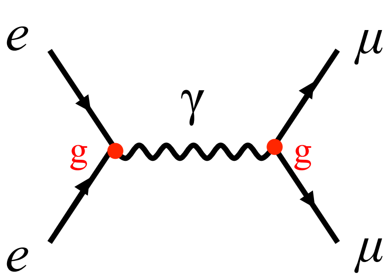


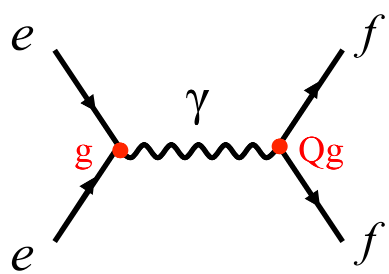


Again, at low energies, the weak interaction contribution is negligible (suppressed by the high mass of the weak gauge bosons) e.g.

$$\frac{1}{q^2} \quad \text{vs.} \quad \frac{1}{q^2 - M^2c^2}$$

dependence of the propagator.





Here Q is the electric charge of the fermion f (≠ 1 for quarks!)

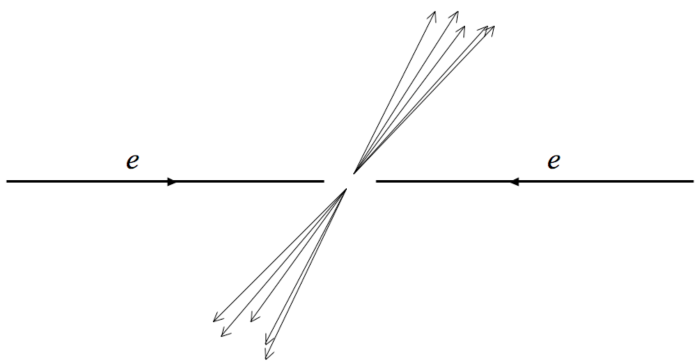
additional factor of Q^2 in $|\mathcal{M}|^2$

Here $f = u,d,s,c,b,t, \mu,\tau$ (also e, but for that there are other diagrams). No coupling to neutrinos.

Quark Confinement

Quarks (which are coloured) cannot exist freely. They therefore "hadronize" to produce "jets" of particles in the final state. This is a complication, but we can still calculate the amplitude for $e^+e^- \rightarrow hadrons$ if we assume that the hard (e.g. high q^2) scattering process decouples from the hadronization process which takes place at a very different energy scale, in which case:

$$\sigma\left(e^+e^- \rightarrow hadrons\right) = \sum_{q=u,d,s,c,b,t} \sigma\left(e^+e^- \rightarrow q\bar{q}\right)$$

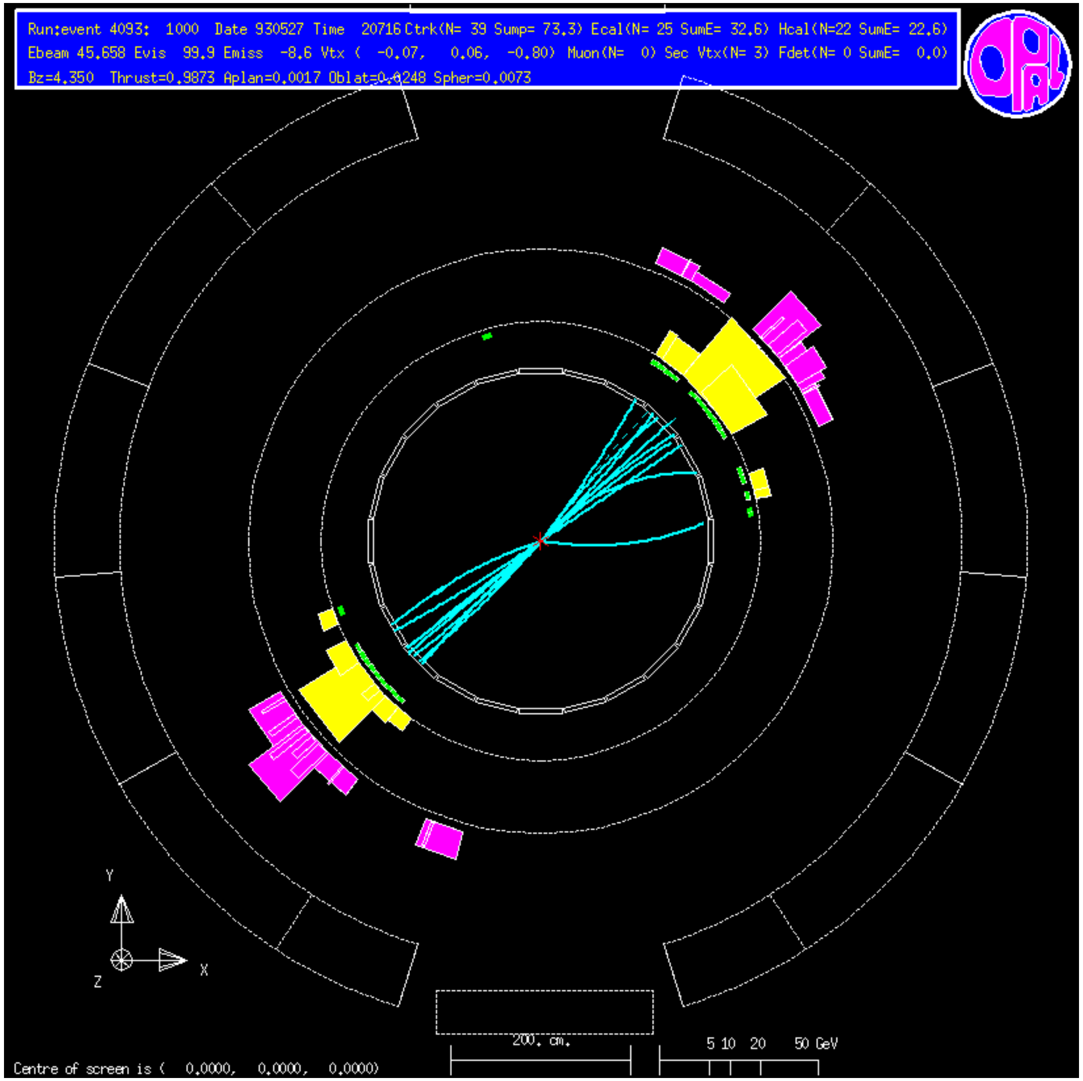


Note that in the above expression the sum is over all quark flavours that are kinematically accessible at the specified centre-of-mass energy: σ = σ(E) (of course). Recall that, in CM frame:

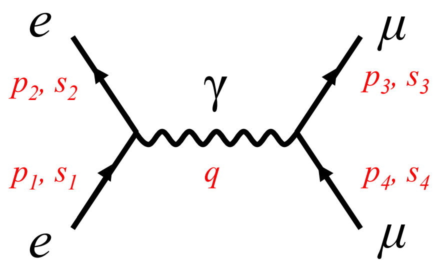
$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\left|\mathcal{M}\right|^2}{\left(E_1 + E_2\right)^2} \frac{\left|\vec{p}_f\right|}{\left|\vec{p}_i\right|}$$

For $m_1 = m_2$ this is proportional to E^{-2}.

Two jet final state in e^+e^- collisions



Amplitude $\mathcal{M}$ for the process $e^+e^- \to \mu^+\mu^-$



Here we need antiparticle spinors for the first time.

$$\int \underbrace{\left[\bar{v}(2)\; ig\gamma^{\mu}\; u(1)(2\pi)^4\,\delta^4(p_1+p_2-q)\right]}_{\text{electron line}}\frac{-ig_{\mu\nu}}{q^2}\underbrace{\left[\bar{u}(3)\; ig\gamma^{\nu}\; v(4)(2\pi)^4\,\delta^4(q-p_3-p_4)\right]}_{\text{muon line}}\frac{d^4q}{(2\pi)^4}$$

$$\mathcal{M} = -\frac{g^2}{(p_1+p_2)^2}\left[\bar{v}(2)\gamma^{\mu}u(1)\right]\left[\bar{u}(3)\gamma_{\mu}v(4)\right] \qquad |\mathcal{M}|^2 = \left[\frac{g^2}{(p_1+p_2)^2}\right]^2\left[\bar{v}(2)\gamma^{\mu}u(1)\right]\left[\bar{u}(3)\gamma_{\mu}v(4)\right]\left[\bar{v}(2)\gamma^{\nu}u(1)\right]^*\left[\bar{u}(3)\gamma_{\nu}v(4)\right]^*$$

Recall that we had $\sum_{\textit{all spins}}\left[\bar{u}(a)\Gamma_1 u(b)\right]\left[\bar{u}(a)\Gamma_2 u(b)\right]^* = Tr\left(\Gamma_1(\not{p}_b+m_b c)\bar{\Gamma}_2(\not{p}_a+m_a c)\right)$ with $\bar{\Gamma}_2 \equiv \gamma^0\Gamma_2^{\dagger}\gamma^0$

and that the sign of the mass term changes in the case of anti-particle spinors.

Should be anti-particle spinor

$$\sum_{\textit{all spins}}\left[\bar{u}(3)\gamma_{\mu}v(4)\right]\left[\bar{u}(3)\gamma_{\nu}u(4)\right]^* = Tr\left(\gamma_{\mu}(\not{p}_4-Mc)\gamma^0\gamma_{\nu}^{\dagger}\gamma^0(\not{p}_3+Mc)\right) = Tr\left(\gamma_{\mu}(\not{p}_4-Mc)\gamma_{\nu}(\not{p}_3+Mc)\right)$$

$$\sum_{\textit{all spins}}\left[\bar{v}(2)\gamma^{\mu}u(1)\right]\left[\bar{v}(2)\gamma^{\nu}u(1)\right]^* = Tr\left(\gamma^{\mu}(\not{p}_1+mc)\gamma^{\nu}(\not{p}_2-mc)\right)$$

where $m{=}m_e,\ M{=}m_\mu$

Remember, we average over incoming spins and sum over outgoing spins:

$$\left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{4}\frac{g^4}{\left(p_1+p_2\right)^4} \underbrace{Tr\left(\gamma^\mu\left(\not p_1 + mc\right)\gamma^\nu\left(\not p_2 - mc\right)\right)Tr\left(\gamma_\mu\left(\not p_4 - Mc\right)\gamma_\nu\left(\not p_3 + Mc\right)\right)}$$

Look at trace part:

$$= \left[Tr\left(\gamma^\mu \not p_1 \gamma^\nu \not p_2\right) - \left(mc\right)^2 Tr\left(\gamma^\mu\gamma^\nu\right)\right]\left[Tr\left(\gamma_\mu \not p_4 \gamma_\nu \not p_3\right) - \left(Mc\right)^2 Tr\left(\gamma_\mu\gamma_\nu\right)\right]$$

since terms with three γ matrices have Trace = 0 [Rule 10] and Tr($\gamma^\mu\gamma^\nu$)=4$g^{\mu\nu}$

$$\not p_1 = p_{1\lambda}\gamma^\lambda \quad \not p_2 = p_{1\sigma}\gamma^\sigma \qquad\qquad \not p_4 = p_{4\lambda}\gamma^\lambda \quad \not p_3 = p_{3\sigma}\gamma^\sigma$$

$$= \left[p_{1\lambda}p_{2\sigma}\underbrace{Tr\left(\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^\sigma\right)}_{\text{rule 13}} - 4\left(mc\right)^2 g^{\mu\nu}\right]\left[p^{4\lambda}p^{3\sigma}\underbrace{Tr\left(\gamma_\mu\gamma_\lambda\gamma_\nu\gamma_\sigma\right)}_{\text{rule 13}} - 4\left(Mc\right)^2 g_{\mu\nu}\right]$$

$$= \left[4p_{1\lambda}p_{2\sigma}\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\nu}g^{\lambda\sigma} + g^{\mu\sigma}g^{\lambda\nu}\right) - 4\left(mc\right)^2 g^{\mu\nu}\right]\left[4p^{4\lambda}p^{3\sigma}\left(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\nu}g_{\lambda\sigma} + g_{\mu\sigma}g_{\lambda\nu}\right) - 4\left(Mc\right)^2 g_{\mu\nu}\right]$$

$$= 16\left[p_1^\mu p_2^\nu - \left(p_1 \cdot p_2\right)g^{\mu\nu} + p_2^\mu p_1^\nu - \left(mc\right)^2 g^{\mu\nu}\right]\left[p_{4\mu}p_{3\nu} - \left(p_4 \cdot p_3\right)g_{\mu\nu} + p_{3\mu}p_{4\nu} - \left(Mc\right)^2 g_{\mu\nu}\right]$$

$$16\left[p_1^\mu p_2^\nu-\left(p_1\cdot p_2\right)g^{\mu\nu}+p_2^\mu p_1^\nu-\left(mc\right)^2 g^{\mu\nu}\right]\left[p_{4\mu}p_{3\nu}-\left(p_4\cdot p_3\right)g_{\mu\nu}+p_{3\mu}p_{4\nu}-\left(Mc\right)^2 g_{\mu\nu}\right]$$

$$\begin{aligned}=16\Big[\quad &\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)-\left(p_1\cdot p_2\right)\left(p_4\cdot p_3\right)+\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)-\left(p_1\cdot p_2\right)\left(Mc\right)^2\\
&-\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)+4\left(p_1\cdot p_2\right)\left(p_4\cdot p_3\right)-\left(p_1\cdot p_2\right)\left(p_3\cdot p_4\right)+4\left(p_1\cdot p_2\right)\left(Mc\right)^2\\
&+\left(p_2\cdot p_4\right)\left(p_1\cdot p_3\right)-\left(p_1\cdot p_2\right)\left(p_4\cdot p_3\right)+\left(p_2\cdot p_3\right)\left(p_1\cdot p_4\right)-\left(p_1\cdot p_2\right)\left(Mc\right)^2\\
&-\left(mc\right)^2\left(p_4\cdot p_3\right)+4\left(mc\right)^2\left(p_4\cdot p_3\right)-\left(mc\right)^2\left(p_3\cdot p_4\right)+4\left(mc\right)^2\left(Mc\right)^2\quad\Big]\end{aligned}$$

$$=16\left[2\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+2\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)+2\left(Mc\right)^2\left(p_1\cdot p_2\right)+2\left(mc\right)^2\left(p_3\cdot p_4\right)+4\left(mc\right)^2\left(Mc\right)^2\right]$$

$$=32\left[\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)+\left(Mc\right)^2\left(p_1\cdot p_2\right)+\left(mc\right)^2\left(p_3\cdot p_4\right)+2\left(mc\right)^2\left(Mc\right)^2\right]$$

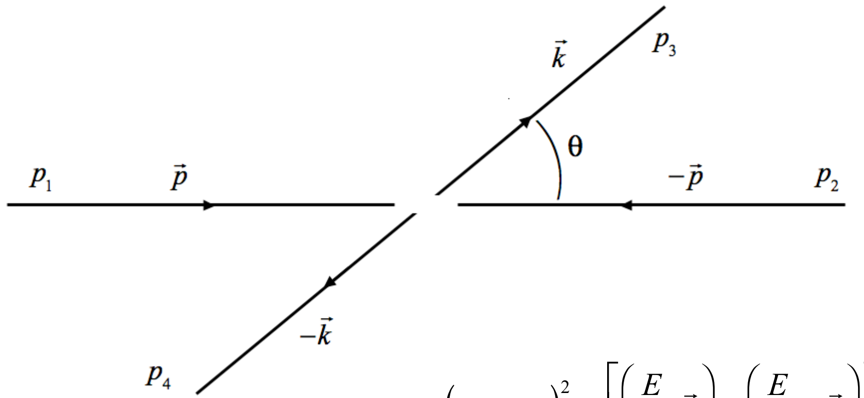
$$\left\langle\left|\mathcal{M}\right|^2\right\rangle=\frac{32}{4}\frac{g^4}{\left(p_1+p_2\right)^4}\left[\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)+\left(Mc\right)^2\left(p_1\cdot p_2\right)+\left(mc\right)^2\left(p_3\cdot p_4\right)+2\left(mc\right)^2\left(Mc\right)^2\right]$$

$$\left\langle\left|\mathcal{M}\right|^2\right\rangle=8\left(\frac{g^2}{\left(p_1+p_2\right)^2}\right)^2\left[\left(p_1\cdot p_3\right)\left(p_2\cdot p_4\right)+\left(p_1\cdot p_4\right)\left(p_2\cdot p_3\right)+\left(Mc\right)^2\left(p_1\cdot p_2\right)+\left(mc\right)^2\left(p_3\cdot p_4\right)+2\left(mc\right)^2\left(Mc\right)^2\right]$$

Compare eqn. 7.129 for $e^-\mu^-$ → $e^-\mu^-$ with $q^2=(p_1+p_2)^2$ and p_2 swapped with p_3.

Look at $e^+e^- \rightarrow \mu^+\mu^-$ in the CM Frame

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\left(\frac{g^2}{(p_1+p_2)^2}\right)^2 \left[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)+(Mc)^2(p_1\cdot p_2)+(mc)^2(p_3\cdot p_4)+2(mc)^2(Mc)^2\right]$$



Note that $|\vec{p}| \neq |\vec{k}|$; this is not elastic scattering. However, $E_e = E_\mu \equiv E$.

$$(p_1+p_2)^2 = \left[\left(\frac{E}{c},\vec{p}\right)+\left(\frac{E}{c},-\vec{p}\right)\right]^2 = \left(\frac{2E}{c}\right)^2 = \frac{4E^2}{c^2}$$

$$(p_1\cdot p_2) = \left(\frac{E}{c},\vec{p}\right)\cdot\left(\frac{E}{c},-\vec{p}\right) = \frac{E^2}{c^2}+p^2 \quad \left[\text{with } p^2 \equiv |\vec{p}|^2\right]$$

$$(p_3\cdot p_4) = \left(\frac{E}{c},\vec{k}\right)\cdot\left(\frac{E}{c},-\vec{k}\right) = \frac{E^2}{c^2}+k^2 \quad \left[\text{with k}^2 \equiv |\vec{k}|^2\right]$$

$$(p_1\cdot p_4) = \left(\frac{E}{c},\vec{p}\right)\cdot\left(\frac{E}{c},-\vec{k}\right) = \frac{E^2}{c^2}+\vec{p}\cdot\vec{k} = \frac{E^2}{c^2}+pk\cos\theta$$

$$(p_2\cdot p_3) = \left(\frac{E}{c},-\vec{p}\right)\cdot\left(\frac{E}{c},\vec{k}\right) = \frac{E^2}{c^2}+\vec{p}\cdot\vec{k} = \frac{E^2}{c^2}+pk\cos\theta$$

$$(p_1\cdot p_3) = \left(\frac{E}{c},\vec{p}\right)\cdot\left(\frac{E}{c},\vec{k}\right) = \frac{E^2}{c^2}-pk\cos\theta$$

$$(p_2\cdot p_4) = \left(\frac{E}{c},-\vec{p}\right)\cdot\left(\frac{E}{c},-\vec{k}\right) = \frac{E^2}{c^2}-pk\cos\theta$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = 8\left(\frac{g^2}{(p_1+p_2)^2}\right)^2 \left[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)+(Mc)^2(p_1\cdot p_2)+(mc)^2(p_3\cdot p_4)+2(mc)^2(Mc)^2\right]$$

$$= 8\frac{g^4c^4}{16E^4}\left[\left(\frac{E^2}{c^2}-pk\cos\theta\right)^2+\left(\frac{E^2}{c^2}+pk\cos\theta\right)^2+(mc)^2\left(\frac{E^2}{c^2}+k^2\right)+(Mc)^2\left(\frac{E^2}{c^2}+p^2\right)+2(mc)^2(Mc)^2\right]$$

$$= \frac{1}{2}\frac{g^4}{(E/c)^4}\left[\frac{2E^4}{c^4}+2p^2k^2\cos^2\theta+(mc)^2\frac{E^2}{c^2}+(mc)^2k^2+(Mc)^2\frac{E^2}{c^2}+(Mc)^2p^2+2(mc)^2(Mc)^2\right]$$

$$= \frac{1}{2}\frac{g^4}{(E/c)^4}\left[\frac{2E^4}{c^4}+2\left(\frac{E^2}{c^2}-m^2c^2\right)\left(\frac{E^2}{c^2}-M^2c^2\right)\cos^2\theta+(mc)^2\frac{E^2}{c^2}+(mc)^2\left(\frac{E^2}{c^2}-M^2c^2\right)\right.$$
$$\left.+(Mc)^2\frac{E^2}{c^2}+(Mc)^2\left(\frac{E^2}{c^2}-m^2c^2\right)+2(mc)^2(Mc)^2\right]$$

$$= \frac{1}{2}\frac{g^4}{(E/c)^4}\left[\frac{2E^4}{c^4}+2\left(\frac{E^4}{c^4}-\frac{E^2}{c^2}m^2c^2-\frac{E^2}{c^2}M^2c^2+(mc)^2(Mc)^2\right)\cos^2\theta\right.$$
$$+(mc)^2\frac{E^2}{c^2}+(mc)^2\frac{E^2}{c^2}\cancel{-(mc)^2(Mc)^2}+(Mc)^2\frac{E^2}{c^2}$$
$$\left.+(Mc)^2\frac{E^2}{c^2}\cancel{-(mc)^2(Mc)^2}\cancel{+2(mc)^2(Mc)^2}\right]$$

$$\left\langle |\mathcal{M}|^2 \right\rangle = g^4 \left\{ 1+\cos^2\theta - \frac{(mc)^2\cos^2\theta}{(E/c)^2} - \frac{(Mc)^2\cos^2\theta}{(E/c)^2} + \frac{(mc)^2(Mc)^2}{(E/c)^4} + \frac{(mc)^2}{(E/c)^2} + \frac{(Mc)^2}{(E/c)^2} \right\}$$

$$= g^4 \left\{ 1 + \frac{(mc^2)^2}{E^2} + \frac{(Mc^2)^2}{E^2} + \left[1 - \frac{(mc^2)^2}{E^2} - \frac{(Mc^2)^2}{E^2} + \frac{(mc^2)^2(Mc^2)^2}{E^4} \right]\cos^2\theta \right\}$$

$$= g^4 \left\{ 1 + \frac{(mc^2)^2}{E^2} + \frac{(Mc^2)^2}{E^2} + \left[1 - \left(\frac{mc^2}{E}\right)^2 \right]\left[1 - \left(\frac{Mc^2}{E}\right)^2 \right]\cos^2\theta \right\}$$ [Griffiths 8.4]

Using $\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\left\langle |\mathcal{M}|^2 \right\rangle}{(2E)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}$ [Griffiths 6.42] and $\frac{|\vec{p}_f|}{|\vec{p}_i|} = \frac{\sqrt{1-(Mc^2/E)^2}}{\sqrt{1-(mc^2/E)^2}}$ [basic kinematics] try it

yields the differential cross-section. For the total cross-section, need to integrate over θ, φ.

Integrate $\left\langle |\mathcal{M}|^2 \right\rangle$ (from above) since that contains all the angular dependence:

$$= 2\pi g^4 \int \left\{ 1 + \frac{(mc^2)^2}{E^2} + \frac{(Mc^2)^2}{E^2} + \left[1 - \left(\frac{mc^2}{E}\right)^2 \right]\left[1 - \left(\frac{Mc^2}{E}\right)^2 \right]\cos^2\theta \right\} \sin\theta\, d\theta$$

$$= 2\pi g^4 \left\{ \left(1 + \frac{\left(mc^2\right)^2}{E^2} + \frac{\left(Mc^2\right)^2}{E^2} \right) \int \sin\theta d\theta + \left[1 - \left(\frac{mc^2}{E} \right)^2 \right] \left[1 - \left(\frac{Mc^2}{E} \right)^2 \right] \underbrace{\int \cos^2\theta \sin\theta d\theta}_{= -\left.\frac{\cos^3\theta}{3}\right|_0^{\pi} = \frac{2}{3}} \right\}$$

$$= 2\pi g^4 \left\{ 2\left(1 + \frac{\left(mc^2\right)^2}{E^2} + \frac{\left(Mc^2\right)^2}{E^2} \right) + \frac{2}{3} \left[1 - \left(\frac{mc^2}{E} \right)^2 \right] \left[1 - \left(\frac{Mc^2}{E} \right)^2 \right] \right\}$$

$$= 4\pi g^4 \left\{ \left(1 + \frac{\left(mc^2\right)^2}{E^2} + \frac{\left(Mc^2\right)^2}{E^2} \right) + \frac{1}{3} \left[1 - \left(\frac{mc^2}{E} \right)^2 \right] \left[1 - \left(\frac{Mc^2}{E} \right)^2 \right] \right\}$$

$$\sigma = \left(\frac{\hbar c}{8\pi} \right)^2 \frac{1}{4E^2} 4\pi g^4 \left\{ \left(1 + \frac{\left(mc^2\right)^2}{E^2} + \frac{\left(Mc^2\right)^2}{E^2} \right) + \frac{1}{3} \left[1 - \left(\frac{mc^2}{E} \right)^2 \right] \left[1 - \left(\frac{Mc^2}{E} \right)^2 \right] \right\} \frac{\left|\vec{p}_f\right|}{\left|\vec{p}_i\right|}$$

$$= \left(\frac{\hbar c}{8\pi} \right)^2 \frac{\pi}{3E^2} g^4 \left\{ \left(3 + 3\frac{\left(mc^2\right)^2}{E^2} + 3\frac{\left(Mc^2\right)^2}{E^2} \right) + \left[1 - \left(\frac{mc^2}{E} \right)^2 \right] \left[1 - \left(\frac{Mc^2}{E} \right)^2 \right] \right\} \frac{\sqrt{1 - \left(Mc^2/E\right)^2}}{\sqrt{1 - \left(mc^2/E\right)^2}}$$

$$=\left(\frac{\hbar c}{8\pi}\right)^2 \frac{\pi}{3E^2} g^4 \left\{\left(3+3\frac{\left(mc^2\right)^2}{E^2}+3\frac{\left(Mc^2\right)^2}{E^2}\right)+\left[1-\left(\frac{mc^2}{E}\right)^2\right]\left[1-\left(\frac{Mc^2}{E}\right)^2\right]\right\}\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$=\left(\frac{\hbar c}{8\pi}\right)^2 \frac{\pi}{3E^2} g^4 \left\{\left(3+3\frac{\left(mc^2\right)^2}{E^2}+3\frac{\left(Mc^2\right)^2}{E^2}\right)+1-\left(\frac{mc^2}{E}\right)^2-\left(\frac{Mc^2}{E}\right)^2+\left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right\}\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$=\left(\frac{\hbar c}{8\pi}\right)^2 \frac{\pi}{3E^2} g^4 \left\{4+2\left(\frac{mc^2}{E}\right)^2+2\left(\frac{Mc^2}{E}\right)^2+\left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right\}\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$=\left(\frac{\hbar c}{8\pi}\right)^2 \frac{4\pi}{3E^2} g^4 \left\{1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2+\frac{1}{4}\left(\frac{mc^2}{E}\right)^2\left(\frac{Mc^2}{E}\right)^2\right\}\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$=\left(\frac{\hbar c}{8\pi}\right)^2 \frac{4\pi}{3E^2} g^4 \left\{\left(1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right)\right\}\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{4\pi}{3E^2} g^4 \left\{ \left(1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right)\right\} \frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

Using g^2=4$\pi\alpha$ this becomes

$$\sigma = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{4\pi}{3E^2} (4\pi\alpha)^2 \left\{ \left(1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right)\right\} \frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$= \frac{1}{64\pi^2} \frac{\hbar^2 c^2 4\pi}{3E^2} (4\pi\alpha)^2 \left\{ \left(1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right)\right\} \frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}$$

$$= \left(\frac{\hbar c\alpha}{E}\right)^2 \frac{\pi}{3} \left\{ \left(1+\frac{1}{2}\left(\frac{mc^2}{E}\right)^2\right)\left(1+\frac{1}{2}\left(\frac{Mc^2}{E}\right)^2\right)\right\} \underbrace{\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}}$$

Note that this factor represents an energy threshold at $E=Mc^2$.

[like Griffiths 8.5]

Note that for differently charged particles in the final state, instead of g^4 (or α^2), we have Q^2g^4 where Q is the charge of the particle (e.g. instead of a factor of g from each vertex, get a factor of g from the electron vertex and a factor of Qg from the final-state vertex: this is squared in the cross-section).

$$\sigma\left(e^+e^- \to f\bar{f}\right) = \left(\frac{\hbar c Q_f \alpha}{E}\right)^2 \frac{\pi}{3}\left\{\left(1+\frac{1}{2}\cancel{\left(\frac{mc^2}{E}\right)^2}\right)\left(1+\frac{1}{2}\cancel{\left(\frac{Mc^2}{E}\right)^2}\right)\right\}\cancel{\frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}}}$$ [as before $f = u,d,s,c,b,t,\mu,\tau$]

Consider the high energy limit, $E >> Mc^2$. In that case we obtain $\sigma\left(e^+e^- \to f\bar{f}\right) = \frac{\pi}{3}\left(\frac{\hbar c Q_f \alpha}{E}\right)^2$.

Now consider the ratio of the cross-section for the production of hadrons in e^+e^- collisions relative to the cross-section for producing a muon pair:

$$R(E) = \frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)}$$

This is a function of energy because the quarks that are kinematically accessible (for the process in the numerator) depends on the energy. As the energy rises we pass various thresholds for the production of new quark-antiquark pairs in the final state.

$$R(E) = 3\sum_{f=u,d,s,c,b,t} Q_f^2$$

Here we need to sum over all kinematically accessible quark-antiquark final states. Where does factor of three come from?

Consider CM energies around 2 GeV where only u,d,s can contribute:

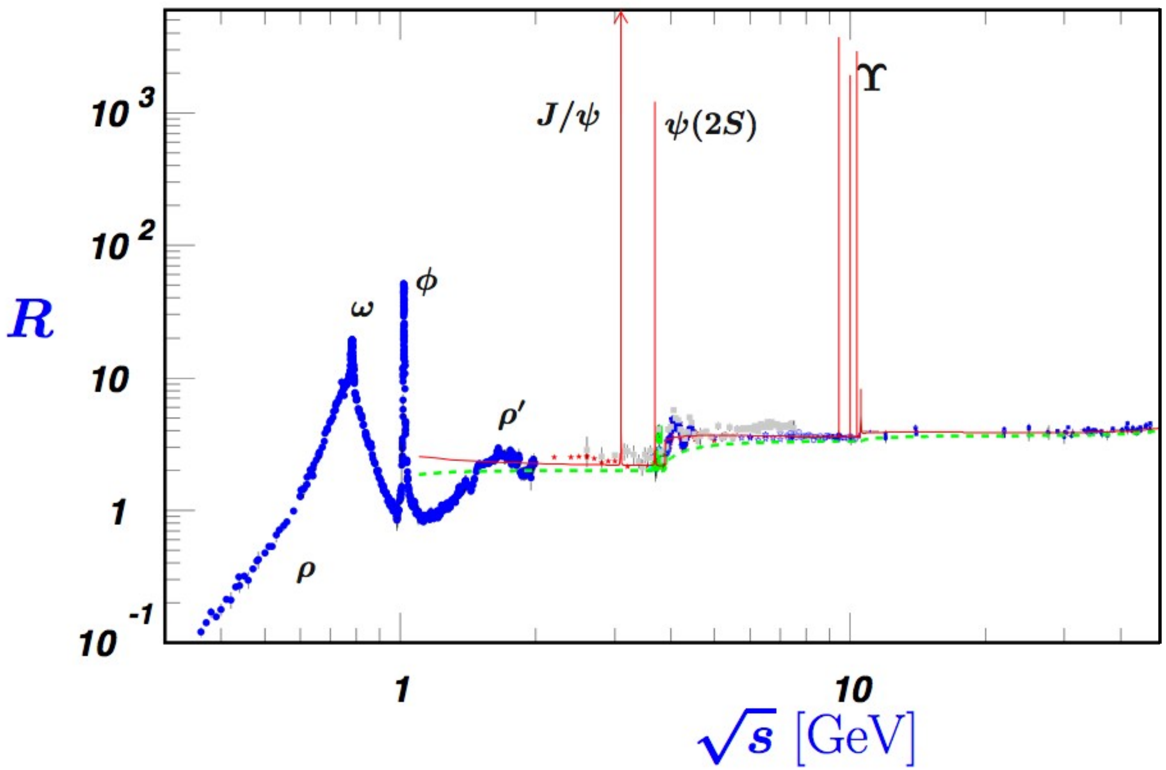
$$R = 3\sum_{f=u,d,s} Q_f^2 = 3\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2$$

At CM energies around 4 GeV we are above the threshold for charm quark$\left(e.g\ c\overline{c}\right)$production:

$$R = 3\sum_{f=u,d,s,c} Q_f^2 = 3\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right] = \frac{10}{3}$$

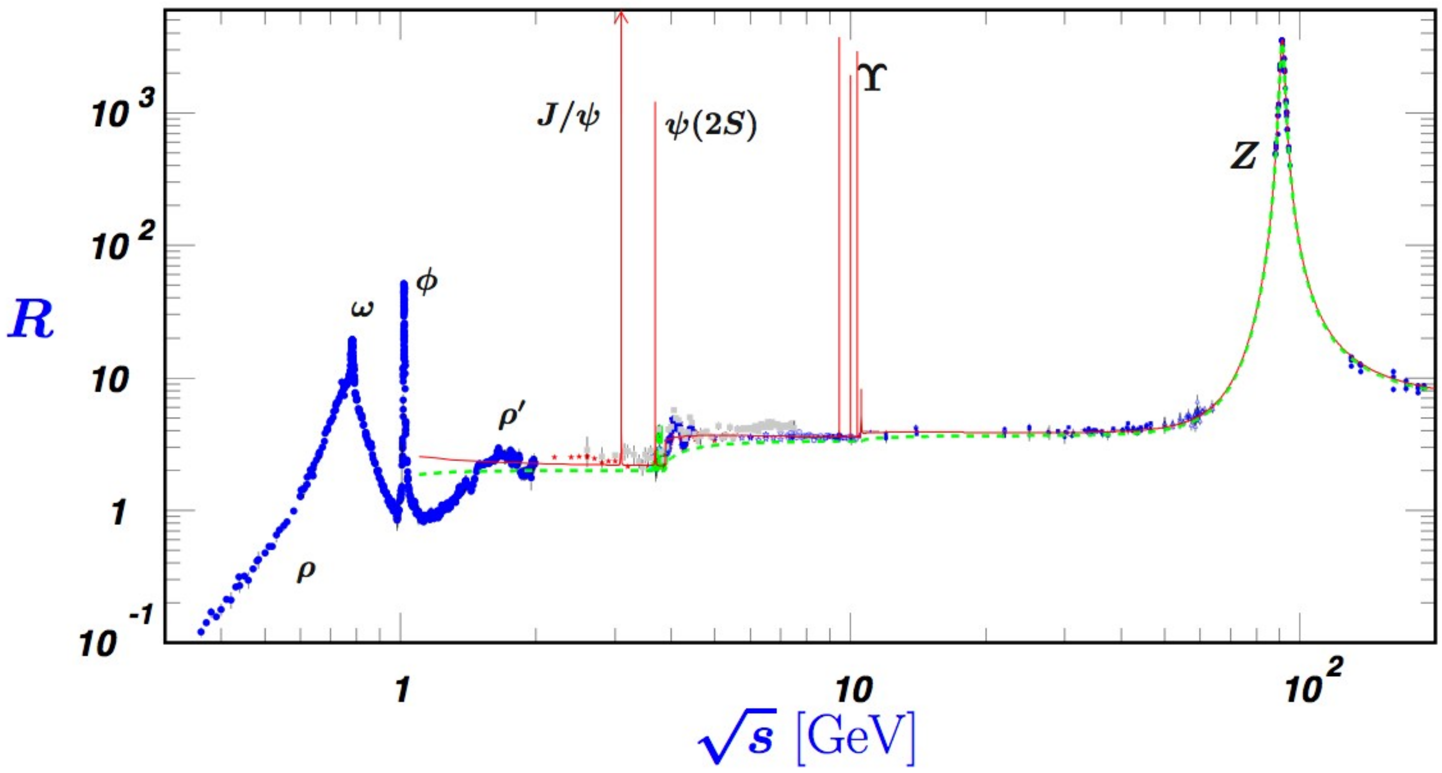
At CM energies around 10 GeV we are above the threshold for b quark production:

$$R = 3\sum_{f=u,d,s,c} Q_f^2 = 3\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = \frac{11}{3}$$



10^3
10^2
10
1
10^{-1}
R
J/ψ
$\psi(2S)$
Υ
ω
ϕ
ρ'
ρ
1
10
$\sqrt{s}$ [GeV]

Z^0 propagator is $$-i\frac{\left(g_{\mu\nu}-q_\mu q_\nu / M_Z^2c^2\right)}{q^2-M_Z^2c^2}$$



For E >> mc^2 the denominator (for the Z contribution) begins to look like E^2-$(M_Z)^2c^2$ which blows up at E = $(M_Z)c^2$. We will see that this "pole" does not cause problems since M_Z has a width associated with it. We will see how this is handled. The photon propagator goes like ~ q^{-2} so becomes small at very high energies. However, at very very high energies (that is, for q^2 >> $(M_Z)^2$) the contributions from the photon and the Z become similar.

Angular Distributions in $e^+e^- \to f\,\overline{f}$

A few slides back, we had:

$$\frac{d\sigma}{d\Omega}\left(e^+e^- \to f\,\overline{f}\right) = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{\left\langle\left|\mathcal{M}\right|^2\right\rangle}{\left(2E\right)^2}\frac{\left|\vec{p}_f\right|}{\left|\vec{p}_i\right|} \quad \text{with} \quad \frac{\left|\vec{p}_f\right|}{\left|\vec{p}_i\right|} = \frac{\sqrt{1-\left(Mc^2/E\right)^2}}{\sqrt{1-\left(mc^2/E\right)^2}} \quad \text{and}$$

$$\left\langle\left|\mathcal{M}\right|^2\right\rangle = g^4 \underbrace{\left\{1+\frac{\left(mc^2\right)^2}{E^2}+\frac{\left(Mc^2\right)^2}{E^2}+\left[1-\left(\frac{mc^2}{E}\right)^2\right]\left[1-\left(\frac{Mc^2}{E}\right)^2\right]\cos^2\theta\right\}}_{=1+\cos^2\theta \text{ in the high-energy limit}}$$

So the angular distribution of the final state (spin-1/2) fermions should be of the form $1+\cos^2\theta$ at energies well above the relevant fermion mass.

Measurements of these angular distributions test the predictions of QED.

Angular Distributions in $e^+e^- \to f\bar{f}$

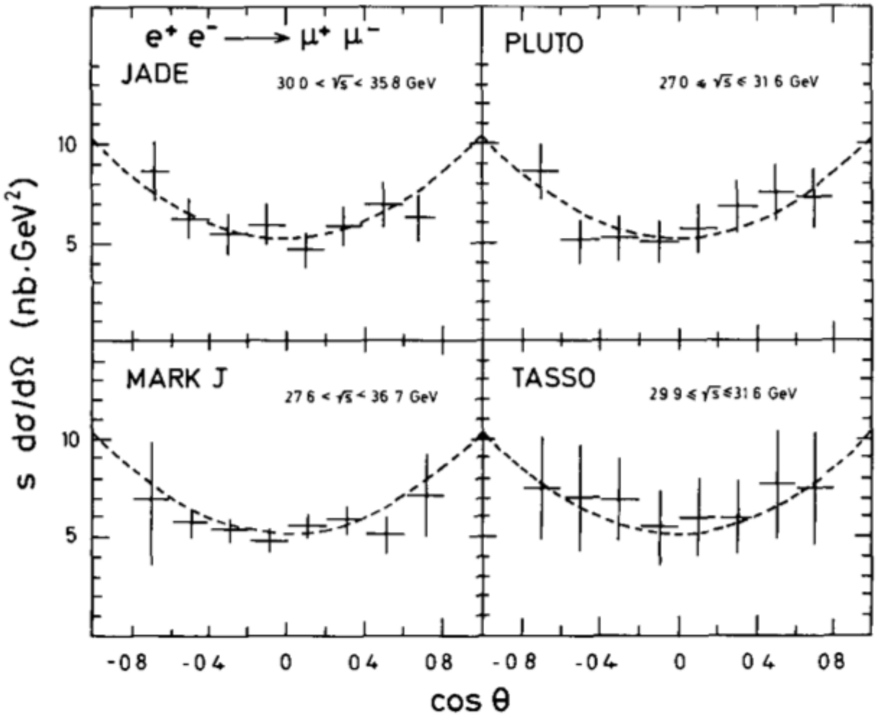


Fig. 9. Differential cross section for the reaction $e^+e^- \to \mu^+\mu^-$. The dashed curves show the QED prediction. The data are corrected for radiative effects and hadronic vacuum polarization

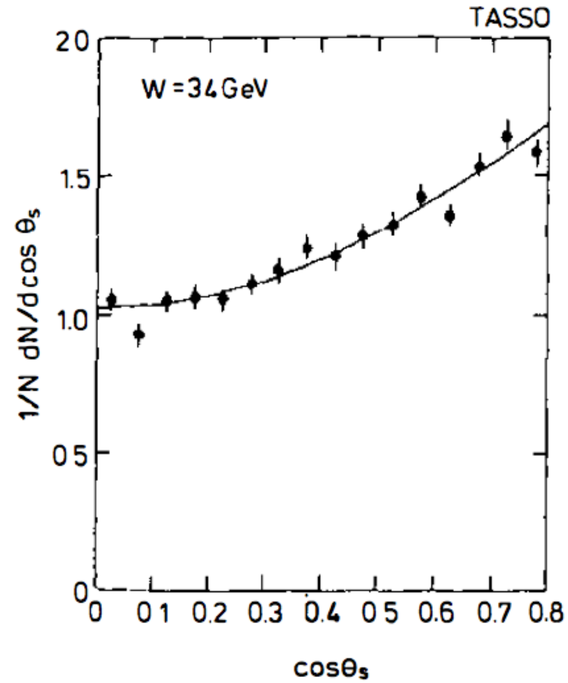
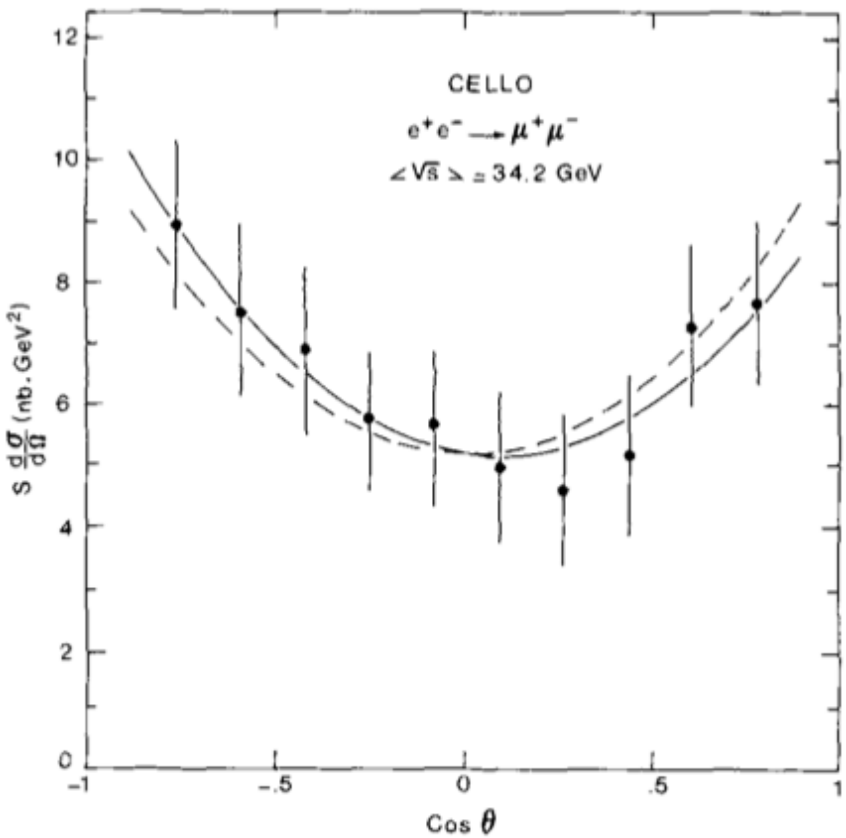


Figure 8 Angular distributions of jets in e^+e^- annihilation measured by TASSO at PETRA, compared with the expected angular distribution for spin-1/2 quarks $(1+\cos^2\theta)$.

Caveat

There's an important caveat here. Experimentally one cannot just measure the part of the interaction that is due to QED. As energies increase, the weak interaction also contributes:



This introduces a so-called forward-backward asymmetry in the angular distribution.

Dotted line is pure QED (symmetric).
Solid line is electroweak theory, for which the predicted angular distribution is asymmetric. We will see the reason for this when we discuss the weak interaction.

Fig. 3. Differential cross section with respect to the polar angle for $\langle\sqrt{S}\rangle = 34.2$ GeV, corrected for acceptance, trigger efficiency, Bhabha contamination and QED asymmetry contribution. The following cuts are applied: momentum >8 GeV and acolinearity ≤ 10 degrees. A fit of the data to the function

$$f(\cos(\theta)) = p(1+\cos^2(\theta)+q\cos(\theta))$$

is superimposed (full line). The QED prediction is also shown (dashed line)