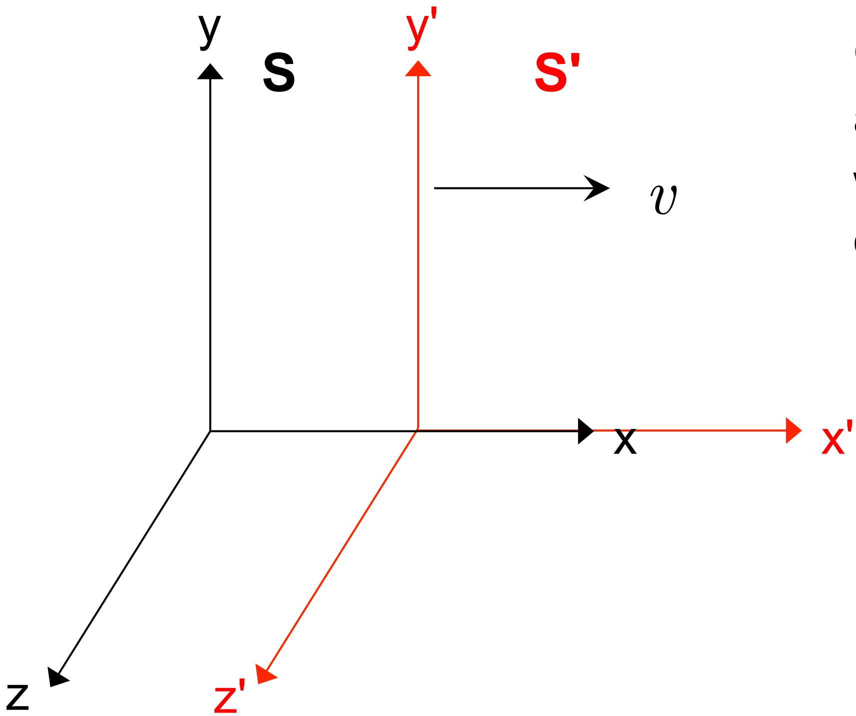
Phy489 Lecture 4

Special Relativity: Lorentz Transformations



not accelerating

Consider two *inertial* reference frames S and S' having a relative velocity v (which we will align along the x axis). If an event occurs in S at space-time coordinates (t,x,y,z) then in S' we have:

$$x' = \gamma(x - vt) \qquad\qquad x = \gamma(x' + vt')$$

$$y' = y \qquad\qquad y = y'$$

$$z' = z \qquad \textbf{OR} \qquad z = z'$$

$$t' = \gamma(t - \frac{v}{c^2}x) \qquad\qquad t = \gamma(t' + \frac{v}{c^2}x')$$

In each case the Lorentz "boost" factor γ is given by $\gamma \equiv \dfrac{1}{\sqrt{1 - v^2/c^2}}$

Usual to also define dimensionless velocity $\beta \equiv v/c \quad \gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

Problem 3.1 (derives the S'→S expressions)

Starting with the expression for the S→ S' transformation

$$x' = \gamma(x - \upsilon t)$$

$$y' = y$$

$$z' = z$$

These two are trivial

$$t' = \gamma(t - \frac{\upsilon}{c^2}x)$$

$$x' = \gamma x - \gamma \upsilon t \quad \rightarrow \quad x = \frac{x'}{\gamma} + \upsilon t \quad = \quad \frac{x'}{\gamma} + \upsilon\left(\frac{t'}{\gamma} + \frac{\upsilon}{c^2}x\right)$$

$$t' = \gamma t - \gamma \frac{\upsilon}{c^2}x \quad \rightarrow \quad t = \frac{t'}{\gamma} + \frac{\upsilon}{c^2}x \quad = \quad \frac{t'}{\gamma} + \frac{\upsilon}{c^2}\left(\frac{x'}{\gamma} + \upsilon t\right)$$

collecting *x* terms on the LHS……….

$$x\left(1 - \frac{\upsilon^2}{c^2}\right) = \frac{x'}{\gamma} + \frac{\upsilon t'}{\gamma} \qquad \frac{x}{\gamma^2} = \frac{x'}{\gamma} + \frac{\upsilon t'}{\gamma} \quad \rightarrow \quad x = \gamma(x' + \upsilon t')$$

Similarly for *t*

Consequences of Special Relativity

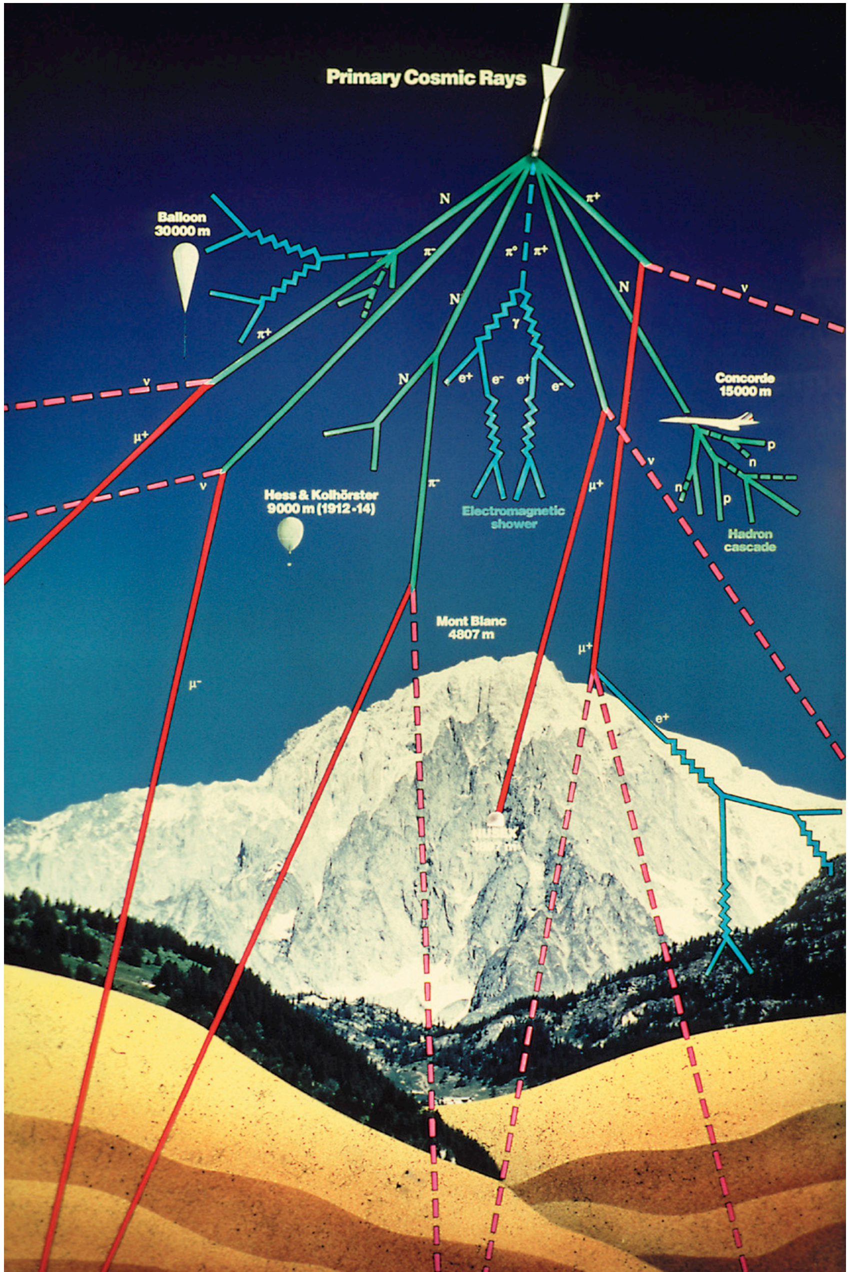
- Relativity of simultaneity: events that are simultaneous in one reference frame are NOT necessarily simultaneous in another.
- Lorentz contraction (along direction of motion): an object of length L’ (in reference frame S’ frame) appears shorter by a factor of γ to an observer in S. L = L’/γ, where $\gamma > 1$.
- Time dilation: time runs more slowly in a moving reference frame. That is, a clock in S’ appears (to an observer in S) to be running slowly, by a factor of γ.

Only one of these effects is important for us: time dialation.

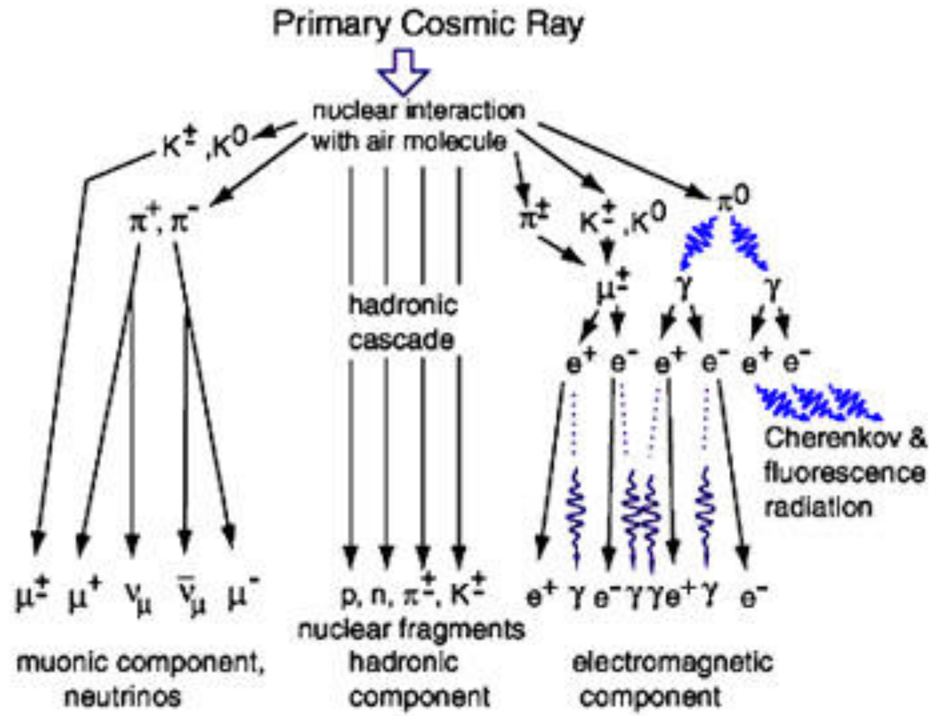
Note that particle lifetimes are defined in their rest frames (there is no other self consistent manner in which to define these - this is the one reference frame upon which all observers can agree). This is called the “proper lifetime” τ. The length of time a particle exists in a reference frame in which it is moving is then given by $\gamma\tau$.

Reference Frames for Scattering and Decays

- We will really deal with only a few situations:
  - Particle decays in the particle rest frame;
  - Particle decays in flight (in the "lab" frame, *i.e.* of the observer);
  - Scattering in the centre-of-mass (CM) frame;
  - Scattering in the lab frame (typically "fixed target" scattering where one of the particles in the initial state is at rest).



Primary Cosmic Rays
Balloon 30000 m
Concorde 15000 m
Hess & Kolhörster 9000 m (1912-14)
Electromagnetic shower
Hadron cascade
Mont Blanc 4807 m



Primary Cosmic Ray
nuclear interaction with air molecule
K±, K0
π+, π-
hadronic cascade
π±
K±, K0
π0
μ±
γ
γ
e+ e- e+ e- e+ e-
Cherenkov & fluorescence radiation
μ± μ+ νμ ν̄μ μ-
p, n, π±, K±
nuclear fragments
e+ γ e- γ γ e+ γ e-
muonic component, neutrinos
hadronic component
electromagnetic component

Simple Illustration: Griffiths Problem 3.4

Cosmic ray muons are produced in the high atmosphere (say 8000*m*) and travel towards the earth at close to the speed of light (say 0.998*c*):

a) Given the lifetime of the muon (τ ~ 2.2μs) would one expect these muons to reach the surface of the earth?

Classically, the muons would (on average) travel a distance

$$d = \left(2.2\times10^{-6}\,s\right)\bullet\left(0.998\right)\bullet(3\times10^{8}\,m/s) = 660\,m$$

So would NOT (in general) make it the 8000*m* to the earth's surface.

Relativistically, the muon lifetime in the observer's rest frame is increased by a factor of γ (=15.8 for v = 0.998*c*), so the distance traveled would be:

$$d = 15.8\bullet\left(2.2\times10^{-6}\,s\right)\bullet\left(0.998\right)\bullet(3\times10^{8}\,m/s) = 10{,}400\,m$$

And one would expect most of the muons to make it to the earths surface (remember, τ represents the mean lifetime, so there is a distribution).

An alternative Point of View

Picture this from the point of view of the decaying muons (so imagine we are in its rest frame):

In this reference frame the muon has a lifetime of 2.2 μ*s*.

However, in this reference frame the distance to the earth is Lorentz contracted by a factor of γ (15.8).

So the particle travels an average of 660*m*, but the distance to the earth is now only 8000*m*/15.8 = 506 *m*, so this is enough to reach the earth.

The conclusion is the same in both reference frames, as it must be.

As a point of interest, v = 0.998*c* corresponds to E_μ ~ 1.7 GeV which is not particularly energetic. So even relatively low-energy particles are typically traveling at close to the speed of light.

[$E_\mu = \gamma m_\mu c^2$ ~ 15.8 (105.6 MeV) ~ 1.7 GeV]

Four Vectors

Vectors in coordinate space are defined as three component objects with particular transformation properties under coordinate transformations.

In special relativity, we are dealing with 4-dimensional space-time. We define four vectors to be four component objects having particular transformation properties under space-time coordinate transformations (Lorentz transformations).

Consider the position-time four vector $$x^\mu = (x^0, x^1, x^2, x^3) \equiv \underbrace{(ct, x, y, z)}$$

Components must have the same units

Re-write the Lorentz transformation in this notation:

again, for velocity along x (here x^1).

$$(x^0)' = \gamma(x^0 - \beta x^1)$$

$$(x^1)' = \gamma(x^1 - \beta x^0)$$

$$(x^2)' = x^2$$

$$(x^3)' = x^3$$

Note the symmetry when the equations are written in this form

More on notation

Can write this as $$(x^\mu)' = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \qquad \mu = 0,1,2,3$$

2nd rank tensor

Can write Λ in 4x4 matrix form
(here for a Lorentz transformation along the x axis)

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Write this in more compact form using the Einstein summation convention in which summation over repeated indices is assumed:

$$(x^\mu)' = \Lambda^\mu_\nu x^\nu$$ (sum over ν assumed)

Lorentz Invariants

One of the most critical concepts in relativistic kinematics is that of relativistic invariants (Lorentz invariants) which are quantities that are the same regardless of which reference frame they are evaluated in.

Consider the quantity $I = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$

Re-write this in terms of the coordinates of the S' reference frame

$$
\begin{aligned}
I &= (\gamma x^{0'} + \gamma\beta x^{1'})^2 - (\gamma x^{1'} + \gamma\beta x^{0'})^2 - (x^{2'})^2 - (x^{3'})^2 \\
&= \gamma^2 (x^{0'})^2 + \gamma^2\beta^2 (x^{1'})^2 \boxed{+ 2\gamma^2\beta (x^{0'})(x^{1'})} \quad \text{cancel} \\
&\quad - \gamma^2 (x^{1'})^2 - \gamma^2\beta^2 (x^{0'})^2 \boxed{- 2\gamma^2\beta (x^{0'})(x^{1'})} - (x^{2'})^2 - (x^{3'})^2 \\
&= \gamma^2 (1-\beta^2)(x^{0'})^2 - \gamma^2 (1-\beta^2)(x^{1'})^2 - (x^{2'})^2 - (x^{3'})^2 \\
&= (x^{0'})^2 - (x^{1'})^2 - (x^{2'})^2 - (x^{3'})^2 = I'
\end{aligned}
$$

reminder

$$\boxed{\gamma^2 (1-\beta^2) = \frac{1}{(1-\beta^2)}(1-\beta^2) = 1}$$

$I = I'$ Same result in either reference frame. I is a Lorentz invariant

Notation

Calculations in relativistic kinematics rely heavily on the use of Lorentz invariant quantities, as we shall see.

To write this invariant in more compact form we define the metric $g_{\mu\nu}$.
This is a second-rank tensor that can be represented by a 4x4 matrix:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$I = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu} x^{\mu} x^{\nu} = g_{\mu\nu} x^{\mu} x^{\nu}$$

Summation convention

x^{μ} is referred to as the contravariant position-time four vector.

Using the metric $g_{\mu\nu}$ define the covariant position-time four-vector: $x_{\mu} = g_{\mu\nu} x^{\nu}$

We can then write our invariant compactly as $I = x_{\mu} x^{\mu}$.

We will use this notation extensively

Other Four Vectors

x^μ is the archetypal four-vector.

Any four-component object a^μ is called a four-vector if it transforms like x^μ under Lorentz transformations.

$$(a^\mu)' = \Lambda^\mu_\nu a^\nu \quad \text{With } \Lambda \text{ as defined earlier } = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a_\mu = g_{\mu\nu} a^\nu$$

$$a^\mu = g^{\mu\nu} a_\nu$$

Technically, these are the elements of g^{-1}, but in this case $g^{-1} = g$.

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Scalar Products

For any two four-vectors a^{μ} and b^{μ} the product $a_{\mu}b^{\mu}$ is INVARIANT.

You should prove this to yourself

This is referred to as the scalar product of a and b.

$$a \bullet b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a_{\mu} b^{\mu} = a^0 b^0 - \vec{a} \bullet \vec{b}$$

$$a = (a^0, \vec{a}) \qquad b = (b^0, \vec{b})$$

$$a^2 = a \bullet a = a_{\mu} a^{\mu} = (a^0)^2 - |\vec{a}|^2$$

$$a^{\mu} \begin{cases} \text{is timelike if} & a^2 > 0 \\ \text{is spacelike if} & a^2 < 0 \\ \text{is lightlike if} & a^2 = 0 \end{cases}$$

Energy-momentum four-vectors

Read text (section 3.3) for description of “proper velocity”, which is used in the development of the energy-momentum four-vector:

$$\vec{\eta} = \gamma \vec{\mathrm{v}} \equiv \frac{\mathrm{d}\vec{\mathrm{x}}}{\mathrm{d}\tau}$$

proper time

This forms part of a four vector $\eta^{\mu} = \left(\eta^{0}, \vec{\eta}\right)$ with $\eta^{0} = \gamma c$.

$$\eta_{\mu}\eta^{\mu} = \gamma^2 (c^2 - \mathrm{v}_\mathrm{x}^2 - \mathrm{v}_\mathrm{y}^2 - \mathrm{v}_\mathrm{z}^2) = \gamma^2 c^2 (1 - \mathrm{v}^2 / c^2) = c^2$$

(which is invariant)

Define the relativistic three momentum as $\vec{p} = m\vec{\eta}$ (in analogy to $\vec{p} = m\vec{\mathrm{v}}$)

Can also show this to be part of a four-vector: $p^{\mu} = m\eta^{\mu} = (\gamma mc, \gamma m\vec{\mathrm{v}})$

The relativistic energy is $E = \gamma mc^2$ where m is the rest mass of the particle.

And so we may write: $p^{\mu} = \left(\frac{E}{c}, p_x, p_y, p_z\right)$ $\quad p_{\mu}p^{\mu} = \frac{E^2}{c^2} - \left|\vec{p}\right|^2 = m^2 \eta_{\mu}\eta^{\mu} = m^2 c^2$

which is invariant

More on relativistic energy and momentum

Note that the expression $E = \gamma mc^2$ applies only to massive particles.

In the case of massless particles (photons for instance) $E = |\vec{p}|c$

Recall that for photons $E = h\nu = \dfrac{hc}{\lambda}$

The (de Broglie) wavelength associated to a particle with momentum p is given by

$$\lambda = \frac{h}{p} \quad \Rightarrow \quad E = h\nu = \frac{hc}{\lambda} = \frac{hc}{h/p} = pc = |\vec{p}|c$$

Collisions (in a generic form)

Discuss different classes of collisions: look classically, then relativistically:

Consider the process $A+B \rightarrow C+D$ *(or $AB \rightarrow CD$)*

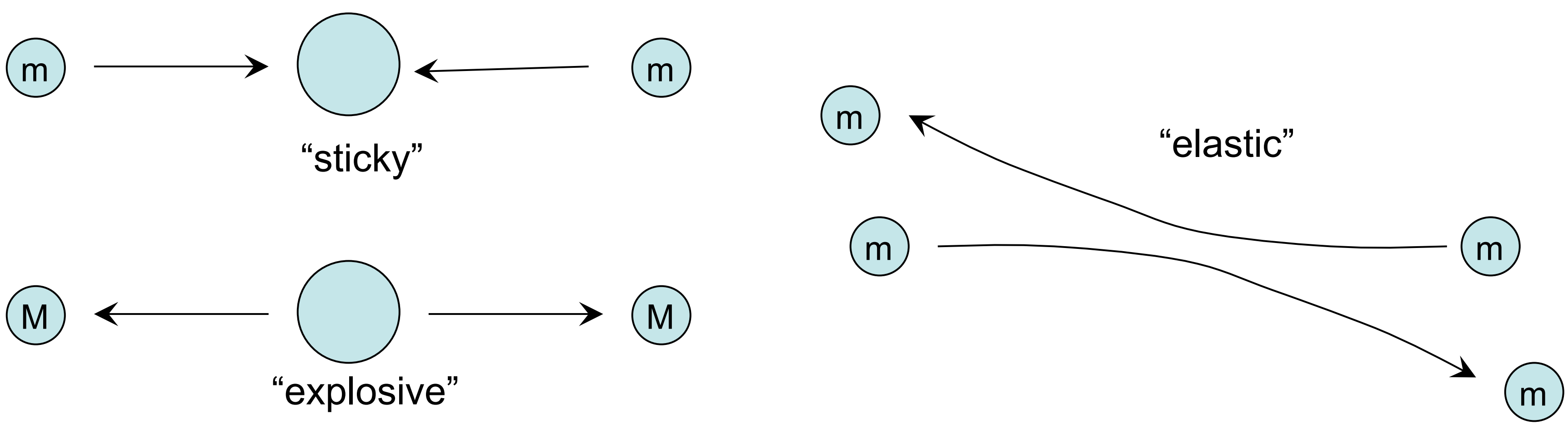
Classically what quantities are conserved in this collision process ?

a) Mass: $M_A + M_B = M_C + M_D$

b) Momentum: $\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D$ (*i.e.* three-momentum)

c) Kinetic energy MAY be conserved (elastic collision)

Consider three "types" of processes:



Collisions, relativistically

Relativistically, the conserved quantities are somewhat different than in the classical case:

a) Energy and momentum are always conserved

b) Kinetic energy MAY or MAY NOT be conserved

c) Mass MAY or MAY NOT be conserved

In a given process, these are either both conserved or both violated

In terms of the categories just introduced we have:

1) "Sticky": Mass ↑, Kinetic energy ↓

2) "Explosive": Mass ↓, Kinetic energy ↑

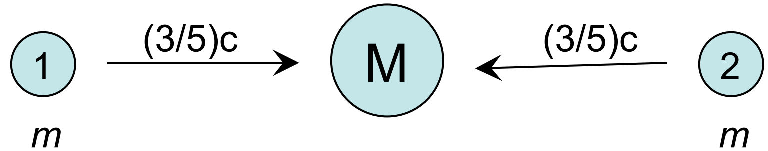
These are referred to as inelastic processes

3) "Elastic": Mass, Kinetic energy are both conserved

Look at some trivial examples (Ex. 3.1 and 3.2 from the text)

Griffiths Example 3.1

"Sticky"



Conservation of energy: $E_1 + E_2 = E_M = 2E_m$

Conservation of momentum: $\vec{p}_1 = -\vec{p}_2$

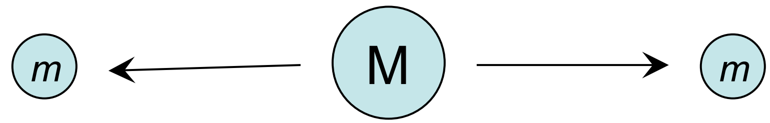
$$\left.\begin{array}{l} \text{Final energy is } E = Mc^2 \\ \\ \text{Initial energy is } 2E_m = 2\gamma mc^2 \end{array}\right\} \quad M = 2\gamma m$$

$$M = \frac{2m}{\sqrt{1-(3/5)^2}} = \frac{2m}{\sqrt{16/25}} = \frac{5}{2}m$$

which is > 2*m* (as it must be)

Griffiths Example 3.2

"Explosive"



Conservation of energy: $Mc^2 = 2\gamma mc^2 \rightarrow \quad M = 2\gamma m = \dfrac{2m}{\sqrt{1-\mathrm{v}^2/c^2}}$

$$\frac{M^2}{4m^2} = \frac{1}{1-\mathrm{v}^2/c^2} \qquad 1-\mathrm{v}^2/c^2 = \frac{4m^2}{M^2} \qquad \mathrm{v}^2/c^2 = 1-\frac{4m^2}{M^2}$$

thus, $\mathrm{v} = c\sqrt{1-\dfrac{4m^2}{M^2}} = c\sqrt{1-\left(\dfrac{2m}{M}\right)^2}$

Note that this makes sense only for *M* > *2m*

Note that in four-vector notation, for example 3.2 we would write the total four-momentum in the initial and final states and require the invariant p^2 to be the same before and after:

$$\left.\begin{array}{lll} \text{Initial state:} & (Mc,0,0,0) & p^2 = M^2c^2 \\ \text{Final state:} & (2\gamma mc,0,0,0) & p^2 = 4\gamma^2 m^2 c^2 \end{array}\right\} \text{Yields } M=2\gamma m \text{ as before}$$

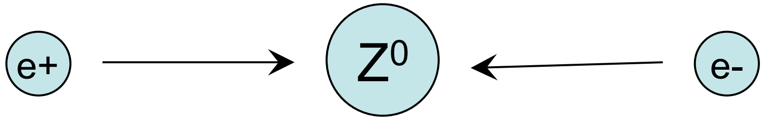
e.g. $\left[\, (\gamma mc, \vec{p}) + (\gamma mc, -\vec{p}) \,\right]$

Advantage of using the four-vector notation is not obvious in these rather simple exercises. Here we have the advantage of a centre-of-mass reference frame and a symmetric system. Look at Exercise 3.3 to see an example where the four-vector notation simplifies the problem. Looks at the decay $\pi^+ \to \mu^+ \nu_\mu$ (in which the final state particles have different masses).

A note on "Sticky" processes

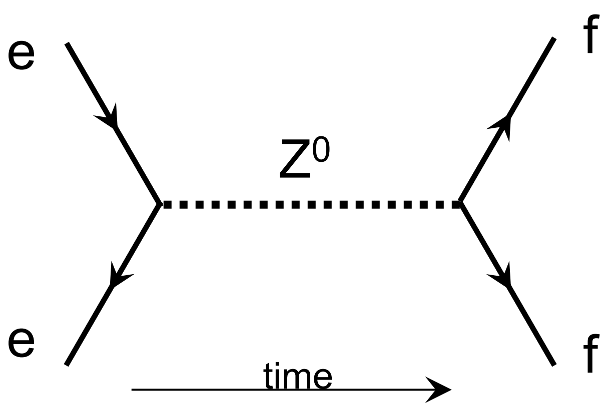
Relativistically an example of a "sticky" process is the production of some new particle in a collision. Often this is followed by the decay of the particle so this could represent the initial part of a scattering process.

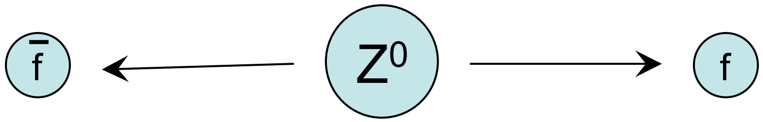
Consider the process $e^+e^- \to Z^0$



In the CM frame, if the electrons each carry energy 45.6 GeV electrons, the Z^0 ($M \sim 91.2\ \mathrm{GeV}/c^2$) is produced at rest. The 45.6 GeV of mostly kinetic energy from each beam goes into the mass of the Z^0.

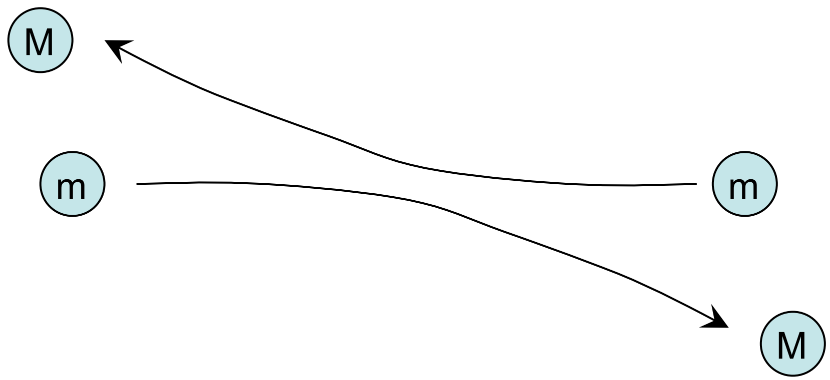
The Z^0 lifetime is very short. It almost immediately decays into any final state that it couples to which is kinematically accessible (so any fermion anti-fermion pair $f\bar{f}$, with $m_f < M(Z^0)/2$).





We will calculate the amplitude for this process later in the course.

The cartoon version of this “inelastic” process (in analogy to the earlier sketches) is



which is a combination of

However, note that we can also have an inelastic process that does not involve the formation of some intermediate state: