Adjoint Method

Qinya Liu

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1 General Principle

Define Hilbert space $\mathcal{F}$, $\mathbf{S} \in \mathcal{F}$.

Define a functional φ on $\mathcal{F}: \mathbf{S} \to \varphi(\mathbf{S})$.

$$\delta\varphi = (\nabla_{\mathbf{S}}\varphi,\ \delta\mathbf{S})$$

Define another Hilbert space $\mathcal{C}$, $\mathbf{m} \in \mathcal{C}$, and a linear operator L: $\mathcal{C} \to \mathcal{F}$ such that

$$\delta\mathbf{S} = L\,\delta\mathbf{m}$$

then

$$\begin{aligned}\delta\varphi &= (\nabla_{\mathbf{S}}\varphi,\ L\,\delta\mathbf{m}) \\ &= \langle L^*\,\nabla_{\mathbf{S}}\varphi,\ \delta\mathbf{m}\rangle\end{aligned}$$

where L^* is the adjoint operator of L. Formally,

$$\nabla_{\mathbf{m}}\varphi = L^*\,\nabla_{\mathbf{S}}\varphi$$

Note: $\nabla_{\mathbf{S}}$ is the input adjoint source, L is the forward Green's operator, and L^* is the corresponding adjoint operator.

2 Application to Seismology

2.1 Introduction

For one earthquake, we have N observations:

$\mathbf{D} = \left[\mathbf{d}(\mathbf{x}^1, t), \ldots, \mathbf{d}(\mathbf{x}^I, t), \ldots, \mathbf{d}(\mathbf{x}^N, t)\right]$

and for model $\mathbf{m}(\mathbf{x})$, we have synthetics

$\mathbf{S}(\mathbf{m}) = \left[\mathbf{s}(\mathbf{m}, \mathbf{x}^1, t), \ldots, \mathbf{s}(\mathbf{m}, \mathbf{x}^I, t), \ldots, \mathbf{s}(\mathbf{m}, \mathbf{x}^N, t)\right]$

and $\mathbf{D}, \mathbf{S} \in \mathcal{F}$.

Define measure of misfit in $\mathcal{F}$:

$$\begin{aligned} \varphi &= \frac{1}{2}\left(\mathbf{S} - \mathbf{D},\ \ \mathbf{S} - \mathbf{D}\right) \\ &= \frac{1}{2}\sum_{I=1}^{N} \int \left[s_i(\mathbf{m}, \mathbf{x}^I, t) - d_i(\mathbf{x}^I, t)\right]^2 \, dt, \end{aligned}$$

and

$$\begin{aligned} \delta\varphi &= \left(\nabla_{\mathbf{S}}\varphi,\ \delta\mathbf{S}\right) \\ &= \sum_{I=1}^{N} \int \left[s_i(\mathbf{m}, \mathbf{x}^I, t) - d_i(\mathbf{x}^I, t)\right] \delta s_i(\mathbf{m}, \mathbf{x}^I, t)\, dt. \end{aligned}$$

Note that $\nabla_{\mathbf{S}}\varphi$ will become the back-propagated adjoint source.

Suppose we can express the variations of $\mathbf{s}$ as

$$\delta \mathbf{s}_i(\mathbf{m}, \mathbf{x}^I, t) = \int_0^t \int G_{ij}(\mathbf{x}^I, t - t', \mathbf{x}')\, \delta f_j(\mathbf{x}', t')\, d\mathbf{x}' dt',$$

where $\delta f_j(\mathbf{x}', t')$ is the equivalent perturbation in the force term due to the variations in the model parameters $\mathbf{m}$. Therefore,

$$\delta\varphi = \sum_{I=1}^{N} \int_0^T \left[s_i(\mathbf{m}, \mathbf{x}^I, t) - d_i(\mathbf{x}^I, t)\right]$$
$$\left\{\int_0^t \int G_{ij}(\mathbf{x}^I, t - t', \mathbf{x}')\, \delta f_j(\mathbf{x}', t')\, d\mathbf{x}' dt'\right\} dt.$$

Do integration over t and summation over I first

$$\delta\varphi = \int \int_0^T \delta f_j(\mathbf{x}', t')$$
$$\left\{\int_{t'}^T dt \sum_{I=1}^{N} G_{ij}(\mathbf{x}^I, t - t', \mathbf{x}')\, \left[s_i(\mathbf{m}, \mathbf{x}^I, t) - d_i(\mathbf{x}^I, t)\right]\right\} dt' d\mathbf{x}'.$$

Define

$$\begin{aligned} & P_j(\mathbf{x}', t') \\ & = \int_{t'}^T \sum_{I=1}^{N} G_{ij}(\mathbf{x}^I, t - t', \mathbf{x}')\, \left[s_i(\mathbf{m}, \mathbf{x}^I, t) - d_i(\mathbf{x}^I, t)\right] dt \\ & = \int_{t'}^T \sum_{I=1}^{N} G_{ji}(\mathbf{x}', t - t', \mathbf{x}^I)\, \left[s_i(\mathbf{m}, \mathbf{x}^I, t) - d_i(\mathbf{x}^I, t)\right] dt \\ & = \int_0^{T-t'} \sum_{I=1}^{N} G_{ji}(\mathbf{x}', T - t' - t, \mathbf{x}^I)\, \left[s_i(\mathbf{m}, \mathbf{x}^I, T - t) - d_i(\mathbf{x}^I, T - t)\right] dt. \end{aligned}$$

Define force field

$$F_i(\mathbf{x}, t) = \sum_{I=1}^{N} \left[s_i(\mathbf{m}, \mathbf{x}^I, T - t) - d_i(\mathbf{x}^I, T - t)\right] \delta(\mathbf{x} - \mathbf{x}^I),$$

and adjoint field associated with this force field:

$$s_j^\dagger(\mathbf{x}', t') = \int_0^{t'} \int G_{ji}(\mathbf{x}', t' - t, \mathbf{x})\, F_i(\mathbf{x}, t) d\mathbf{x} dt$$

then

$$P_j(\mathbf{x}', t') = s_j^\dagger(\mathbf{x}', T - t')$$

and

$$\delta\varphi = \int \left[\int_0^T \delta f_j(\mathbf{x}', t') s_j^\dagger(\mathbf{x}', T - t')\, dt' \right] d\mathbf{x}'.$$

Note: A useful formula is $\int [f(t) * g(t)] h(t) dt = \int f(t)[g(t) * h(-t)] dt$.

2.2 Waveform Tomography

In waveform tomography, $\mathbf{m}$ denotes the perturbation in structural parameters, including ρ and $\mathbf{c}$.

$\mathbf{s}(\mathbf{m}, \mathbf{x}, t)$ satsifies the wave equation

$$\rho\, \partial_t^2 \mathbf{s} = \nabla \cdot (\mathbf{c} : \nabla \mathbf{s}) + \mathbf{F_0} \qquad \text{in } V$$

$$\mathbf{n} \cdot (\mathbf{c} : \nabla \mathbf{s}) = 0 \qquad \text{on } \Sigma$$

and

$$\rho\, \partial_t^2 \delta\mathbf{s} = \nabla \cdot (\mathbf{c} : \nabla \delta\mathbf{s}) - \delta\rho\, \partial_t^2 \mathbf{s} + \nabla \cdot (\delta\mathbf{c} : \nabla \mathbf{s})$$

$$\mathbf{n} \cdot (\mathbf{c} : \nabla \delta\mathbf{s}) = -\mathbf{n} \cdot (\delta\mathbf{c} : \nabla \mathbf{s})$$

Using the symmetry of $\mathbf{c}$ and define (a little bit tricky with b.c.)

$$\delta\mathbf{f} = -\delta\rho\, \partial_t^2 \mathbf{s} - (\nabla \mathbf{s} : \delta\mathbf{c}) \cdot \nabla$$

Invoke Green's function, we obtain

$$\delta\mathbf{s}_i(\mathbf{m}, \mathbf{x}^I, t) = \int_0^t \int \delta f_j(\mathbf{x}', t')\, G_{ij}(\mathbf{x}^I, t - t', \mathbf{x}')\, d\mathbf{x}' dt'$$

Substitute in the expression for $\delta\varphi$,

$$\delta\varphi = \int \left\{ \int_0^T \left[-\delta\rho\, \partial_t^2 \mathbf{s} - (\nabla \mathbf{s} : \delta\mathbf{c}) \cdot \nabla \right] |_{(\mathbf{x}', t')} \cdot \mathbf{s}^\dagger(\mathbf{x}', T - t') dt' \right\} d\mathbf{x}'$$

Then we can define ρ and $\mathbf{c}$ kernels:

$$\begin{aligned}
K_\rho(\mathbf{x}) &= -\rho(\mathbf{x}) \int_0^T [\partial_t^2 \mathbf{s}(\mathbf{x},t) \cdot \mathbf{s}^\dagger(\mathbf{x}, T-t')] dt' \\
K_{c_{jklm}}(\mathbf{x}) &= -c_{jklm}(\mathbf{x}) \int_0^T [\epsilon_{jk}(\mathbf{x},t)\epsilon^\dagger_{lm}(\mathbf{x}, T-t)]\, dt
\end{aligned}$$

and $\delta\varphi$ evolves into

$$\delta\varphi = \int \left[K_\rho(\mathbf{x})\delta \ln \rho(\mathbf{x}) + K_{c_{jklm}}(\mathbf{x})\delta \ln c_{jklm}(\mathbf{x})\right] d\mathbf{x}$$

A simpler version

$$\delta\varphi = \int \left[K_{\rho'}(\mathbf{x})\delta \ln \rho(\mathbf{x}) + K_\alpha(\mathbf{x})\delta \ln \alpha(\mathbf{x}) + K_\beta(\mathbf{x})\delta \ln \beta(\mathbf{x})\right] d\mathbf{x}$$

Where

$$\begin{aligned}
K_\mu(\mathbf{x}) &= -2\mu(\mathbf{x}) \int_0^T \mathbf{D}(\mathbf{x},t) : \mathbf{D}^\dagger(\mathbf{x}, T-t)\, dt \\
K_\kappa(\mathbf{x}) &= -\kappa(\mathbf{x}) \int_0^T [\nabla \cdot \mathbf{s}(\mathbf{x},t)][\nabla \cdot \mathbf{s}^\dagger(\mathbf{x}, T-t)]\, dt \\
K_{\rho'} &= K_\rho + K_\kappa + K_\mu \\
K_\beta &= 2\left(K_\mu - \frac{4\mu}{3\kappa} K_\kappa\right) \\
K_\alpha &= 2\left(1 + \frac{4\mu}{3\kappa}\right) K_\kappa
\end{aligned}$$

Notice that these K kernels will have the same units as φ/V.

2.3 Travel-time Tomography

Let the misfit function be

$$\delta\varphi = \frac{1}{2}\sum_{I=1}^{N}\sum_{i=1}^{L_I} \tau_i^2(\mathbf{x}^I, \mathbf{m})$$

where I is the number of receivers, L_I is the number of wave packets for the Ith receiver, and $\tau_i(\mathbf{x}^I, \mathbf{m})$ is the time shift bwtween the data and the synthetics for the ith wave packet of the Ith receiver for model $\mathbf{m}$. Use the formula derived in the 'Travel-time tomography' notes

$$\begin{aligned}
\delta\varphi &= \sum_I \sum_{L_I} \tau_i(\mathbf{x}^I, \mathbf{m})\delta\tau_i(\mathbf{x}^I, \mathbf{m}) \\
&= \sum \tau_i(\mathbf{x}^I, \mathbf{m})N_i \int_0^T w_i(t)\dot{s}(\mathbf{x}^I, t)w_i(t)\delta s(\mathbf{x}^I, t)dt \\
&= \sum \tau_i(\mathbf{x}^I, \mathbf{m})N_i \int_0^T w_i^2(t)\dot{s}(\mathbf{x}^I, t)[\int_0^t \int G_{.j}(\mathbf{x}^I, t - t', \mathbf{x}')\delta f_j(\mathbf{x}', t')d\mathbf{x}'dt']\, dt \\
&= \int_0^T \int \delta f_j(\mathbf{x}', t')[\int_0^{T-t'} \sum \tau_i(\mathbf{x}^I, \mathbf{m})N_i w_i^2(T - t)\dot{s}(\mathbf{x}^I, T - t)G_{j.}(\mathbf{x}', T - t - t', \mathbf{x}^I)\, dt]d\mathbf{x}'dt' \\
&= \int_0^T \int \delta f_j(\mathbf{x}', t')s_j^\dagger(\mathbf{x}', T - t')\, d\mathbf{x}'dt
\end{aligned}$$

Where $\mathbf{s}^\dagger(\mathbf{x}', t')$ is the adjoint field associated with the adjoint source $\sum_{I,L_I} N_i\tau_i(\mathbf{x}^I, \mathbf{m})w_i^2(T - t)\dot{s}(\mathbf{x}^I, T - t)$, and $N_i = (\int [w(t)\dot{s}(t)]^2\, dt)^{-1}$. This means that the back-propagated signal is the windowed the synthetics with proper normalization and weighted with travel-time anomaly.

3 Application to Source Inversion

For an earthquake, we have N observations:

$\mathbf{D} = \left[\mathbf{d}(\mathbf{x}^1, t), \ldots, \mathbf{d}(\mathbf{x}^I, t), \ldots, \mathbf{d}(\mathbf{x}^N, t)\right]$

and for force field $\mathbf{F}(\mathbf{x}, t, \mathbf{f})$, we have synthetics

$\mathbf{S}(\mathbf{f}) = \left[\mathbf{s}(\mathbf{f}, \mathbf{x}^1, t), \ldots, \mathbf{s}(\mathbf{f}, \mathbf{x}^I, t), \ldots, \mathbf{s}(\mathbf{f}, \mathbf{x}^N, t)\right]$, which satisify the wave equations

$$\rho\, \partial_t^2 \mathbf{s} = \nabla \cdot (\mathbf{c} : \nabla \mathbf{s}) + \mathbf{F}(\mathbf{x}, t, \mathbf{f})$$

and

$$\delta \mathbf{s}_i(\mathbf{f}, \mathbf{x}^I, t) = \int_0^t \int G_{ij}(\mathbf{x}^I, t - t', \mathbf{x}')\, \delta F_j(\mathbf{x}', t', \mathbf{f})\, d\mathbf{x}' dt'$$

Follow the same deduction as before:

$$\delta\varphi = \int \left[\int_0^T \delta F_j(\mathbf{x}', t', \mathbf{f}) s_j^\dagger(\mathbf{x}', T - t')\, dt' \right] d\mathbf{x}'$$

In practice, another version is more useful:

$$\delta\varphi = \int \left[\int_0^T \delta \dot{\mathbf{F}}_j(\mathbf{x}', t', \mathbf{f}) \mathbf{I}_{s}{}_j^\dagger(\mathbf{x}', T - t')\, dt' \right] d\mathbf{x}'$$

where we have defined $\dot{\mathbf{F}} = \partial_t \mathbf{F}$, and $\mathbf{I}_s = \int \mathbf{s} dt$.

We study the cases of point force, point moment-tensor source and finite fault.

3.1 Point Source

Point force at location x_0 can be expressed as:

$$F_j(\mathbf{x}, t, \mathbf{f}) = f_j g(t)\delta(\mathbf{x} - \mathbf{x}_0)$$

where f_j is the amplitude of the force and also the 'model' parameter to solve for; $g(t)$ is the normalized source time function.

$$\delta F_j(\mathbf{x}, t, \mathbf{f}) = \delta f_j g(t)\delta(\mathbf{x} - \mathbf{x}_0)$$

Therefore

$$\delta\varphi = \left[\int_0^T g(t) s_j^{\dagger}(\mathbf{x}_0, T - t)\, dt\right] \delta f_j$$

3.2 Moment-Tensor source

If the moment tensor for a point source $\mathbf{x}_0$ is given by:

$$M_{jk}(\mathbf{x}, t, \mathbf{m}) = M_{jk} g(t)\delta(\mathbf{x} - \mathbf{x}_0),$$

then the corresponding force can be expressed as:

$$F_j(\mathbf{x}, t, \mathbf{m}) = -\partial_k M_{jk} = -M_{jk} g(t)\partial_k \delta(\mathbf{x} - \mathbf{x}_0)$$

and

$$\delta\varphi = \left[-\int\int_0^T g(t) s_j^{\dagger}(\mathbf{x}, T - t)\partial_k \delta(\mathbf{x} - \mathbf{x}_0)\, dt dx\right] \delta M_{jk}$$

Integration by parts and assume $\mathbf{x}_0$ is not on any boundaries, then

$$\begin{aligned}\delta\varphi &= \left[\int_0^T g(t)\partial_k s_j^\dagger(\mathbf{x}_0, T-t)\,dt\right]\delta M_{jk} \\ &= \left[\int_0^T g(t)\epsilon_{jk}^\dagger(\mathbf{x}_0, T-t)\,dt\right]\delta M_{jk}\end{aligned}$$

Therefore, in order to compute the Fréchet derivative of the misfit function with respect to moment tensor elements, we just need to convolve the source time function $g(t)$ with the adjoint strain $\epsilon_{jk}(\mathbf{x}_0, t)$ at the source location, which can be done on the fly in the numerical simulations.

If we are also interested in resolving the location x_0, then we have:

$$\begin{aligned}\delta\varphi &= \left[\int_0^T \epsilon_{jk}^\dagger(\mathbf{x}_0, T-t)g(t)\,dt\right]\delta M_{jk} \\ &+ \left[\int_0^T \partial_i\left(M_{jk}\epsilon_{jk}^\dagger(\mathbf{x}_0, T-t)\right)g(t)\,dt\right]\delta x_i^0\end{aligned}$$

In practice, we will compute

$$\int_0^T \epsilon_{jk}^\dagger(\mathbf{x}_0, T-t)g(t)\,dt$$

and

$$\int_0^T \left(M_{jk}\epsilon_{jk}^\dagger(\mathbf{x}_0, T-t)\right)g(t)\,dt$$

'on the fly' for the source element, from which we obtain the Fréchet derivatives with respect to moment-tensor elements and source location at the end of time loop.

3.3 Finite Fault

For a finite fault with known geometry, mesh it into rectangulars sub-elements, and define the basis function associated with each rectangular $P_{IJ}(\mathbf{x})$. However, it is tricky if one also wants to resolve the source time function $g(t)$. Note that $g(t=\infty) \neq 0$, which means we cannot parameterize the time domain with finite number of basis functions. A natural choice is to look at the $\dot{\mathbf{F}}(\mathbf{x}, t)$ (i.e. $\dot{g}(t)$), for which we select the time window to resolve the time history, and the associated basis function $B_\sigma(t)$ for each time block.

3.3.1 Force

$$\dot{F}_j(\mathbf{x}, t) = \sum_{I,J,\sigma} f_j^{IJ\sigma} P_{IJ}(\mathbf{x}) B_\sigma(t)$$

and

$$\delta\varphi = \sum_{IJ\sigma} \left[\int \int_0^T P_{IJ}(\mathbf{x}) B_\sigma(t) I_{s_j}^{\dagger}(\mathbf{x}, T-t)\, dt dx \right] \delta f_j^{IJ\sigma}$$

Notice that the Fréchet derivatives for $\delta f_j^{IJ\sigma}$ is actually taking the appropriate time and spatial slice of $I_{s_j}^{\dagger}(\mathbf{x}, T-t)$.

3.3.2 Moment Density Tensor

Let the moment-rate density tensor function be

$$\dot{m}_{jk}(\mathbf{x}, t) = \sum_{I,J,\sigma} m_{jk}^{IJ\sigma} P_{IJ}(\mathbf{x}) B_\sigma(t)$$

and

$$\delta\varphi = \sum_{IJ\sigma} \left[\int \int_0^T P_{IJ}(\mathbf{x}) B_\sigma(t) I_{\epsilon_{jk}}^\dagger(\mathbf{x}, T-t)\, dtdx \right] \delta m_{jk}^{IJ\sigma}$$

Therefore the the Fréchet derivatives for $m_{jk}^{IJ\sigma}$ is given by the appropriate time and spatial slice of $I_{\epsilon_{jk}}^\dagger(\mathbf{x}, T-t)$. One thing to bear in mind is that $m_{jk}^{IJ\sigma}$ is actually the discretized version of the 'moment-rate density function', not the 'moment-density function'.