QUANTUM FIELD THEORY

SEP. 23 Units; Translational invariance, Rotational invariance and Lorentz invariance for a single free particle.

25 Position operator; Violation of Causality; Pair Production; Fock Space; Occupation number representation; SHO; Oscillator-like formalism for fock space.

30 Causality, observables and quantum fields; Constructing the quantum field from Fock space operators axiomatically; Translational invariance; Lorentz invariance and Relativistic normalisation; Field constructed satisfies Klein-Gordon equation.

OCT. 2 Constructing Fock space from the quantum field axiomatically; The Method of the Missing Box, Classical Particle Mechanics, Quantum Particle Mechanics, Classical Field Theory, Quantum Field Theory; Quantum field from free scalar theory.

7 Hamiltonian recovered in free scalar theory up to infinite constant; Normal ordering; Symmetries and Conservation laws, Noether's theorem; Noether's theorem in field theory, conserved currents; ambiguity in currents; Energy-momentum tensor.

OCT. 9 Lorentz transformations; Angular momentum conservation; Internal symmetries; SO(2) internal symmetry; Charged field; SO(n) internal symmetry.

14 Lorentz transformation properties of conserved quantities; Discrete symmetries; $\phi \to -\phi$; Charge conjugation; Parity; Ambiguity of choice of parity; Time reversal; Unitary and anti-unitary operators. angular momentum; Dilatations.

16 Scattering theory overview; Low budget sgattering theory; Turning on and off function; Schrodinger picture, Heisenberg picture and interaction picture; Evolution operator; Time ordered product; Three models; Wick's theorem.

21 Diagrammatic perturbation theory in Model 3; Vertex in model 1; Connected diagrams; Thm: Σ all Wick diagrams $=: e^{\Sigma \text{ connected Wick diagrams}}$; Model 1 Solved; Model 2 begun.

23 Model 2 finished; Vacuum energy c.t.; S matrix is 1; Ground state energy; Yukawa potential; Ground state wave function; Model 3 and Mass renormalization; Renormalized perturbation theory

28 Feynman diagrams in Model 3; Feynman rules in model 3; A catalog of all Feynman diagrams in model 3 to $O(g^2)$; Scattering amplitude at $O(g^2)$; Direct and exchange Yukawa potentials

OCT. 30 "Nucleon"-anti"nucleon" scattering at $O(g^2)$; Energy eigenstate picture; Meson-"nucleon" scattering; "Nucleon"-anti"nucleon" annihilation; Assembling the amplitudes for various processes into one amplitude; Mandelstam variables; Mandelstam-Kibble plot; ΛCPT; crossing symmetry; Phase space and the S matrix; $\frac{\text{Differential tran. prob.}}{\text{unit time}}$.

NOV. 4 Applications of $\frac{\text{Differential tran. prob.}}{\text{unit time}}$; Decay; Cross sections, Flux; Final state phase space simplified for two bodies; $\frac{d\sigma}{d\Omega}$; Optical theorem; Final state phase space for three bodies; Feynman diagrams with external lines off the mass shell; they could be an internal part of a larger diagram.

6 Fourier transform of the new blob; A second interpretation of Feynman diagrams with lines off the mass shell; they are the coefficients of ρ^n in $\langle 0|s|0\rangle_\rho$; A third interpretation of the blob; they are the fourier transform of the VEV of a string of Heisenberg fields; Reformulation of scattering theory; S matrix elements without the turning on and off function; LSZ formula stated.

13 LSZ formula proved; A second look at Model 13 and its renormalization; Renormalization conditions.

NOV. 18 Perturbative determination of a c.t.; Problems with derivative couplings; Rephrasing renormalization conditions in terms of Green's functions; Lehmann-Kallen spectral representation for the propagator; Rephrasing renormalization conditions in terms of 1PI functions.

20 Perturbative determination of c.t.; Corrections to external lines in the computation of S matrix elements; One loop correction to meson self energy; Feynman's trick for combining 2 denominators; Shift to make denominator so(3,1) invariant; Wick rotation to make denominator O(4) invariant; Integral tables for convergent combinations; Self energy at one loop studied; Combining lots of denominators; The shift in the general case to reduce any multi-loop integral to an integral over Feynman parameters.

25 Rephrasing coupling constant renormalization in terms of a 1PI function; Experimental significance of the definition; Renormalization versus the infinities; Renormalizable lagrangians; Unstable particles, Decay process.

DEC. 2 Unstable particles, lifetime, method of stationary phase; where it begins again; Lorentz transformation laws of fields; Equivalent representations; Reducible reps; The finite dimensional inequivalent irreducible representations of SO(3); Unitarity; Complex conjugation; Direct product; Projection operators and reducibility.

DEC. 4 Parametrizing the Lorentz group; Commutation relations for the [illegible] generators; decomposition into two sets obeying SO(3) commutation relations; The catalog; Complex conjugation properties; Tensor product properties; Restriction to SO(3); The vector; Rank 2 tensors; Spinors.

9 Lagrangian made of two component spinors; Solution of Weyl equations of motion; Weyl particles; Dirac Lagrangian; four-component spinors; Weyl basis

11 Dirac basis; Plane wave solutions of the Dirac equation; Pauli's theorem; Dirac adjoint; Pauli-Feynman notation; Parity; Bilinears; Orthogonality; Completeness; Summary.

16 Canonical quantization of the Dirac Lagrangian.

Class: Why not use Feynman's lecture notes?

Gell-Mann: Because Feynman uses a different method than we do.

Class: What is Feynman's method?

Gell-Mann: You write down the problem. Then you look at it and you think. Then you write down the answer.

DEC. 18 Perturbation theory for spinors; Time ordered product; Wick's theorem; Calculation of the contraction (propagator); Wick diagrams; Feynman diagrams; Matrix multiplication; Spin averages and spin sums.

JAN. 6 Parity for spinors; Fermion and antifermion have opposite intrinsic parity; Charge conjugation; Majorana basis; Charge conjugation properties of fermion bilinears; Decay of ortho and para positronium $U_P U_C = U_C U_P (-1)^{N_F}$; PT.

8 Effect of PT on states; Proof of CPT theorem in perturbation theory; Renormalization of spinor theories; Propagator.

13 1PI part of propagator; Spectral representation for propagator; $\Sigma'(\not{p})$ to $O(g^2)$ in meson-nucleon theory; Coupling constant renormalization; Is renormalization sufficient to eliminate ∞'s.

15 Regularization; Regulator fields; Dim'l regularization; Minimal subtraction; BPHZ renormalizability; Renormalization and symmetry; Renormalization of composite operators.

SEPT. 23

In NRQM, rotational invariance simplifies scattering problems. Why does the addition of relativity, the addition of L.I., complicate quantum mechanics?

The addition of relativity is necessary at energies $E \gtrsim mc^2$. At these energies

$$p+p \rightarrow p+p+\pi^0$$

is possible. At slightly higher energies

$$p+p \rightarrow p+p+p+\bar{p}$$

can occur. The exact solution of a high energy scattering problem necessarily involves many particle processes.

You might think that for a given E, only a finite number, even a small number, of processes actually contribute, but you already know from NRQM that that isn't true.

$$H \rightarrow H+\delta V \qquad \delta E_0 = \langle 0|\delta V|0\rangle + \sum_n \frac{|\langle 0|\delta V|n\rangle|^2}{E_0 - E_n} + \ldots$$

Intermediate states of all energies contribute, suppressed by energy denominators.

For calculations of high accuracy effects at low energy, relativistic effects of order $\left(\frac{v}{c}\right)^2$ can be included. Intermediate states with extra particles will contribute corrections of order

$$\frac{E}{mc^2} \sim \frac{mv^2}{mc^2} = \left(\frac{v}{c}\right)^2$$

(Typical energies in problem → E; Typical energy denominator → mc^2)

As a general conclusion:

the corrections of relativistic kinematics and the corrections from multiparticle intermediate states are comparable; the addition of relativity forces you to consider many-body problems. We can't even solve the zero-body problem. (It is a phenomenal fluke that relativistic kinematic corrections for the Hydrogen atom work. If the Dirac equation is used without considering multi-particle intermediate states, corrections of $O\left(\frac{v}{c}\right)^2$ can be obtained. This is a fluke caused by some unusually low electrodynamic matrix elements.)

We will see that you cannot have a consistent relativistic picture without pair production.

UNITS

$\hbar = c = 1$

Sometimes $1 = -1 = 2\pi$ and $\frac{1}{2\pi} = \frac{1}{\!\!\!-} =$ "one-bar"

Because we're doing relativistic quantum mechanics.

$[m] = [E] = [T^{-1}] = [L^{-1}]$

$(1 \text{ fermi})^{-1} \approx 197 \text{ GeV}$

We say things like the inverse Compton wavelength of the proton is "1 GeV."

LORENTZ INVARIANCE

Every Lorentz transformation is the product of an element of the connected Lorentz group, $SO(3,1)$, and $1, P, T,$ or PT. By Lorentz invariance we mean $SO(3,1)$.

P: reflects all three space components

T: time reversal

Metric convention: $+---$.

THEORY OF A SINGLE FREE SPINLESS PARTICLE OF MASS μ

The components of momentum form a complete set of commuting variables.

Momentum operator $\vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$

- State of a spinless particle is completely specified by its momentum.

Normalization $\langle \vec{k}|\vec{k}'\rangle = \delta^{(3)}(\vec{k}-\vec{k}')$

The statement that this is a complete set of states, that there are no others, is

$$1 = \int d^3k\, |\vec{k}\rangle\langle\vec{k}|$$

$$|\psi\rangle = \int d^3k\, \psi(\vec{k})|\vec{k}\rangle \qquad \psi(\vec{k}) \equiv \langle\vec{k}|\psi\rangle$$

(IF we were doing NRQM, we'd finish describing the theory by giving the Hamiltonian, and thus the time evolution:

$$H|\vec{k}\rangle = \frac{|\vec{k}|^2}{2\mu}|\vec{k}\rangle .\ )$$

We take $H|\vec{k}\rangle = \sqrt{|\vec{k}|^2+\mu^2}\,|\vec{k}\rangle$

$$\equiv \omega_{\vec{k}}|\vec{k}\rangle$$

That's it, the theory of a single free spinless particle, made relativistic.

How do we know this theory is L.I.? Just because it contains one relativistic formula, it is not necessarily relativistic. The theory is not manifestly L.I..

The theory is manifestly rotationally and translationally invariant. Let's be more precise about this.

Translational Invariance

Given a four-vector, a, specifying a translation (active), there should be a linear operator, $U(a)$, satisfying

$$U(a)U(a)^\dagger = 1 \quad (1) \quad \text{to preserve probability amplitudes}$$

$$U(0) = 1 \quad (2)$$

$$U(a)\,U(b) = U(a+b) \quad (3)$$

The U satisfying these is $U(a) = e^{iP\cdot a}$ where $P = (H, \vec{P})$.

(This lecture is in pedagogical, not logical order. The logical order would be to state

(a) That we want to set up a translationally invariant theory of a spinless particle. The theory would contain unitary translation operators $U(\vec{a})$.

(b) Define $P^i = -i \left.\frac{\partial U}{\partial a^i}\right|_{\vec{a}=0}$ (by (3) $[P_i, P_j] = 0$; (by (1) $\vec{P} = \vec{P}^\dagger$)

(d) Define $H = \sqrt{\vec{p}^2 + \mu^2}$, thus giving the time evolution

(c) Declare P^i to be a complete set and classify the states by measuring)

4'

More translation invariance

$$U(a)|0\rangle = |a\rangle \qquad U(a) = e^{iP\cdot a}$$

where $|0\rangle$ here means state centered at zero
and $|a\rangle$ means state centered at a.

$$\mathcal{O}(x+a) = U(a)\,\mathcal{O}(x)\,U(a)^\dagger$$

$$\langle a|\,\mathcal{O}(x+a)\,|a\rangle = \langle 0|\,\mathcal{O}(x)\,|0\rangle \quad \checkmark$$

Non-relativistic reduction

$$U(\vec{a}) = e^{-i\vec{P}\cdot\vec{a}}$$

$$e^{-i\vec{P}\cdot\vec{a}}\,|\vec{q}\rangle = |\vec{q}+\vec{a}\rangle$$

Something hard to digest, but correct:

$$\hat{\vec{q}}\,e^{-i\vec{P}\cdot\vec{a}}\,|\vec{q}\rangle = (\vec{q}+\vec{a})\,|\vec{q}+\vec{a}\rangle$$

$$e^{i\vec{P}\cdot\vec{a}}\,\hat{\vec{q}}\,e^{-i\vec{P}\cdot\vec{a}}\,|\vec{q}\rangle = (\vec{q}+\vec{a})\,|\vec{q}\rangle$$

$$\Rightarrow \quad e^{i\vec{P}\cdot\vec{a}}\,\hat{\vec{q}}\,e^{-i\vec{P}\cdot\vec{a}} = \hat{\vec{q}} + \vec{a}$$

The $\hat{\vec{q}}$ operator is not an operator localized at $\vec{0}$. No reason for these to look alike

Looks like the opposite of

$$e^{-i\vec{P}\cdot\vec{a}}\,\mathcal{O}(\vec{x})\,e^{i\vec{P}\cdot\vec{a}} = \mathcal{O}(\vec{x}+\vec{a})$$

Rotational Invariance

Given an $R \in SO(3)$, there should be a $U(R)$ satisfying

$$U(R)\,U(R)^{+} = 1 \qquad (1)$$

$$U(1) = 1 \qquad (2)$$

$$U(R_1)\,U(R_2) = U(R_1 R_2) \qquad (3)$$

Furthermore denote $|\psi'\rangle = U(R)|\psi\rangle$

$$\langle\psi'|\vec{P}|\psi'\rangle = R\langle\psi|\vec{P}|\psi\rangle$$

for any $|\psi\rangle$ i.e. $U(R)^{+}\,\vec{P}\,U(R) = R\vec{P}$ (4)

and $U(R)^{+} H U(R) = H$ (5)

A $U(R)$ is given by satisfying all these properties

$$U(R)\,|\vec{k}\rangle = |R\vec{k}\rangle$$

That (2) and (3) are satisfied is trivial.

Proof that (1) is satisfied

$$U(R)\,U(R)^{+} = U(R)\int d^3k\; |\vec{k}\rangle\langle\vec{k}|\; U(R)^{+}$$

$$= \int d^3k\; |R\vec{k}\rangle\langle R\vec{k}| \qquad k' = Rk \quad d^3k' = d^3k$$

$$= \int d^3k'\; |\vec{k}'\rangle\langle\vec{k}'| = 1 \quad \checkmark\checkmark$$

Proof that (4) is satisfied

$$U(R)^\dagger \vec{P}\, U(R) = \underbrace{U(R)^{-1}}_{\text{by (1)}} \vec{P}\, U(R)^{-1\,\dagger}$$

↓ by (2) and (3)

$$= U(R^{-1})\vec{P}\, U(R^{-1})^\dagger$$

$$= U(R^{-1})\vec{P}\int d^3k\, |\vec{k}\rangle\langle\vec{k}|\, U(R^{-1})^\dagger$$

$$= U(R^{-1})\int d^3k\, \vec{k}\, |\vec{k}\rangle\langle\vec{k}|\, U(R^{-1})^\dagger$$

$$= \int d^3k\, \vec{k}\, |R^{-1}\vec{k}\rangle\langle R^{-1}\vec{k}|$$

$k = Rk'$, $d^3k = d^3k'$

$$= \int d^3k'\, R\vec{k}'\, |\vec{k}'\rangle\langle\vec{k}'|$$

$$= R\vec{P} \quad \surd\surd$$

You supply proof of (5)

This is the template for studying L.I.

Suppose a silly physicist took

$$|\vec{k}\rangle_S = \sqrt{1+k_z^2}\,|\vec{k}\rangle$$

$$S\langle\vec{k}|\vec{k}'\rangle_S = (1+k_z^2)\,\delta^3(\vec{k}-\vec{k}')$$

$$1 = \int d^3k \frac{1}{1+k_z^2}\, |\vec{k}\rangle_S\, S\langle\vec{k}|$$

If he took

$$U_S(R)\,|\vec{k}\rangle_S = |R\vec{k}\rangle_S$$

His proofs of (1), (4) and (5) would break down because

$\frac{d^3k}{1+k_z^2} \neq \frac{d^3k'}{1+k_z'^2}$ i.e. $\frac{d^3k}{1+k_z^2}$ is not a rotationally invariant measure!

Lorentz Invariance

$\langle \vec{k} | \vec{k}' \rangle = \delta^{(3)}(\vec{k}-\vec{k}')$ is a silly normalization for Lorentz invariance

d^3k is not a Lorentz invariant measure.

We want a Lorentz invariant measure on the hyperboloid $k^2 = \mu^2 \quad k^0 > 0$.

d^4k is a Lorentz invariant measure.

Restrict it to the hyperboloid by multiplying it by a Lorentz invariant

$$d^4k \; \delta(k^2-\mu^2)\,\theta(k^0)$$

* Think of the δ fn as a fn of k^0 and use the general formula $\delta(f(k^0)) = \sum_{\text{zeroes of } f,\, k_i} \frac{\delta(k^0-k_i)}{|f'(k_i)|}$ *

this yields the measure on the hyperboloid

$$\frac{d^3k}{2\omega_{\vec{k}}}$$

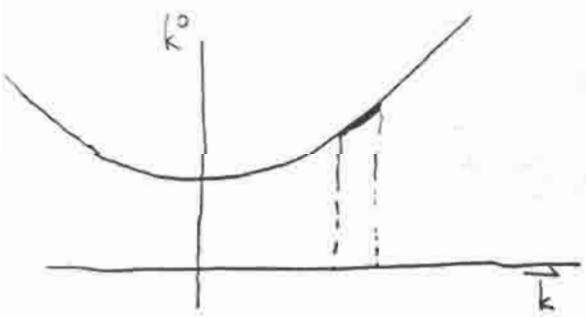
So we take

$k = (\omega_{\vec{k}}, \vec{k}) \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2+\mu^2}$

$$|k\rangle = \sqrt{(2\pi)^3}\sqrt{2\omega_{\vec{k}}}\,|\vec{k}\rangle$$

↑ relativistically normalized

↑ So factors of 2π come out right in the Feynman rules a few months from now



(Using $\int d^3x\, \delta(\vec{x}^2-R^2)$ can get $\frac{R}{2}\sin\theta\, d\theta\, d\varphi$)

Looks like factor multiplying d^3k ought to get larger as $|\vec{k}|$ gets larger. This is an illusion, caused by graphing on Euclidean paper. This is the same illusion as in the twin paradox. The moving twin's path looks longer, but in fact, its proper time is shorter.

Now the demonstration of Lorentz invariance:

Given any Lorentz transformation Λ define

$$U(\Lambda)|k\rangle = |\Lambda k\rangle$$

$U(\Lambda)$ satisfies

$$U(\Lambda)U(\Lambda)^\dagger = 1 \qquad (1)$$

$$U(1) = 1 \qquad (2)$$

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2) \qquad (3)$$

$$U(\Lambda)P\,U(\Lambda) = \Lambda P \qquad (4)$$

The proofs of these are exactly like the proofs of rotational invariance, using

$$1 = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} |k\rangle\langle k| \qquad \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} = \frac{d^3k'}{(2\pi)^3 2\omega_{\vec{k}'}}$$

We have a fairly complete theory except we still don't know where anything is.

We need a position operator, satisfying

$$\vec{X} = \vec{X}^\dagger \qquad (1) \qquad \text{it is observable}$$

$$R\vec{X} = U(R)^\dagger \vec{X}\, U(R) \qquad (2)$$

$$e^{i\vec{P}\cdot\vec{a}}\,\vec{X}\,e^{-i\vec{P}\cdot\vec{a}} = \vec{X} + \vec{a} \qquad (3)$$

Take $\frac{\partial}{\partial a_i}$ of (3) to get $i[P_i, X_j] = \delta_{ij}$

Determination of $\vec{X}$ in position space

$\Psi(\vec{k}) \equiv \langle \vec{k} | \psi \rangle \qquad \langle \vec{k} | \vec{P} | \psi \rangle = \vec{k}\,\Psi(\vec{k})$

$$\langle \vec{k} | \vec{X} | \psi \rangle = i \frac{\partial}{\partial \vec{k}} \Psi(\vec{k}) + \vec{k} F(|\vec{k}|^2) \Psi(\vec{k})$$

↑ to satisfy inhomogeneous part of commutation relation

↑ an arbitrary vector commuting with the $\vec{P}$'s, a complete set, can be written this way

The extra arbitrary vector can be eliminated by redefining the phases of the states. In the momentum state basis let

$$|\vec{k}\rangle \to |k\rangle_N = e^{iG(|\vec{k}|^2)} |\vec{k}\rangle$$

(This is a unitary transformation, call it U. In effect, $U^\dagger \vec{X} U$ is our new position operator)

where $\vec{\nabla} G(|\vec{k}|^2) = \vec{k} F(|\vec{k}|^2)$

The only formula this affects in all that we have done so far is the expression for $\langle \vec{k} | \vec{X} | \psi \rangle$. With the redefined states, it is

$${}_N\langle \vec{k} | \vec{X} | \psi \rangle_N = \int d^3k' \; {}_N\langle \vec{k} | \vec{k}' \rangle \langle \vec{k}' | \vec{X} | \psi \rangle_N$$

$$e^{-iG(|\vec{k}|^2)} \delta^3(k-k') \left[i \frac{\partial}{\partial \vec{k}'} + \vec{k}' F(|\vec{k}'|^2) \right] e^{iG(|\vec{k}'|^2)} \Psi(\vec{k}')$$

$$= \int d^3k' \, e^{-iG(|\vec{k}'|^2)} \delta^3(\vec{k}-\vec{k}') \left\{ \left[-\frac{\partial}{\partial \vec{k}'} G(|\vec{k}'|^2) + \vec{k}' F(|\vec{k}'|^2) \right] e^{iG} \Psi + e^{iG} \frac{\partial}{\partial \vec{k}'} \Psi \right\}$$

$$= i \frac{\partial}{\partial \vec{k}} \Psi(\vec{k})$$

Up to an unimportant phase definition, we have shown that the obvious definition for $\vec{X}$ is the unique definition, and we have done it without using L.I. or the explicit form of H.

SEPT. 15

It is possible to measure a particles position in our theory, $\vec{x}$ is an observable, and this leads to a conflict with causality.

Introduce position eigenstates

$$\langle \vec{k}|\vec{x}\rangle = \frac{1}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{x}}$$

By translational invariance and superposition, we could easily get the evolution of any initial configuration from this calculation.

We will evaluate $\langle \vec{x}|e^{-iHt}|\vec{x}=0\rangle$ and see that our particle has a nonzero amplitude to be found outside its forward light cone

$$\langle \vec{x}|e^{-iHt}|\vec{x}=0\rangle \qquad H|\vec{k}\rangle = \omega_{\vec{k}}|\vec{k}\rangle$$

$$= \int d^3k \langle \vec{x}|\vec{k}\rangle\langle \vec{k}|e^{-iHt}|\vec{x}=0\rangle$$

$$= \int d^3k \frac{1}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-i\omega_{\vec{k}}t} \qquad \omega_{\vec{k}} = \sqrt{\vec{k}^2+\mu^2}$$

$$= \int_0^\infty \frac{k^2 dk}{(2\pi)^3} \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, e^{ikr\cos\theta} e^{-i\omega_k t} \qquad k=|\vec{k}| \quad r=|\vec{x}|$$

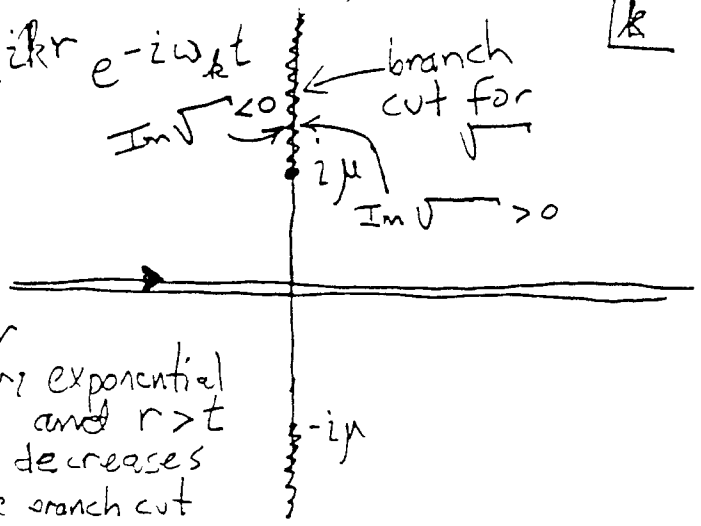
$$\omega_k = \sqrt{k^2+\mu^2}$$

$\int_0^\pi d\theta \sin\theta = \int_{-1}^{1} d(\cos\theta)$ $\qquad \int_0^{2\pi} d\varphi = 2\pi$

These steps can be applied to the F.T. of any function of $|\vec{k}|$

$$= \frac{1}{(2\pi)^2}\frac{1}{ir}\int_0^\infty k^2 dk \left(e^{ikr}-e^{-ikr}\right) e^{-i\omega_k t}$$

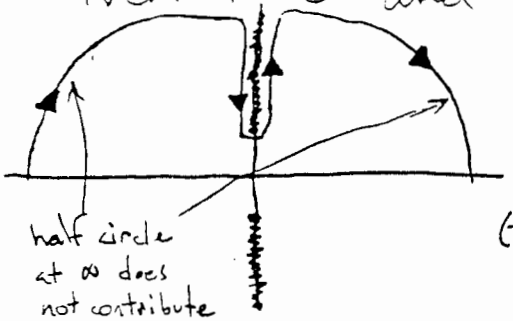
$$= \frac{-i}{(2\pi)^2 r}\int_{-\infty}^{\infty} k^2 dk\, e^{ikr} e^{-i\omega_k t}$$



$e^{-i\sqrt{k^2+\mu^2}}$ is a growing exponential as you go up the left side of the upper branch cut, and a decreasing exponential on the left side. Given $r>0$ and $r>t$ the product $e^{ikr}e^{-i\omega_k t}$ decreases exponentially as you go up the branch cut

SEPT. 25

Given $r>0$ and $r>t$ deform the contour to



The integral becomes

$$\frac{-i}{(2\pi)^2 r}\int_{\mu}^{\infty}(-z^2)\,d(iz)\,e^{-zr}\left[e^{\sqrt{z^2-\mu^2}\,t}-e^{-\sqrt{z^2-\mu^2}\,t}\right] \qquad k=iz$$

$$= -\frac{1}{2\pi^2 r}e^{-\mu r}\int_{\mu}^{\infty} z^2\,dz\,e^{(z-\mu)r}\sinh\left(\sqrt{z^2-\mu^2}\,t\right)$$

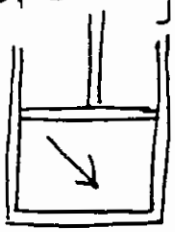
The integrand is positive definite, the integral is nonzero.

We can measure a particle's position in this theory. We can trap it in a box of arbitrarily small size, and we can release it and detect it outside of its forward light cone. The particle can travel faster than light and thus it can move backwards in time, with all the associated paradoxes.

Admittedly, the chance that the particle is found outside the forward light cone falls off exponentially as you get further from the light cone, and that makes it extremely unlikely that I could go back and convince my mother to have an abortion, but if it is at all possible, it is still an unacceptable contradiction.

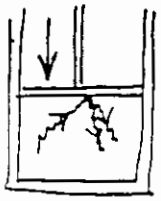
In practice, how does this affect atomic physics? Not at all, because we never tried to localize articles to spaces of order their Compton wavelength when doing atomic physics. We say that the electron is in the TV picture tube and there is not much chance that it is actually out in the room with you.

In principle, the ability to localize a single particle is a disaster, how does nature get out of it.



Particle trapped in container with reflecting walls

If the particle is localized to a space with dimensions on the order of L the uncertainty in the particle's momentum is $\sim \frac{1}{L}$. In the relativistic regime this tells us that the uncertainty in the particles energy is $\sim \frac{1}{L}$. As L gets less than $\frac{1}{\mu}$ states with more than one particle are energetically accessible. If the box contained a photon and the walls were mirrors the photon would pick up energy as it reflected off the descending mirror, it could turn into two photons as it reflected.



If we try to localize a particle in a box with dimensions smaller or on the order of a Compton wavelength it is unknown whether what we have in the box is 1 particle, 3 particles, 27 particles or 0 particles.

Relativistic causality is inconsistent with a single particle theory. The real world evades the conflict through pair production. This strongly suggests that the next thing we should do is develop a multi-partied theory.

SEPT. 25

The space we construct is called Fock space

Any number of one type of free spinless mesons

This formalism is also used in thermodynamics with the grand canonical ensemble. Particle number instead of being fixed, fluctuates around a value determined by the chemical potential.

Basis for single particle states $|\vec{k}\rangle$

$\langle \vec{k}'|\vec{k}\rangle = \delta^{(3)}(\vec{k}-\vec{k}')$ $\quad H|\vec{k}\rangle = \omega_{\vec{k}}|\vec{k}\rangle$ $\quad \vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$

This is the same as last time, except now this is just part of the basis.

Two particle states $|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$

↑ indistinguishability, Bose statistics

$$\langle \vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2' \rangle = \delta^{(3)}(\vec{k}_1-\vec{k}_1')\delta^{(3)}(\vec{k}_2-\vec{k}_2') + \delta^{(3)}(\vec{k}_1-\vec{k}_2')\delta^{(3)}(\vec{k}_2-\vec{k}_1')$$

$H|\vec{k}_1, \vec{k}_2\rangle = (\omega_{\vec{k}_1}+\omega_{\vec{k}_2})|\vec{k}_1, \vec{k}_2\rangle$ $\quad \vec{P}|\vec{k}_1, \vec{k}_2\rangle = (\vec{k}_1+\vec{k}_2)|\vec{k}_1, \vec{k}_2\rangle$

etc.

Also need a no particle state $|0\rangle$

$\langle 0|0\rangle = 1$ $\quad H|0\rangle = 0$ $\quad \vec{P}|0\rangle = 0$

↑ not part of a continuum

The vacuum is unique, it must satisfy $U(\Lambda)|0\rangle = |0\rangle$. All observers agree that the state with no particles is the state with no particles.

Completeness relation

$$1 = |0\rangle\langle 0| + \int d^3k\, |\vec{k}\rangle\langle\vec{k}| + \frac{1}{2!}\int d^3k_1 d^3k_2 |k_1 k_2\rangle\langle k_1 k_2| + \cdots$$

↑ to avoid double counting. Alternatively, just check that this works on $|\vec{k}, \vec{k}'\rangle$.

Now we could proceed by setting up equations for wave functions. To specify a state, a wave function contains a number, a function of three variables, a function of six variables, etc. Interactions involving a change in particle number will connect a function of six variables to a function of nine variables. This would be a mess.

We need a better description. As a pedagogical device, we will work in a periodic cubical box of side L for a while. Since a translation by L does nothing, the momenta must be restricted to allowed values

$$\vec{k} = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L} \right)$$

satisfying $\vec{k}\cdot(0,0,L) = 2n_z\pi$, $\vec{k}\cdot(0,L,0) = 2n_y\pi$, $\vec{k}\cdot(L,0,0) = 2n_x\pi$

Dirac deltas become Kronecker deltas and integrals become sums

$$\delta^{(3)}(\vec{k}-\vec{k}') \to \delta_{\vec{k}\vec{k}'} \qquad \int d^3k \to \sum_{\vec{k}}$$

$$\langle \vec{k} | \vec{k}' \rangle = \delta_{\vec{k}\vec{k}'} \qquad \langle \vec{k}_1, \vec{k}_2 | \vec{k}_1', \vec{k}_2' \rangle = \delta_{\vec{k}_1\vec{k}_1'}\delta_{\vec{k}_2\vec{k}_2'} + \delta_{\vec{k}_1\vec{k}_2'}\delta_{\vec{k}_2\vec{k}_1'}$$

Occupation number representation

Each basis state corresponds to a single function

$$|\vec{k}_1\rangle \leftrightarrow n(\vec{k}) = \delta_{\vec{k}\vec{k}_1}$$

$$|\vec{k}_1, \vec{k}_2\rangle \leftrightarrow n(\vec{k}) = \delta_{\vec{k}\vec{k}_1} + \delta_{\vec{k}\vec{k}_2}$$

$$|0\rangle \leftrightarrow n(\vec{k}) = 0$$

Given a function $n(\vec{k})$ in the occupation number description, we write the state

$|n(\cdot)\rangle$ Inner product $\langle n(\cdot)|n'(\cdot)\rangle = \prod_{\vec{k}} n(\vec{k})!\,\delta_{n(\vec{k})n'(\vec{k})}$

no argument to emphasize that the state depends on the whole function n, not just its value for one specific $\vec{k}$.

Define an occupation number operator

$$N(\vec{k})\,|n(\cdot)\rangle = n(\vec{k})\,|n(\cdot)\rangle$$

$$H = \sum_{\vec{k}} \omega_{\vec{k}}\, N(\vec{k}) \qquad \vec{P} = \sum_{\vec{k}} \vec{k}\, N(\vec{k})$$

This is a better formalism, but it could still use improvement. It would be nice to have an operator formalism that did not have any wave functions at all.

Note that H for our system has the form it would have if the system we were dealing with was actually a bunch of harmonic oscillators. The two systems are completely different. In ours the $N(\vec{k})$ tells how many particles are present with a given momentum. In a system of oscillators, $N(\vec{k})$ gives the excitation level of the oscillator labelled by $\vec{k}$.

Review of the simple harmonic oscillator

No physics course is complete without a lecture on the simple harmonic oscillator. We will review the oscillator using the operator formalism. We will then exploit the formal similarity to Fock space to get an operator formulation of our multi-particle theory.

$$H = \frac{1}{2}\omega\left[p^2 + q^2 - 1\right] \qquad [p, q] = -i$$

If $[p, A] = [q, A] = 0$ then $A = \lambda I$ (I: identity, λ: c-number)

This is all we need to get the spectrum.

Define raising and lowering operators

$$a \equiv \frac{q + ip}{2} \qquad a^\dagger = \frac{q - ip}{2} \qquad H = \omega a^\dagger a$$

$$[a, a^\dagger] = 1$$

$$[H, a^\dagger] = \omega a^\dagger \qquad [H, a] = -\omega a$$

$$H a^\dagger |E\rangle = a^\dagger H |E\rangle + \omega a^\dagger |E\rangle$$

$$= (E + \omega) a^\dagger |E\rangle$$

$a^\dagger$ is the raising operator

$$H a |E\rangle = (E - \omega) a |E\rangle$$

Because,

$$\langle\psi|H|\psi\rangle = \omega\langle\psi|a^\dagger a|\psi\rangle$$

$$= \omega \||a|\psi\rangle\|^2 \geq 0$$

Ladder of states

$\uparrow a^\dagger$ $E + 2\omega$

$\uparrow a^\dagger$ $E + \omega$

$\uparrow a^\dagger$ E

$\downarrow a$ $E - \omega$

the ladder of states must stop going down or else E becomes negative. The only way this can happen is if $a|E_0\rangle = 0$. Then $H|E_0\rangle = 0$

The lowest state of the ladder, having $E_0=0$ is denoted $|0\rangle$. The higher states are made by

$a^{\dagger n}|0\rangle \propto |n\rangle \qquad H|n\rangle = n\omega|n\rangle$

Get normalizations right $\quad a^\dagger|n\rangle = c_n|n+1\rangle$

$|c_n|^2 = \langle n|aa^\dagger|n\rangle = n+1 \Rightarrow \quad c_n = \sqrt{n+1}$

$a|n\rangle = d_n|n-1\rangle \qquad |d_n|^2 = n \quad d_n = \sqrt{n}$

Now we use $[P,A]=[Q,A]=0 \Rightarrow A=\lambda I$ to show that this ladder built from a state with $E_0=0$ is in fact the whole space. we do this by considering the projector, $\mathcal{P}$, onto the states in the ladder. Since a and $a^\dagger$ keep you within the ladder

$$[a,\mathcal{P}] = [a^\dagger,\mathcal{P}] = 0 \Rightarrow [P,\mathcal{P}] = [Q,\mathcal{P}] = 0$$
$$\Rightarrow \mathcal{P} = \lambda I$$

The projector onto the ladder is proportional to the identity. There is nothing sides the states we have found

Now we will apply this to Fock space

Define creation and annihilation operators for each momentum, $a^\dagger_{\vec{k}}, a_{\vec{k}}$, satisfying

$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'} \qquad [a_{\vec{k}}, a_{\vec{k}'}] = 0 \qquad [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0$

Hilbert space built by acting on $|0\rangle$ with strings of creation operators. $\quad |\vec{k}\rangle = a^\dagger_{\vec{k}}|\,\rangle$

$a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_3}|0\rangle = |\vec{k}_1, \vec{k}_2, \vec{k}_3\rangle \qquad a_{\vec{k}}|0\rangle = 0$

$H = \sum_{\vec{k}} \omega_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}} \qquad \vec{P} = \sum_{\vec{k}} \vec{k}\, a^\dagger_{\vec{k}} a_{\vec{k}}$ (IF $\forall_{\vec{k}} \quad 0=[a_{\vec{k}},A]=[a^\dagger_{\vec{k}},A]$

This tells us there are no other degrees of freedom $\Rightarrow A = \lambda I$

We've laid out a compact formalism for Fock space. Let's drop the box normalization and see if it is working.

$$[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^{(3)}(\vec{k}-\vec{k}') \quad [a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}]$$

$$H = \int d^3k\, \omega_{\vec{k}}\, a^\dagger_{\vec{k}} a_{\vec{k}} \qquad \vec{P} = \int d^3k\, \vec{k}\, a^\dagger_{\vec{k}} a_{\vec{k}}$$

Check energy and normalization of one particle states.

$$\langle \vec{k}'|\vec{k}\rangle = \langle 0| a_{\vec{k}'} a^\dagger_{\vec{k}} |0\rangle = \langle 0|[a_{\vec{k}'}, a^\dagger_{\vec{k}}]|0\rangle$$
$$= \delta^{(3)}(\vec{k}-\vec{k}')\langle 0|0\rangle = \delta^{(3)}(\vec{k}-\vec{k}')$$

$$[H, a^\dagger_{\vec{k}}] = \int d^3k'\, \omega_{\vec{k}'} [a^\dagger_{\vec{k}'} a_{\vec{k}'}, a^\dagger_{\vec{k}}] = \omega_{\vec{k}} a^\dagger_{\vec{k}} \Longrightarrow$$
$$H|\vec{k}\rangle = H\, a^\dagger_{\vec{k}}|0\rangle = [H, a^\dagger_{\vec{k}}]|0\rangle = \omega_{\vec{k}}|\vec{k}\rangle$$

$$[\vec{P}, a^\dagger_{\vec{k}}] = \vec{k}\, a^\dagger_{\vec{k}} \Longrightarrow \vec{P}|\vec{k}\rangle = \vec{k}|\vec{k}\rangle$$

Check normalization of two-particle states

$|\vec{k}_1, \vec{k}_2\rangle = a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2}|0\rangle$ using commutation relations check

$$\langle \vec{k}_1', \vec{k}_2' | \vec{k}_1, \vec{k}_2\rangle = \delta^{(3)}(\vec{k}_1-\vec{k}_1')\delta^{(3)}(\vec{k}_2-\vec{k}_2') + \delta^{(3)}(\vec{k}_1-\vec{k}_2')\delta^{(3)}(\vec{k}_2-\vec{k}_1')$$

Mathematical Footnote:

We've been calling the $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ operators. An operator takes any normalizable vector in Hilbert space to another normalizable vector: $A|\psi\rangle$ is normalizable whenever $|\psi\rangle$ is normalizable. Even x in 1-d QM is not an operator. $\int dx\, |f(x)|^2 < \infty \not\Rightarrow \int dx\, |f(x)|^2 x < \infty$. x is an unbounded operator. An unbounded operator has $A|\psi\rangle$ normalizable for a dense set of $|\psi\rangle$. $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ are even more awful than unbounded operators. An extra meson in a plane wave state added to any state is enough to make it nonnormalizable. $a^\dagger_{\vec{k}}$ is an operator valued distribution. You only get something a mathematician would be happy with after integration $\int d^3k\, f(\vec{k})\, a^\dagger_{\vec{k}}$ is acceptable. with f "sufficiently smooth" function.

Any $|\psi\rangle$ is the limit of a sequence in the dense set.

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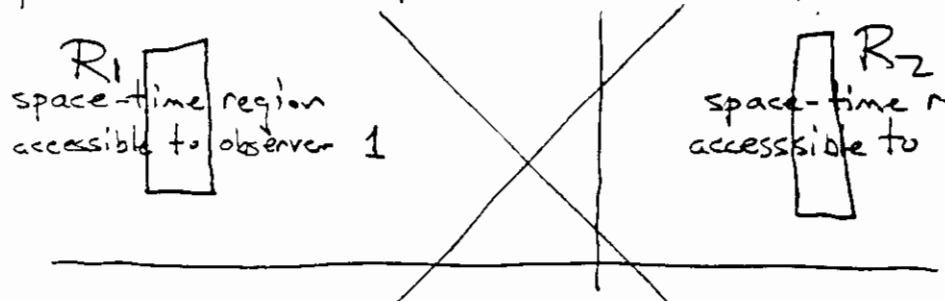
SEPT. 30

In ordinary QM, any hermitian operator operator is observable. This can't be true in relativistic quantum mechanics. Imagine two experimenters that are at space-like separation. If $x_1 \in R_1$ and $x_2 \in R_2$ then $(x_1 - x_2)^2 < 0$



Suppose observer 2 has an electron in his lab and she measures σ_x. If observer 1 can measure σ_y of that electron it will foul up observer 2's experiment. Half the time when she remeasures σ_x it will have been flipped. This tells her that observer 1 has made a measurement. This is faster than light communication, an impossibility. It is a little hard to mathematically state the obvious experimental fact that I can't measure the spin of an electron in the Andromeda galaxy. We don't have any way of localizing particles, any position operator, yet. We can make a mathematical statement in terms of observables:

If O_1 is an observable that can be measured in R_1 and O_2 is an observable that can be measured in R_2 and R_1 and R_2 are space-like separated then they commute:

$$[O_1, O_2] = 0$$

Observables are attached to space time points. A given observer cannot measure all observables, only the ones associated with his or her region of space-time.

It is not possible even in principle, for everyone to measure everything. Out of the hordes of observerbales, only a restricted set can be measured, a space-time region. Localization of measurements is going to substitute for localization of particles.

℞ The attachment of observables to space-time points has no analog in NRQM, nor in the classical theory of a single particle, relativistic or nonrelativistic, but it does have an analog in classical field theory. In electromagnetism, there are six observables at each point: $E_x(x) = E_x(\vec{x}, t)$, $E_y(x)$, $E_z(x)$ $B_x(x)$, $B_y(x)$, $B_z(x)$. We can't design an apparatus here that measures the E_x field now in the Andromeda galaxy.

In classical field theory these observables are numbers. In quantum mechanics, observables are given by operators. The fields will become quantum fields, an operator for each spacetime point. We can see in another way that the electric field is going to have to become a quantum field: How would you measure an electric field. You mount a charged ball, pith ball, to some springs and see how much the springs stretch! The location of the ball is given by an equation like:

$$q\ddot{x} = E_x(x)$$ ← Might be $\int d^3x\, f(x) E_x(x)$ where $f(x)$ gives some suitable average over the pith ball.

The amount the ball moves is related to the $\vec{E}$ field, and if the world is quantum mechanical, x must be an operator, and so E_x must be an operator.

We don't have a proof, but what is strongly suggested, is that QM and relativistic causality force us to introduce quantum fields. In fact, relativistic QM is practically synonymous with quantum field theory.

We will try to build our observables from a complete commuting set of quantum fields.

$\phi^a(x)$ operator valued functions of space-time.

$a = 1, \ldots, N$

Observables in a region R will be built out of $\phi^a(x)$ with $x \in R$. Observables in space-like separated regions will be guaranteed to commute if

(1) $\left[\phi^a(x), \phi^b(y)\right] = 0$ whenever $(x-y)^2 < 0$

We are going to construct our fields out of the creation and annihilation ops. These five conditions will determine them:

(2) $\phi^a(x) = \phi^{a\dagger}(x)$ hermition, observable

and that they have proper translation and Lorentz transformation properties

(3) $e^{-iP\cdot a}\,\phi^a(x)\,e^{iP\cdot a} = \phi^a(x-a)$

(4) $U(\Lambda)^\dagger\,\phi^a(x)\,U(\Lambda) = \phi^a(\Lambda^{-1}x)$

If this were not a scalar field, there would be extra factors here reflecting a change of basis as well as a change of argument

and finally, a simplifying assumption, that the fields are a linear combination of a_k and $a_k^\dagger$ (if that doesn't work, we'll try quadratic functions.)

(5) $\phi^a(x) = \int d^3k \left[F_k^a(x)\,a_{\vec{k}} + G_k^a(x)\,a_{\vec{k}}^\dagger\right]$

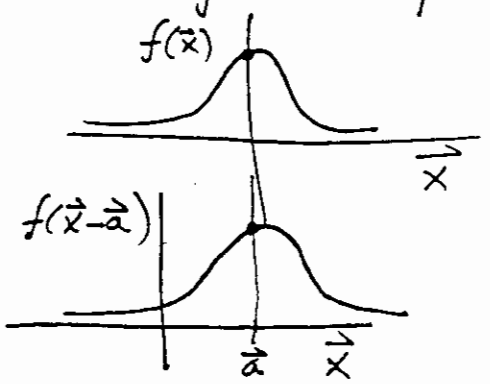
SEPT. 30 We can think of our unitary transformations in two ways; as transformations on the states $|\psi\rangle \to U|\psi\rangle$, or as transformations on the operators $A \to U^\dagger A U$. NOT BOTH!

4

What's embodied in assumption (3):

Given $U(\vec{a})$ the unitary operator of space translation by $\vec{a}$ $\left(U(\vec{a}) = e^{-i\vec{P}\cdot\vec{a}}\right)$ the translation of a state $|\psi\rangle$ is a state $|\psi'\rangle = U(\vec{a})|\psi\rangle$. Suppose the value of some observable, like charge density is

$$f(\vec{x}) = \langle\psi|\rho(\vec{x})|\psi\rangle$$



then it should be that

$$\langle\psi'|\rho(\vec{x})|\psi'\rangle = f(\vec{x}-\vec{a}).$$

Rewrite the first equation with $\vec{x} \to \vec{x}-\vec{a}$

$$f(\vec{x}-\vec{a}) = \langle\psi|\rho(\vec{x}-\vec{a})|\psi\rangle$$

Equate $\langle\psi|\rho(\vec{x}-\vec{a})|\psi\rangle = \langle\psi'|\rho(\vec{x})|\psi'\rangle$

$$= \langle\psi|e^{i\vec{P}\cdot\vec{a}}\rho(\vec{x})e^{-i\vec{P}\cdot\vec{a}}|\psi\rangle$$

A hermitian operator is determined by its expectation values

$$e^{i\vec{P}\cdot\vec{a}}\rho(\vec{x})e^{-i\vec{P}\cdot\vec{a}} = \rho(\vec{x}-\vec{a})$$

(3) is just the full relativistic form of this equation. The equation with $a = (t, \vec{0})$ is just the time evolution for Heisenberg fields.

For the exact same reason as $x-a$ appears in the RHS of (3), $\Lambda^{-1}x$ appears in the RHS of (4). (4) gives the Lorentz transformation properties of a scalar field. This is not much of an assumption. We can get fields transforming as vectors or tensors by taking derivatives of the ϕ's. Out of vector or tensor fields, we could make scalars.

In order to apply condition (4), it is nice to have the discussion phrased in terms of relativistically normalized creation and annihilation operators.

Recall the relativistically normalized one particle states

$$|k\rangle = (2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}\,|\vec{k}\rangle$$

Introduce $\alpha^+(k) = (2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}\,a_{\vec{k}}$ $\quad \alpha^+(k)|0\rangle = |k\rangle$

Multiparticle states are made by

$$\alpha^+(k_1)\cdots\alpha^+(k_n)|0\rangle = |k_1,\ldots,k_n\rangle$$

The Lorentz transformation properties of the states are $\quad U(\Lambda)|0\rangle = |0\rangle,$

$$U(\Lambda)|k_1,\ldots,k_n\rangle = |\Lambda k_1,\ldots,\Lambda k_n\rangle$$

and $\quad U(a) = e^{iP\cdot a}$ is found from

$P^\mu|0\rangle = 0$, $\quad P^\mu|k_1,\ldots,k_n\rangle = (k_1+\cdots+k_n)^\mu|k_1,\ldots,k_n\rangle$

On pages 4 and 8 of the Sept. 23 lecture we set up the criteria that $U(\Lambda)$ and $U(a)$ must satisfy. At the time our Hilbert space consisted only of the one particle part of the whole Fock space we have now. You should check that the criteria are satisfied in fockspace.

We can determine Lorentz transformation and translation properties of the $\alpha^+(k)$. Consider,

$$\begin{aligned} &U(\Lambda)\alpha^+(k)U(\Lambda)^+|k_1,\ldots,k_n\rangle \\ &= U(\Lambda)\alpha^+(k)|\Lambda^{-1}k_1,\ldots,\Lambda^{-1}k_n\rangle \\ &= U(\Lambda)|k,\Lambda^{-1}k_1,\ldots,\Lambda^{-1}k_n\rangle = |\Lambda k, k_1,\ldots,k_n\rangle \end{aligned}$$

That is $U(\Lambda)\alpha^+(k)U(\Lambda)^+|k_1,\dots,k_n\rangle$
$= \alpha^+(\Lambda k)|k_1,\dots,k_n\rangle$

$|k_1,\dots,k_n\rangle$ is an arbitrary state in our complete basis so we have determined it's action completely

$$U(\Lambda)\alpha^+(k)U(\Lambda)^+ = \alpha^+(\Lambda k)$$

Similarly, or by taking the adjoint of this equation

$$U(\Lambda)\alpha(k)U(\Lambda)^+ = \alpha(\Lambda k)$$

An analagous derivation shows that

$$e^{iP\cdot x}\alpha^+(k)e^{-iP\cdot x} = e^{ik\cdot x}\alpha^+(k)$$
$$e^{iP\cdot x}\alpha(k)e^{-iP\cdot x} = e^{-ik\cdot x}\alpha(k)$$

Now to construct the field ϕ (if there is more than one we'll label them when we've found them) satisfying all 5 conditions. First we'll satisfy condition (5) except we'll write the linear combination of $a_{\vec{k}}$ and $a_{\vec{k}}^+$ in terms of our new $\alpha(\vec{k})$ and $\alpha^+(\vec{k})$

It would be stupid not to use the L.I. measure.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\left[f_k(x)\,\alpha(k) + g_k(x)\,\alpha^+(k)\right]$$

By (3), $\phi(x) = e^{iP\cdot x}\phi(0)e^{-iP\cdot x}$, that is,

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\left[f_k(0)e^{iP\cdot x}\alpha(k)e^{-iP\cdot x} + g_k(0)e^{iP\cdot x}\alpha^+(k)e^{-iP\cdot x}\right]$$
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\left[f_k(0)e^{-ik\cdot x}\alpha(k) + g_k(0)e^{ik\cdot x}\alpha^+(k)\right]$$

We have found the x dependence of $f_k(x)$ and $g_k(x)$ now we will use (4) to get their k dependence. A special case of (4) is

$$\phi(0) = U(\Lambda)\,\phi(0)\,U(\Lambda)^\dagger$$

$$\int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[f_k(0)\,\alpha(k) + g_k(0)\,\alpha^\dagger(k) \right]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \Big[f_k(0) \underbrace{U(\Lambda)\,\alpha(k)\,U(\Lambda)^\dagger}_{\alpha(\Lambda k)} + g_k(0)\,\underbrace{U(\Lambda)\,\alpha^\dagger(k)\,U(\Lambda)^\dagger}_{\alpha^\dagger(\Lambda k)} \Big]$$

change variables $k \longrightarrow \Lambda^{-1}k$
Measure is unchanged

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[f_{\Lambda^{-1}k}(0)\,\alpha(k) + g_{\Lambda^{-1}k}(0)\,\alpha^\dagger(k) \right]$$

The coefficients of $\alpha(k)$ and $\alpha^\dagger(k)$ must be unchanged

$$\Rightarrow \quad f_k(0) = f_{\Lambda^{-1}k}(0) \quad \text{and} \quad g_k(0) = g_{\Lambda^{-1}k}(0)$$

k ranges all over the mass hyperboloid ($k^0 > 0$ sheet), but a Lorentz transformation can turn any of these k's into any other. So $f_k(0)$ and $g_k(0)$ are constants, independent of k.

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[f\, e^{-ik\cdot x}\,\alpha(k) + g\, e^{ik\cdot x}\,\alpha^\dagger(k) \right]$$

We have two linearly independent solutions of conditions (3), (4) and (5), the coefficients of the complex constants f and g. We'll name them. (Switching back to our old creation and annihilation ops.)

$$\phi^+(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\, a_{\vec{k}}\, e^{-ik\cdot x} \qquad \phi^-(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\, a^\dagger_{\vec{k}}\, e^{ik\cdot x}$$

Note $\phi^-(x) = \phi^+(x)^\dagger$

$\pm$ convention is bananas, but it was est'd by Heisenberg and Pauli 50 years ago.

Now we'll apply hermiticity. Two independent combinations satisfying (2) are

$$\phi(x) = \phi^+(x) + \phi^-(x) \quad \text{and} \quad \phi(x) = \frac{1}{i}\left(\phi^+(x) - \phi^-(x)\right)$$

These are two independent cases of the most general choice satisfying (2):

$$\phi(x) = e^{i\theta}\phi^+(x) + e^{-i\theta}\phi^-(x)$$

Now to satisfy (1). There are three possible outcomes of trying to satisfy (1).

Possibility A: Both of the above combinations are OK. We have two fields ϕ^1 and ϕ^2 commuting with themselves and each other at spacelike separation. In this possibility $\phi^+(x)$ and $\phi^-(x)$ commute with each other at spacelike separation.

Possibility B: Only one combination is acceptable. It is of the form $\phi = e^{i\theta}\phi^+ + e^{-i\theta}\phi^-$. While θ may be arbitrary, only one θ is acceptable.

Possibility C: The program crashes, and we could weaken (5) or think harder.

Let's first calculate some commutators. Using their expansions in terms of the a_k and $a_k^\dagger$ and the commutation relations for a_k and $a_k^\dagger$ we find

$$\left[\phi^+(x), \phi^+(y)\right] = 0 = \left[\phi^-(x), \phi^-(y)\right]$$

and

$$\left[\phi^+(x), \phi^-(y)\right] = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} e^{-ik\cdot(x-y)} \equiv \Delta_+(x-y, \mu^2)$$

also $\left[\phi^-(x), \phi^+(y)\right] = -\Delta_+(x-y)$ or just $\Delta_+(x-y)$

Δ_+ is manifestly Lorentz invariant $\Delta_+(\Lambda x) = \Delta_+(x)$.

Possibility A runs only if $\Delta_+(x-y)=0$ for $(x-y)^2<0$. We have encountered a similar integral when we were looking at the evolution of one particle position eigenstates. In fact,

$$i\partial_0 \Delta_+(x-y)=\int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot(x-y)}$$

is the very integral we studied, and we found that it did <u>not</u> vanish when $(x-y)^2<0$.

Possibility A is DEAD

On to possibility B. Take $\phi(x)=e^{i\theta}\phi^+(x)+e^{-i\theta}\phi^-(x)$ and calculate

$$[\phi(x),\phi(y)]=\Delta_+(x-y)-\Delta_+(y-x)$$

θ dependence drops out

Does this vanish when $(x-y)^2<0$. Yes, and we can see this without any calculations. A space-like vector has the property that it can be turned into minus itself by a (connected) Lorentz transformation. This and the fact that Δ_+ is Lorentz invariant, tells us that $[\phi(x),\phi(y)]=0$ when $x-y$ is space-like. We can choose θ arbitrarily, but we can't choose more than one θ. We'll choose $\theta=0$. Any phase could be absorbed into the $a_{\vec{k}}$ and $a^+_{\vec{k}}$.

Possibility B is ALIVE, and we don't have to go on to C.

We have our free scalar field of mass μ

$$\phi(x)=\phi^+(x)+\phi^-(x)=\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\left[a_{\vec{k}}e^{-ik\cdot x}+a^+_{\vec{k}}e^{ik\cdot x}\right]$$

Our field satisfies an equation (show using $k^2=\mu^2$)

$$(\Box+\mu^2)\,\phi(x)=0 \qquad \Box=\partial^\mu\partial_\mu$$

This is the Heisenberg equation of motion for the field. It is called the Klein-Gordon equation. If we had quantized the electromagnetic field it would have satisfied Maxwell's equations.

Actually, Schrodinger first wrote down the Klein-Gordon equation. He got it at the same time as he got the Schrodinger equation:

$$i\partial_0\psi=-\frac{1}{2\mu}\nabla^2\psi$$

This equation is by starting with $E=\frac{\vec{p}^2}{2m}$ noting that plane waves $E=\omega$ ($\hbar=1$) and $p^i=k^i$ and for $\omega=i\partial_0$ and $k^i=\frac{1}{i}\partial_i$

Schrodinger was no dummy, he knew about relativity, so he also obtained

$$(\Box+\mu^2)\,\phi(x)=0 \quad \text{from} \quad p^2=\mu^2$$

He immediately saw that something was wrong with the equation though. The equation has both positive and negative energy solutions. For a free particle the energies of its possible states are unbounded below! This is a disgusting relativistic generalization of a single particle wave equation, but with 50 years hindsight we see that this is no problem a field that can create and destroy particles.

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OCT. 2

We have constructed the quantum field. It is the object observables are built from. More than that though, we can reconstruct the entire theory from the quantum field. The structure we built in the last three lectures is rigid. We can make the top story the foundation. Suppose we started with a quantum field satisfying (as our ϕ does),

(1) $\phi(x) = \phi^\dagger(x)$ hermiticity

(2) $(\Box + \mu^2)\phi(x) = 0$ K-G equation

(3) $[\phi(x), \phi(y)] = \Delta_+(x-y) - \Delta_+(y-x)$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right]$$

(4) $U(\Lambda)^\dagger \phi(x) U(\Lambda) = \phi(\Lambda^{-1}x)$
$U(a)^\dagger \phi(x) U(a) = \phi(x-a)$
} ϕ is a scalar field

(5) $\phi(x)$ is a complete set of operators

i.e. if $\forall_x \; [A, \phi(x)] = 0 \Rightarrow A = \lambda I$

then from these properties, we could reconstruct the creation and annihilation operators and the whole theory. Conversely, all these properties follow from the expression for $\phi(x)$ in terms of the creation and annihilation operators. The two beginning points are logically equivalent.

OCT. 2 2

Defining $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$ and recovering their properties

Property (2) of ϕ, that ϕ is a solution of the Klein-Gordon equation, tells us that

$$\phi(x) = \int d^3k \left(\alpha_{\vec{k}} e^{-ik\cdot x} + \beta_k e^{ik\cdot x} \right) \quad \left(k^0 = \omega_{\vec{k}} = \sqrt{k^2 + \mu^2} \right)$$

This is because any solution of the Klein-Gordon equation can be expanded in a complete set of solutions of the K.-G. eqn., and the plane wave solutions are a complete set. Because ϕ is an operator, the coefficients in the expansion, $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$, are operators. Because of property (1) $\alpha_{\vec{k}} = \beta^{\dagger}_{\vec{k}}$.
Just to please my little heart, let's define

(These funny factors will make the commutation relations for the $a^{\dagger}_{\vec{k}}$ and $a_{\vec{k}}$ come out nice.)

$$\alpha_{\vec{k}} = \frac{a_{\vec{k}}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \qquad \beta_{\vec{k}} = \alpha^{\dagger}_{\vec{k}} = \frac{a^{\dagger}_{\vec{k}}}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}$$

Then the expansion is

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{ik\cdot x} \right)$$

This defines implicitly the $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$. To find their commutation relations we'll first have to solve for them. Note that

$$\phi(\vec{x}, 0) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + a^{\dagger}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

while

$$\dot{\phi}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(a_{\vec{k}} (-i\omega_{\vec{k}}) e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} (i\omega_{\vec{k}}) e^{ik\cdot x} \right)$$

and

$$\dot{\phi}(\vec{x}, 0) = \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_{\vec{k}}}{2}} \left(-i a_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + i a^{\dagger}_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}} \right)$$

OCT. 2 2'

The coefficient of $\frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$ in the expansion for $\phi(\vec{x},0)$ is $\frac{1}{\sqrt{2\omega_{\vec{k}}}}(a_{\vec{k}} + a^{\dagger}_{-\vec{k}})$ therefore $\frac{1}{\sqrt{2\omega_{\vec{k}}}}(a_{\vec{k}} + a^{\dagger}_{-\vec{k}}) = \int \frac{d^3x}{(2\pi)^{3/2}} \phi(\vec{x},0) e^{-i\vec{k}\cdot\vec{x}}$

The coefficient of $\frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$ in the expansion for $\dot{\phi}(\vec{x},0)$ is $\sqrt{\frac{\omega_{\vec{k}}}{2}}(-i a_{\vec{k}} + i a^{\dagger}_{-\vec{k}})$ therefore $\sqrt{\frac{\omega_{\vec{k}}}{2}}(-i a_{\vec{k}} + i a^{\dagger}_{-\vec{k}}) = \int \frac{d^3x}{(2\pi)^{3/2}} \dot{\phi}(x,0) e^{-i\vec{k}\cdot\vec{x}}$

That's just the inverse fourier transformation applied.

Now I can use these two expressions to solve for $a_{\vec{k}}$. Take $\sqrt{2\omega_{\vec{k}}}$ times the expression for $\frac{1}{\sqrt{2\omega_{\vec{k}}}}(a_{\vec{k}} + a^{\dagger}_{-\vec{k}})$ and add $\frac{2}{\sqrt{\omega_{\vec{k}}}}$ i times the expression for $\sqrt{\frac{\omega_{\vec{k}}}{2}}(-i a_{\vec{k}} + i a^{\dagger}_{-\vec{k}})$ and divide by 2 to get

$$a_{\vec{k}} = \frac{1}{2}\left[\sqrt{2\omega_{\vec{k}}} \int \frac{d^3x}{(2\pi)^{3/2}} \left(\phi(\vec{x},0) + \frac{i}{\omega_{\vec{k}}} \dot{\phi}(\vec{x},0)\right) e^{-i\vec{k}\cdot\vec{x}}\right]$$

Take the hermitian conjugate to get an expression for $a^{\dagger}_{\vec{k}}$.

Having solved for the $a_{\vec{k}}$ and $a^{\dagger}_{\vec{k}}$, we can now find their commutation relations. Write out the double integral for $[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}]$ and use property (3') on the following page (which is weaker than property (3)). You should get $\delta^{(3)}(\vec{k}-\vec{k}')$.

From property (4), you can derive, what $U(a)^{\dagger}\phi(x)\,U(a)$, $U(a)^{\dagger} a_{\vec{k}}\, U(a)$ and $U(\Lambda)^{\dagger} a_{\vec{k}}\, U(\Lambda)$ are. ← It is easier to find the action of Lorentz transformations on $a(k) \equiv [2\pi]^{3/2}\sqrt{2\omega_{\vec{k}}}\, a_{\vec{k}}$

You can also derive and state the analog of property (5).

The properties of the field $\phi(x)$ are actually a little overcomplete. We can weaken one of them, (3), without losing anything. Since ϕ' obeys the K.-G. equation, (2),

$$(\Box_x + \mu^2)[\phi(x), \phi(y)] = 0$$

We see that the commutator obeys the K.-G. equation for any given y. For a given y we only need to give the commutator and the time derivative on one initial time surface, and the K.-G. equation determines its evolution off the surface. So we will weaken (3) to

$$(3') \qquad [\phi(\vec{x},t), \phi(\vec{y},t)] = 0$$
$$[\partial_0 \phi(\vec{x},t), \phi(\vec{y},t)] = -i\delta^{(3)}(\vec{x}-\vec{y})$$

We can easily check that this is the right specialization of property (3) by doing the integrals, which are easy with $x^0 = y^0$. (For a given $y = (\vec{y}, t)$ we have chosen our initial surface to be $x^0 = t$, (3') are called equal time commutation relations.)

$$[\phi(\vec{x},t), \phi(\vec{y},t)] = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left[e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right]$$

Change variables $\vec{k} \to -\vec{k}$ in second term

$$= 0$$

$$[\partial_0 \phi(\vec{x},t), \phi(\vec{y},t)] = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} (-i\omega_{\vec{k}}) \left[e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right]$$

again $\vec{k} \to -\vec{k}$ in second term

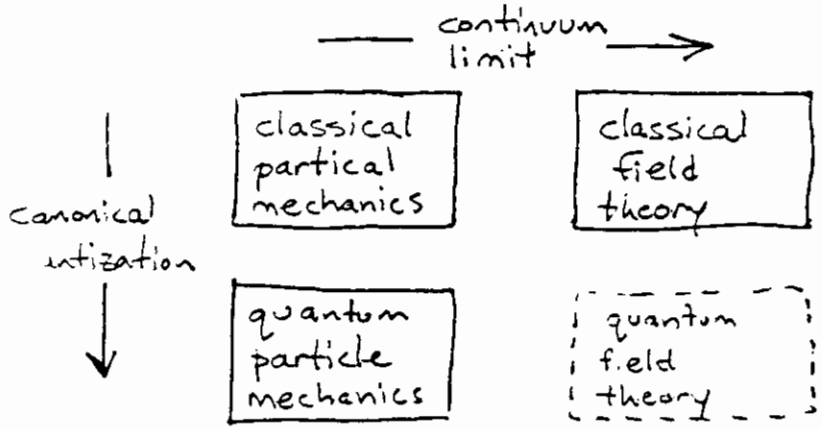
$$= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} (-i\omega_{\vec{k}}) 2 e^{i\vec{k}\cdot(\vec{x}-\vec{y})}$$

$$= -i\delta^{(3)}(\vec{x}-\vec{y})$$

(Reminds us a little of a bad dream in which $[p_a, e^b] = -i\delta_a^b$)

In the remainder of this lecture we are going to develop a completely different approach to quantum field theory. We will obtain a field satisfying properties (1), (2), (3′), (4) and (5), and then we'll stop. We have just shown that given a field satisfying these properties we can recover everything we did in the first three lectures. The new approach is

The method of the missing box



Anyone looking at this diagram can see there is a missing box

Taking the continuum limit essentially is just letting the number of coordinates, degrees of freedom, go to infinity. This is usually done in a cavalier way. It doesn't matter a whole lot if you have a discrete infinity or a continuous infinity. With a continuous infinity you can Fourier transform the coordinates to obtain a discrete set of coordinates.

Canonical quantization is a turn the crank way of getting quantum mechanics, beginning with a classical Hamiltonian.

We'll combine these two standard operations to get the missing box, but first a lightning review of the essential principles of the three boxes we have.

Classical Particle Mechanics

We start with a Lagrangian, a function of the generalized coordinates, and their time derivatives

$$L(q^1,\dots,q^N,\dot q^1,\dots,\dot q^N,t) \qquad q^a(t) \quad a\in 1,\dots,N$$

↑ real number functions of time

e.g. $L=\frac{1}{2}m\dot q^2 - V(q)$

We define the action $S=\int_{t_1}^{t_2} dt\, L$ and apply Hamilton's principle to get the equations of motion. That is, we vary S by arbitrarily varying $\delta q^a(t)$ except at the endpoints

↑ in the 19th century this was called

$$\delta q^a(t_1)=\delta q^a(t_2)=0$$

and demand $0=\delta S$

$$\delta S=\int_{t_1}^{t_2} dt\left[\sum_a\left(\frac{\partial L}{\partial q^a}\delta q^a+\frac{\partial L}{\partial \dot q^a}\delta \dot q^a\right)\right]$$

↓ parts integration

$$=\int_{t_1}^{t_2} dt\sum_a\left(\frac{\partial L}{\partial q^a}-\frac{d}{dt}p_a\right)\delta q^a + \cancel{p_a\,\delta q^a\Big|_{t_1}^{t_2}}$$

vanishes because of endpoint restriction

$$p_a\equiv\frac{\partial L}{\partial \dot q^a}$$

Since the variation $\delta q^a(t)$ is arbitrary

$$\frac{\partial L}{\partial q^a}=\dot p_a \quad \text{for all } a. \qquad \text{e.g. } p=m\dot q \quad \dot p=-\frac{dV}{dq}$$

These are the Euler-Lagrange equations.

Around 1920(!) the Hamiltonian formulation of classical particle mechanics was discovered.

Define $H(p_1,\dots,p_N,q_1,\dots,q_N,t) = \sum_a p_a \dot{q}^a - L$

H must be written in terms of the p's and q's only, not the $\dot{q}$'s. (This is not always possible. The new variables must also be independent, so it is possible to vary them independently. Examples where the p's are not complete and independent are electromagnetism, and a particle on a sphere in $\mathbb{R}^3$. The Lagrangian for the latter system may be taken as

$$L = \frac{1}{2} m \dot{\vec{r}}^2 + \lambda(\vec{r}^2 - a^2) - V(\vec{r})$$

How can I vary that?!?

The equation of motion for the variable λ enforces the constraint. In the passage to the Hamiltonian formulation $p_\lambda = 0$; p_λ is not an independent function. In this system, one way to get to the Hamiltonian description is to first eliminate λ and one more coordinate, taking perhaps θ, φ, polar coordinates for the system, rewriting the Lagrangian, and then trying again.)

Vary the coordinates and momenta

$$dH = \sum_a \left(dp_a \dot{q}^a + \cancel{p_a d\dot{q}^a} - \underbrace{\frac{\partial L}{\partial q^a}}_{\dot{p}_a \text{ by the E-L equations}} dq^a - \cancel{\frac{\partial L}{\partial \dot{q}^a} d\dot{q}^a} \right)$$

Fortunately we didn't have to expand out $d\dot{q}^a$

Read off Hamilton's equations

$$\frac{\partial H}{\partial p_a} = \dot{q}^a \qquad \frac{\partial H}{\partial q^a} = -\dot{p}_a$$

Notice: $\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dH}{dt} = 0$

in that case H is called the energy (which is the name reserved for the conserved quantity resulting from time translation invariance)

Quantum Particle Mechanics

We replace the classical variables p_a, q^a by operator valued functions of time satisfying

$$[q^a(t), q^b(t)] = 0 = [p_a(t), p_b(t)]$$

and

$$[p_a(t), q^b(t)] = -i\delta_a^{\ b} \qquad (\hbar = 1)$$

The p's and q's are hermitian observables. They are assumed to be complete. They determine the Hilbert space. For example 1-d particle mechanics, the space of square integrable functions. A basis set can be the eigenstates of q. p or rather e^{ipx} tells you how to relate the phases of the various eigenstates, and indeed even that the range of q is $\mathbb{R}$ (not $[-1,7]$ or anything else).

The quantum Hamiltonian, which determines the dynamics, is just the classical Hamiltonian except it is now a function of the operator p's and q's. For any A

$$\frac{dA}{dt} = i[H, A] + \frac{\partial A}{\partial t}$$

← if there is explicit dependence on time other than that implicit in the time dependence of the p's and q's

H is the generator of infinitesimal time translations just as in classical mechanics.

H suffers from ordering ambiguities. Because p and q don't commute, it is not clear, and it may matter, whether you write p^2q, qp^2 or pqp. Sometimes this ambiguity can be cleared up using other criteria: for example, in central force problems, we quantize in Cartesian coordinates and then transform to central coordinates.

Heisenberg equations of motion

$$\frac{dq^a}{dt} = i[H, q^a] = i\left(-i\frac{\partial H}{\partial p_a}\right) = \frac{\partial H}{\partial p_a}$$

Similarly, $\frac{dp_a}{dt} = -\frac{\partial H}{\partial q^a}$

This step depends only on the commutation relations for the p's and q's. Fairly easy to see for a polynomial in the p's and q's

This is the motivation for the commutation relations. It is a way of putting the correspondence principle into the theory. The quantum equations resemble the classical equations, at least up to the ordering ambiguities, and any variation due to ordering ambiguities is down by a factor of $\hbar$. We wouldn't expect any general procedure for turning classical theories into quantum theories, motivated only by the correspondence principle, to be able to fix those ambiguities. We can be a little more precise about how the classical equations are actually recovered. In general, the Heisenberg equations of motion for an arbitrary operator A relate one polynomial in p, q, $\dot{p}$ and $\dot{q}$ to another. We can take the expectation value of this equation to obtain (a quantum mechanical average of) an equation between observables. In the classical limit, when fluctuations are small, expectations of products can be replaced by products of expectations, $\langle p^n \rangle \to \langle p \rangle^n$ $\langle pq \rangle \to \langle p \rangle \langle q \rangle$, and this turns our equation among polynomials of quantum operators into an equation among classical variables.

The other two boxes are actually going to go quite quickly, because if you are just a little cavalier, the continuum limit is little more than some new notation.

Classical Field Theory

We have an infinite set of generalized coordinates, real number functions of time, $\phi^a(\vec{x},t)$ labelled by a discrete index, a, and a continuous index, $\vec{x}$.

$$\phi^a(\vec{x},t) \longleftrightarrow q^a(t)$$

$$t \longleftrightarrow t$$

$$a \longleftrightarrow a,\vec{x}$$

It is sometimes a handy mnemonic to think of $t,\vec{x}$ as a generalization of t, but that is not the right way to think about it. For example we are used to giving initial value data at fixed t in CPM. In CFT we don't give initial value data at fixed t and $\vec{x}$, that is obviously incomplete. We give initial value data for fixed t and <u>all</u> $\vec{x}$.

With cowboy boldness, everywhere in CPM we see a sum on a, we'll just replace it with a sum on a and an integral over $\vec{x}$, and everywhere we see $\delta^a{}_b$ we'll replace it with $\delta^a{}_b\,\delta^{(3)}(\vec{x}-\vec{y})$. The Dirac delta function has the exact same properties in integrals that the Kronecker delta has in sums.

The next thing to do is write down Lagrangians. Because a CPM Lagrangian can contain products of q^a with different a, our Lagrangian could contain products of $\phi(\vec{x},t)$ with different $\vec{x}$

$$\int d^3x\, d^3y\, d^3z\, f_{abc}(\vec{x},\vec{y},\vec{z})\,\phi^a(\vec{x},t)\,\phi^b(\vec{y},t)\,\phi^c(\vec{z},t)$$

$$\longleftrightarrow \sum_{a,b,c} f_{abc}\, q^a(t)\, q^b(t)\, q^c(t)$$

But note that a CPM Lagrangian does not contain products at different times.

With an eye to Lorentz invariance, and noticing that Lagrangians are local in time, we will specialize to Lagrangians that are local in space. Also, you know that when taking continuum limits, differences of neighboring variables become spatial derivatives and a combination like

$$\frac{1}{a^2}\left[\rho((n-1)a,t)+\rho((n+1)a,t)-2\rho(na,t)\right]$$

would become $\frac{\partial^2\rho}{\partial x^2}$ in the continuum. But again with an eye to Lorentz invariance, because only first derivatives with respect to time appear in the Lagrangian, we will only consider first derivatives with respect to the x^i. So L has the form

$$L(t)=\int d^3x\,\mathcal{L}\left(\phi^a(x),\partial_\mu\phi^a(x),x\right)$$

And the action

$$S=\int_{t_1}^{t_2}dt\,L(t)=\int d^4x\,\mathcal{L}$$

The Euler-Lagrange equations which come from varying S will be Lorentz covariant if $\mathcal{L}$ is a Lorentz scalar. At three points we have used Lorentz invariance to cut down on the possible forms for L. These have been specializations, not generalizations. Now we'll apply Hamilton's principle

$$0=\delta S \text{ (under arbitrary variations } \delta\phi \text{ satisfying } \delta\phi^a(\vec{x},t_1)=\delta\phi^a(\vec{x},t_2)=0)$$

$$=\sum_a\int d^4x\left(\frac{\partial\mathcal{L}}{\partial\phi^a(\vec{x},t)}\delta\phi^a(\vec{x},t)+\underbrace{\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\phi^a}}_{\equiv\pi^\mu_a}\underbrace{\delta\partial_\mu\phi^a}_{\partial_\mu\delta\phi^a}\right)$$

↓ Do parts integration. As usual, the restrictions on $\delta\phi^a$ make the surface terms at t_1 and t_2 drop out

$$=\sum_a\int d^4x\left[\frac{\partial\mathcal{L}}{\partial\phi_a}-\partial_\mu\pi^\mu_a\right]\delta\phi^a$$

implying $\frac{\partial\mathcal{L}}{\partial\phi_a}=\partial_\mu\pi^\mu_a$ (for all $\vec{x}$ and a) Euler-Lagrange equations

OCT. 2

Π^{μ}_{a} should not be thought of as a four-vector generalization of p_a. The correspondence is

$$\Pi^{0}_{a}(\vec{x},t) \longleftrightarrow p_a(t)$$

In fact Π^{0}_{a} is often just written Π_a.

We should say something about the surface terms at spatial infinity which we ignored when we did our parts integration. One can say "we are only considering field configurations which fall off sufficiently rapidly at spatial infinity that we can ignore surface terms." Alternatively we could work in a box with periodic boundary conditions. Anyway, we'll just be slothful.

A simple example of a possible $\mathcal{L}$:

Most general $\mathcal{L}$ satisfying (3) conditions

(1) Build $\mathcal{L}$ out of one real scalar field, ϕ

$\phi = \phi^*$ (not $\phi = \phi^\dagger$, we're not doing QFT yet)

(2) $\mathcal{L}$ is a Lorentz scalar

(3) $\mathcal{L}$ is quadratic in ϕ and $\partial_\mu \phi$

(1) and (3) are for simplicity. A motivation for (3) is that a quadratic action yields linear equations of motion, the easiest ones to solve. Most general $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2} a \left[\partial_\mu \phi \, \partial^\mu \phi + b \, \phi^2 \right]$$

One of these constants is superfluous; we are always free to rescale ϕ, $\phi \to \phi/\sqrt{\text{fiat}}$. $\mathcal{L}$ becomes

$$\mathcal{L} = \pm \frac{1}{2} \left[\partial_\mu \phi \, \partial^\mu \phi + b \, \phi^2 \right]$$

The Euler Lagrange equations for our example are

$$\pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = \pm \partial^\mu \phi \qquad \text{and}$$

$$\partial_\mu \pi^\mu - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \text{or} \qquad \pm(\partial^\mu \partial_\mu \phi - b\phi) = 0$$

In general the Hamiltonian which was $\sum_a p_a \dot{q}^a - L$ in CPM is

$$H = \sum_a \int d^3x \left(\pi^0_a \, \partial_0 \phi^a - \mathcal{L} \right) = \int d^3x \, \mathcal{H}$$

$$\mathcal{H} = \sum_a \pi^0_a \, \partial_0 \phi^a - \mathcal{L} \qquad \text{is the Hamiltonian density}$$

In our example

$$H = \pm \int d^3x \left[\frac{1}{2} (\pi^0)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 - b\phi^2 \right]$$

Since each of these terms separately can be made arbitrarily large, if the energy is to be bounded below, they each better have a positive coefficient. $\pm$ better be $+$ and b better be

$$b = -\mu^2 \qquad \text{definition of } \mu \geq 0$$

The E-L equation is now $\partial_\mu \partial^\mu \phi + \mu^2 \phi = 0$

Gosh, this is looking familiar now.

We are ready to fill in the last box. We are ready to canonically quantize classical field theory. Since we are not worrying about the passage to an infinite number of variables, this will be little more than a notational change in the canonical quantization of CPM. It would not even have required a notational change if Newton hadn't chosen two ways of writing S for sum, $\int$ and $\sum$.

Quantum Field Theory

We replace the classical variables $\phi^a, \pi^o{}_a$ by quantum operators satisfying

$$[\phi^a(\vec{x},t), \phi^b(\vec{y},t)] = 0 = [\pi^o{}_a(\vec{x},t), \pi^o{}_b(\vec{y},t)]$$

and $$[\pi^o{}_a(\vec{x},t), \phi^b(\vec{y},t)] = -i\delta^b{}_a\,\delta^{(3)}(\vec{x}-\vec{y})$$

The Dirac delta for the continuous index $\vec{x}$ is just the continuum generalization of the Kronecker delta $H = \int d^3x\, \mathcal{H}(\pi^a{}_a, \phi^a, x)$ determines the dynamics as usual. The commutation relations are set up to reproduce the Heisenberg equations of motion. Since this is just a change of notation there is no need to redo any proofs, but let's check that things are working in our example anyway.

$$H = \int d^3x\, \frac{1}{2}\left[(\pi^o)^2 + (\vec{\nabla}\phi)^2 + \mu^2\phi^2\right]$$

$$\partial_o \phi(\vec{y},t) = i[H, \phi(\vec{y},t)] = i(-i)\int d^3x\, \pi^o(\vec{x},t)\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$= \pi^o(\vec{y},t)$$

Similarly, $\partial_o \pi^o(\vec{y},t) = \vec{\nabla}^2\phi - \mu^2\phi$. You need the commutator $[\pi^o(\vec{y},t), \vec{\nabla}\phi(\vec{x},t)]$ which is obtained by taking the gradient of the equal time commutation relation (ETCR).

We have reproduced our quantum field satisfying (1), (2), (3'), (4) and (5). This was accomplished in one lecture rather than three because this is a mechanical method without physical insight. The physical interpretation comes at the end instead of at the beginning. In our first method, the constructive method, we spook each object we introduced to make sure it made sense, and we finally obtained a local observable, the quantum field. We can't put interactions in in this method though, because in the very first steps we had to know the whole spectrum of the theory, and about the only theory we know the exact spectrum for is the free theory. In our magical canonical quantization method it's easy to put in interactions: just let $\mathcal{L} \to \mathcal{L} - \lambda\phi^4$. At the first order in perturbation theory the part of $\lambda\phi^4$ that has two creation and two annihilation operators produces two-into-two scattering! At second order we'll get two-into-four and two-into-six scattering. Looks easy... if there aren't any boson traps... but boy are there going to be boson traps. (← pair production!)

OCT. 7

The simplest example obtained from applying canonical quantization to a free scalar field led to the same theory we got by constructing a local observable painstakingly in the theory we had before.

We'll double check that we get the same Hamiltonian we had in that theory. Evaluate

$$H = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\phi)^2 + \mu^2\phi^2\right] \qquad \pi = \dot{\phi}$$

Write ϕ in terms of its Fourier expansion

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\left[a_{\vec{k}}\, e^{-ik\cdot x} + a^\dagger_{\vec{k}}\, e^{ik\cdot x}\right]$$

Substitution would lead to a triple integral, but the x integration is easily done yielding a δ function, which does one of the k integrals.

$$H = \frac{1}{2}\int \frac{d^3k}{2\omega_{\vec{k}}}\Big\{ a_{\vec{k}}\, a_{-\vec{k}}\, e^{-2i\omega_{\vec{k}}t}\left(-\omega^2_{\vec{k}} + |\vec{k}|^2 + \mu^2\right)$$

$\int d^3x\, e^{+i\vec{k}\cdot\vec{x}}\, e^{+i\vec{k}'\cdot\vec{x}} = (2\pi)^3\delta^{(3)}(\vec{k}+\vec{k}') \quad \vec{k}' = -\vec{k}$

That what multiplies $e^{\pm 2i\omega_{\vec{k}}t}$ is 0 is good. H should not be time-dependent

$$+ a_{\vec{k}}\, a^\dagger_{\vec{k}}\left(\omega^2_{\vec{k}} + |\vec{k}|^2 + \mu^2\right)$$

$$+ a^\dagger_{\vec{k}}\, a_{\vec{k}}\left(\omega^2_{\vec{k}} + |\vec{k}|^2 + \mu^2\right)$$

$$+ a^\dagger_{\vec{k}}\, a^\dagger_{-\vec{k}}\, e^{2i\omega_{\vec{k}}t}\left(-\omega^2_{\vec{k}} + |\vec{k}|^2 + \mu^2\right)\Big\}$$

H had four types of terms: ones creating particles with momentum $\vec{k}$ and $-\vec{k}$, ones destroying particles " " " " " ", ones creating a particle with momentum $\vec{k}$, then destroying one with momentum $\vec{k}$, and ones destroying a particle with momentum $\vec{k}$, then creating one with momentum $\vec{k}$. These all conserve momentum.

$$H = \frac{1}{2}\int d^3k\, \omega_k\left(a_{\vec{k}}\, a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}}\, a_{\vec{k}}\right)$$

Almost, but not quite what we had before.

Using $[a_{\vec{k}}, a^\dagger_{\vec{k}'}] = \delta^3(\vec{k}-\vec{k}')$ gives $H = \int d^3k\, \omega_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\delta^{(3)}(0)\right)$ $\delta^{(3)}(0)$! Can get some idea of the meaning of this by putting the system in a box. H becomes

$$\frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}} \left(a_{\vec{k}} a^\dagger_{\vec{k}} + a^\dagger_{\vec{k}} a_{\vec{k}}\right) = \sum_{\vec{k}} \omega_{\vec{k}} \left(a^\dagger_{\vec{k}} a_{\vec{k}} + \frac{1}{2}\right)$$

In a box we see that this is just a zero point energy. (It is still infinite.) though.) Just as the spectrum of the spectrum of $H = \frac{1}{2}(p^2 + \omega^2 q^2)$ starts at $\frac{1}{2}\omega$, not zero, the spectrum of our Hamiltonian starts at $\frac{1}{2}\sum_{\vec{k}} \omega_{\vec{k}}$, not zero. That this constant is infinite even in a box is because even in a box the field has an infinite number of degrees of freedom. There can be an excitations of modes with arbitrarily short wavelengths. The zero point energy is ultraviolet divergent. In general, infinite systems, like an infinite crystal, also have "infrared divergences." If the energy density of an infinite crystal is changed by a finite amount, the total energy is changed by an infinite amount. Our Hamiltonian for the system in infinite space, would still have an infinity ($\delta^{(3)}(0)$) even if the momentum integral were cut off, say by restricting it to $|\vec{k}| < K$. That part of the infinity is eliminated by putting the system in a box, eliminating long wavelengths. Our system in infinite space has a zero-point energy that is also infrared divergent.

This is no big deal for two reasons.

(a) You can't measure absolute energies, only energy differences, so it's stupid to ask what the zero point energy is. This even occurs in introductory physics. We usually put interaction energies to be zero when particles are infinitely separated, but for some potentials you can't do that and you have to choose your zero another place.

(b) this is just an ordering ambiguity. Just as the quantum Hamiltonian for the harmonic oscillator could be chosen as

$$H = \frac{1}{2}(p + i\omega q)(p - i\omega q)$$

we could reorder our Hamiltonian (see below.).

In general relativity the absolute value of the energy density does matter. Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

couple directly to the energy density T_{00}. Indeed, introducing a change in the vacuum energy density, in a covariant way,

$$T_{\mu\nu} \to T_{\mu\nu} - \lambda g_{\mu\nu}$$

is just a way of changing the cosmological constant, a term introduced by Einstein, and repudiated by him 10 years later. No astronomer has ever observed a nonzero cosmological constant. Our theory is eventually going to be applied to the strong interactions, maybe even to some grand unified theory. Strong interactions have energies typically of 1GeV and a characteristic length of a fermi, 10^{-13}cm. With a cosmic energy density of 10^{39} GeV/cm^3, the universe would be about 1km long according to Einstein's equations. You couldn't even get to MIT without coming back where you started. We won't talk about why the cosmological constant is zero in this course. They don't explain it in any course given at Harvard because nobody knows why it is zero.

OCT. 7

Reordering the Hamiltonian:

Given a set of free fields (possibly with different masses) $\phi_1'(x_1), \ldots \phi_n^n(x_n)$, define the normal ordered product

$$:\phi_1(x_1)\cdots\phi_n(x_n):$$

as the usual product except rearranged so that all the annihilation operators on the right and a *fortiori* all creation operators on the left. No further specification is needed since all annihilation operators commute with one another as do all annihilation operators. This operation is only defined for free fields satisfying the K.-G. equation, which tell us we can write the field in terms of time independent creation and annihilation operators.

Redefine H to be $:H:$. This gets rid of the normal ordering constant.

This is the first infinity encountered in this course. We'll encounter more ferocious ones. We ran into it because we asked a dumb question, a physically uninteresting question about an unobservable quantity. Later on we'll have to think harder about what we've done wrong to get rid of troublesome infinities.

If we wanted to get as quickly as possible to applications of QFT, we'd develop scattering theory and perturbation theory next. But first we are going to get some more exact results from field theory.

What's wrong? $a^\dagger a = aa^\dagger - 1$. Normal order both sides $:a^\dagger a: = :aa^\dagger: - 1$, $0 = -1$. Answer: We don't "normal order" equations. Normal ordering is not derived from the ordinary product any more than the cross product is derived from the scalar product. Normal order cannot accurately be used is a ...

Symmetries and Conservation Laws

We will study the relationship between symmetries (or invariances) and conservation laws in the classical Lagrangian framework. Our derivations will only use classical equations of motion. Hopefully everything will go through all right when Poisson brackets become commutators, since the commutation relations are set up to reproduce the classical equations of motion. In any given case you can check to see if ordering ambiguities or extraterms in commutators screw up the calculations.

Given some general Lagrangian $L(q^a, \dot{q}^a, t)$ and a transformation of the generalized coordinates

$$q^a(t) \to q^a(t, \lambda) \qquad q^a(t, 0) = q^a(t)$$

we can define

$$Dq^a \equiv \frac{\partial q^a}{\partial \lambda}\Big|_{\lambda=0}$$

This is useful because the little transformations will be important.

Some examples:

(1) Space translation of point particles described by n vectors $\vec{r}^{\,a}$ $a = 1, \ldots, n$
The Lagrangian we'll use is

$$\sum_{a=1}^{n} \frac{m_a}{2} |\dot{\vec{r}}^{\,a}|^2 - \sum_{ab} V_{ab}\left(|\vec{r}_a - \vec{r}_b|\right)$$

The transformation is $\vec{r}_a^{\,a} \to \vec{r}^{\,a} + \vec{e}\,\lambda$, all particles in the system moved by $\vec{e}\,\lambda$
$D\vec{r}^{\,a} = \vec{e}$

(2) Time translations for a general system. Given some evolution $q^a(t)$ of the system, the transformed system is a time λ ahead $q^a(t) \to q^a(t+\lambda)$, $Dq^a = \frac{\partial q^a}{\partial t}$.

(3) Rotations in the system of example (1)

$$\vec{r}^{\,a} \to R(\lambda \vec{e})\,\vec{r}^{\,a} \qquad D\vec{r}^{\,a} = \vec{e} \times \vec{r}^{\,a}$$

rotation matrix, angle, axis

(If $\vec{e} = \hat{z}$, $Dx^a = y^a$, $Dy^a = -x^a$, $Dz^a = 0$.)

Most transformations are not symmetries (invariances).

Definition: A transformation is a symmetry iff $DL = \frac{df}{dt}$ for some $F(q^a, \dot{q}^a, t)$. This equality must hold for arbitrary $q^a(t)$ not necessarily satisfying the equations of motion.

Why is this a good definition? Hamilton's principle is

$$0 = \delta S = \int_{t_1}^{t_2} dt\, DL = \int_{t_1}^{t_2} dt\, \frac{df}{dt} = F(t_2) - F(t_1)$$

Thus a symmetry transformation won't affect the equations of motion (we only consider transformations that vanish at the endpoints).

Back to our examples:

(1) $DL = 0$ $\qquad F = 0$

(2) If L has no explicit time dependence, all of its time dependence comes from its dependence on q^a and $\dot{q}^a$, $\frac{\partial L}{\partial t} = 0$, then

$$DL = \frac{dL}{dt} \qquad F = L$$

(3) $F = 0$

Theorem (E. Noether say 'Nota'): For every symmetry there is a conserved quantity.

The proof comes from considering two expressions for DL.

$$DL = \sum_a \left(\frac{\partial L}{\partial q^a} Dq^a + p_a D\dot{q}^a \right) \quad \text{used } p_a \equiv \frac{\partial L}{\partial \dot{q}^a}$$

$$= \sum_a \left(\dot{p}_a Dq^a + p_a D\dot{q}^a \right) \quad \text{used E-L equations}$$

$$= \frac{d}{dt} \sum_a p_a Dq^a \quad \text{by equality of mixed partials } D\dot{q}^a = \frac{d}{dt} Dq^a$$

By assumption $DL = \frac{dF}{dt}$. Subtracting these two expressions for DL we see that the quantity

$$Q = \sum_a p_a Dq^a - F \quad \text{satisfies} \quad \frac{dQ}{dt} = 0$$

(There is no guarantee that $Q \neq 0$, or that for each independent symmetry we'll get another independent Q, in fact the construction fails to produce a Q for gauge symmetries.)

We can write down the conserved quantities in our examples

(1) $\left(p_a = m_a \dot{\vec{r}}^a\right)$ $D\vec{r}^a = \vec{e}$ $F = 0$

$$Q = \sum_a m_a \vec{e} \cdot \dot{\vec{r}}^a = \vec{e} \cdot \sum_a m_a \dot{\vec{r}}^a$$

For each of three independent $\vec{e}$'s we get a conservation law. $\vec{p}$ The momentum

$$\vec{p} = \sum_a m_a \dot{\vec{r}}^a \quad \text{is conserved.} \left(\frac{d\vec{p}}{dt} = 0 \right).$$

Whenever we get conserved quantities from spatial translation invariance, whether or not the system looks anything like a collection of point particles we'll call the conserved quantities the momentum.

(2) $Dq^a = \frac{\partial q^a}{\partial t}$ $\quad F = L$ Note: Q is identical to H

$Q = \sum_a p_a \dot{q}^a - L$ is conserved (when $\frac{\partial L}{\partial t} = 0$)

Whenever we get a conserved quantity from time translation invariance, we'll call the conserved quantity the energy.

(3) $D\vec{r}^a = \vec{e} \times \vec{r}^a = F = 0$

$$Q = \sum_a \vec{p}_a \cdot (\vec{e} \times \vec{r}^a) = \sum_a \vec{e} \cdot (\vec{r}^a_a \times \vec{p}_a)$$

$$= \vec{e} \cdot \sum_a \vec{r}^a_a \times \vec{p}_a \qquad \text{three laws}$$

$\vec{J} = \sum_a \vec{r}^a \times \vec{p}_a$ is conserved $\left(\frac{d\vec{J}}{dt} = 0\right)$

Whenever we get conserved quantities from rotational invariance, we'll call them the angular momentum.

There is nothing here that was not already in the Euler-Lagrange equations. What this theorem provides us with is a turn the crank method for obtaining conservation laws from a variety of theories. Before this theorem, the existence of conserved quantities, like the energy, had to be noticed from the equations of motion in each new theory. This theorem organizes conservation laws. It explains, for example, why a variety of theories, including ones with velocity dependent potentials all have a conserved Hamiltonian, or energy (example (2)).

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Show: $[Q^i, Q^j] = i\epsilon^{ijk} Q^k$ when the Q^i's are generators of an SO(3) internal symmetry. For a general internal symmetry group, the usually quantum Q's recreate the algebra of the generators.

From the conserved quantity, we can reconstruct the symmetry. This can be done in classical mechanics using Poisson brackets. We'll do it in quantum mechanics using commutators.

Assume Dq^b and F depend only on the q^a not the $\dot{q}^a$ so that their expression in the Hamiltonian formulation only depends on the q^a, not the p^a. Then

$$[Q, q^a] = [\Sigma_b p_b Dq^b - F, q^a] = \Sigma_b \underbrace{[p_b, q^a]}_{-i\delta_b^a} Dq^b$$
$$= -i Dq^a$$

This assumption is not at all necessary. Usually the result holds. For example the energy generates time translations even though the assumption doesn't hold.

Symmetries and Conservation Laws in (classical) Field Theory

Field theory is a specialization of particle mechanics. There will be more that is true in field theory. What is this more?

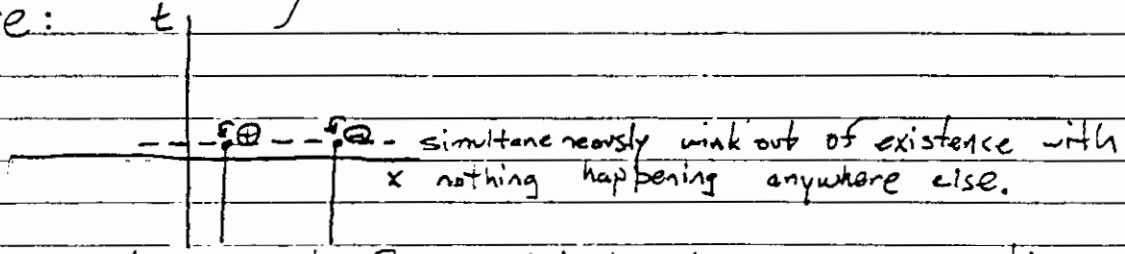
In electromagnetism possesses a conserved quantity, Q, the electric charge. The charge is the integral of the charge density ρ, $Q = \int d^3x \rho(\vec{x}, t)$. There is also a current $\vec{J}$ and there is a much stronger statement of charge conservation than $\frac{dQ}{dt} = 0$. Local charge conservation says

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Integrate this equation over any volume V with boundary S to get

$$\frac{dQ_V}{dt} = -\int dA\, \hat{n} \cdot \vec{J} \quad \text{using Gauss's theorem} \quad \left(Q = \int_V d^3x\, \rho(\vec{x}, t)\right)$$

This equation says you can see the charge change in any volume by watching the current flowing out of the volume. You can't have:



This picture satisfies global charge conservation, but violates local charge conservation. You have to be able to account for the change in charge in any volume, and there would have to be a flow of current in between the two charges. Even if their were not a current, and a local conservation law, we could invoke special relativity to show this picture is inconsistent. In another frame the charges don't disappear simultaneously and for a moment global charge conservation is violated.

Field theory which embodies the idea of local measurements, should have local conservation laws.

Given some Lagrangian density $\mathcal{L}(\phi^a, \partial_\mu \phi^a, x)$ and a transformation of the fields

$$\phi^a(x) \to \phi^a(x, \lambda) \qquad \phi^a(x, 0) = \phi^a(x)$$

we define $$D\phi^a = \frac{\partial \phi^a}{\partial \lambda}\Big|_{\lambda=0}$$

Definition: A transformation is a symmetry iff $D\mathcal{L} = \partial_\mu F^\mu$ for some $F^\mu(\phi^a, \partial_\mu \phi^a, x)$. This equality must hold for arbitrary $\phi^a(x)$ not necessarily satisfying the equations of motion.

We are not using special relativity. F^μ is not necessarily a 4-vector (just some set of 4 objects).

Note that the previous definition can be obtained. To wit, $\mathcal{D}L = \mathcal{D}\int d^3x\,\mathcal{L} = \int d^3x\,\partial_\mu F^\mu = \int d^3x\,\partial_0 F^0 = \frac{d}{dt}F$ where $F = \int d^3x\,F^0$.

Why is this a good definition? Hamilton's principle is

$$0 = \int \delta S = \int d^4x\,\mathcal{D}\mathcal{L} = \int d^4x\,\partial_\mu F^\mu$$
$$= \int d^3x\,\left[F^0(\vec{x},t_2) - F^0(\vec{x},t_1)\right]$$

6 lithe as usual about contributions from spatial infinity.

Thus a symmetry transformation does not affect the equations of motion (we only consider variations that vanish at the endpoints when deriving the equations of motion).

Theorem (maybe this is Noether's theorem?): For every symmetry there is a conserved current.

The proof comes from considering two expressions for $\mathcal{D}\mathcal{L}$

$$\mathcal{D}\mathcal{L} = \sum_a \left(\frac{\partial \mathcal{L}}{\partial \phi^a}\mathcal{D}\phi^a + \pi^\mu_a\,\mathcal{D}\partial_\mu\phi^a\right) \quad \text{used } \pi^\mu_a \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}$$

$$= \sum_a \left(\partial_\mu \pi^\mu_a\,\mathcal{D}\phi^a + \pi^\mu_a\,\mathcal{D}\partial_\mu\phi^a\right) \quad \text{used E-L equations}$$

$$= \partial_\mu \sum_a \pi^\mu_a\,\mathcal{D}\phi^a \quad \text{by equality of mixed partials } \mathcal{D}\partial_\mu\phi^a = \partial_\mu\mathcal{D}\phi^a$$

By assumption $\mathcal{D}\mathcal{L} = \partial_\mu F^\mu$. Subtracting these two expressions for $\mathcal{D}\mathcal{L}$ we see that the four quantities

$$J^\mu = \sum_a \pi^\mu_a\,\mathcal{D}\phi^a \quad \text{satisfy} \quad \partial_\mu J^\mu = 0$$

which implies $\frac{dQ_V}{dt} = -\int_S dA\,\hat{n}\cdot\vec{J}$ ← this equation justifies calling J^0 the density of stuff and $\vec{J}$ the current of stuff

S = boundary of V

The total amount of stuff, Q is independent of time.

There is an ambiguity in the definition of F^μ and J^μ since F^μ can be changed by any χ^μ satisfying $\partial_\mu \chi^\mu = 0$. In the particle mechanics case, F was ambiguous, but only by a time independent quantity. For arbitrary antisymmetric A we can let $F^\mu \to F^\mu + \partial_\nu A^{\mu\nu}$ $(\partial_\mu \partial_\nu A^{\mu\nu} = 0)$. As a result of this change $J^\mu \to J^\mu - \partial_\nu A^{\mu\nu}$

$$Q = \int d^3x J^0 \to Q - \int d^3x \, \partial_i A^i = Q$$

i.e. Q is unchanged, ignoring contributions from spatial infinity as usual.

This is a lot of freedom in the definition of J^μ. For example in a theory with three fields we can let

$$J^\mu \to J^\mu - (\phi^3)^{14} \partial_\nu \left(\partial^\mu \phi^1 \partial^\nu \phi^2 - \partial^\nu \phi^1 \partial^\mu \phi^2 \right)$$

This J^μ is as good as any other. There are 500 papers arguing about which energy-momentum tensor is the right one to use inside a dielectric medium. 490 of them are idiotic. It's like if someone passes you a plate of cookies and you start arguing about which copy is #1 and which is #2. They're all edible! Sometimes one J^μ is more useful in a given calculation than another, for some reason. Instead of arguing that the most convenient J^μ is the "right" one, you should just be happy that you had some freedom of choice.

From cranking Hamilton's principle, we can give another derivation of the relation between symmetries and conserved quantities.

$$\delta S = \delta \int dt \, L = \sum_a p_a \, Dq^a \Big|_{t_1}^{t_2}$$

for an arbitrary variation about a solution of the equations of motion. By assumption of a symmetry

$$\delta S = \int dt \, DL = \int dt \, \frac{df}{dt} \Big|_{t_1}^{t_2}$$ Subtracting we see $\sum_a p_a Dq^a - F$ is time independent.

OCT. 7 c.c. using Hamilton's principle rather than mucking around with the equations of motion 13

[ASIDE: NOETHER'S THEOREM DERIVED AT THE LEVEL OF THE ACTION

If without using the equations of motion, for an arbitrary function of space-time $\alpha(x)$ parametrizing a transformation, to first order in $\alpha(x)$

$$\delta\mathcal{L} = \alpha(x)\,\partial_\mu k^\mu + \partial_\mu \alpha(x)\; j^\mu$$

(Equivalent to $D\mathcal{L} = \partial_\mu k^\mu$ when α is a constant)

Then by Hamilton's principle, for fields satisfying the equations of motion,

$$0 = \int d^4x\, \delta\mathcal{L} = \int d^4x \left(\alpha(x)\,\partial_\mu k^\mu + \partial_\mu \alpha(x)\, j^\mu\right)$$

$$= \int d^4x \left(\alpha(x)\,\partial_\mu k^\mu - \alpha(x)\,\partial_\mu j^\mu\right)$$ let $\alpha(x)$ vanish at ∞ to do part. integration.

So it must be that

$$J^\mu = k^\mu - j^\mu \text{ is conserved}$$

Example Space-time translation invariance

μ plays the same role as ϕ for this type of transformation.

$$\delta\phi = -\epsilon_\nu \partial^\nu \phi \qquad \delta\,\partial_\tau\phi = -\partial_\tau\left(\epsilon_\nu \partial^\nu \phi\right)$$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\,\partial_\tau\phi}\delta\,\partial_\tau\phi$$

$$= -\frac{\partial\mathcal{L}}{\partial\phi}\epsilon_\nu\partial^\nu\phi - \frac{\partial\mathcal{L}}{\partial\,\partial_\tau\phi}\epsilon_\nu\partial^\nu\partial_\tau\phi - \frac{\partial\mathcal{L}}{\partial\,\partial_\tau\phi}\,\partial_\tau\epsilon_\nu\,\partial^\nu\phi$$

$$= -\epsilon_\nu\mathcal{L}^{\mu\nu} - \pi^\mu\,\partial^\nu\phi\;\partial_\mu\epsilon_\nu$$

I read off $k^{\mu\nu} = -g^{\mu\nu}\mathcal{L}$ $\quad j^{\mu\nu} = -\pi^\mu\partial^\nu\phi$

as $T^{\mu\nu} = \pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathcal{L}$ is conserved.]

OCT. 7

13'

Noether's thm. when the variation is a hermitian traceless matrix. (Useful in theories with fields most easily written as matrices.)

Suppose $\delta S = \int d^4x \, \mathrm{Tr}\, \delta\omega \, \partial_\mu F^\mu$ (without using the equations of motion)

$\delta\omega$: Hermitian, traceless, but otherwise arbitrary

F^μ: matrix

Then Hamilton's principle says

$$0 = \int d^4x \, \mathrm{Tr}\, \delta\omega \, \partial_\mu F^\mu$$

Then F^μ is a matrix of conserved currents, except there is no conservation law associated with the trace.

If you like, write this statement as

$$\partial_\mu \left(F^\mu - \frac{1}{N} \mathrm{Id}\, \mathrm{Tr}\, F^\mu \right) = 0$$

N: dimension of F^μ

OCT. 7

Space-time translations in a general field theory

We won't have to restrict ourselves to scalar fields because under space-time translations, an arbitrary field, vector tensor, etc., transforms the same way. Just let the index a also denote vector or tensor indices.

Translations are symmetries as long as $\mathcal{L}$ does not have any explicit dependence on x. It depends on x only through its dependence on ϕ^a and $\partial_\mu \phi^a$, $\frac{\partial \mathcal{L}}{\partial x} = 0$.

$\phi^a(x) \to \phi^a(x + \lambda e)$ e some fixed four vector

$D\phi^a = e^\nu \partial_\nu \phi^a$ $\quad D\mathcal{L} = e^\mu \partial_\mu \mathcal{L} = \partial_\mu (e^\mu \mathcal{L})$

$F^\mu = e^\mu \mathcal{L}$ $\quad J^\mu = \sum_a \pi^\mu{}_a e^\nu \partial_\nu \phi^a - e^\mu \mathcal{L} = e_\nu T^{\mu\nu}$

$T^{\mu\nu} = \sum_a \pi^\mu{}_a \partial^\nu \phi^a - g^{\mu\nu} \mathcal{L}$

There are four conserved currents, four local conservation laws, one for each of the four independent directions we can point e.. i.e. $\partial_\mu T^{\mu\nu} = 0$ since $\partial_\mu J^\mu = 0$ for arbitrary e.

$T^{\mu 0}$ is the current that is conserved as a result of time translation invariance, indeed,

$\mathcal{H} = T^{00} = \sum_a \pi_a \dot{\phi}^a - \mathcal{L}$, $H = \int d^3x \, \mathcal{H}$

T^{00} is the energy density, T^{i0} is the current of energy. $T^{i\mu}$ are the three currents from spatial translation invariance.

$P^i = \int d^3x \, T^{0i}$

T^{0i} = the density of the ith component of momentum

T^{ji} = the jth component of the current of the ith component of momentum.

For a scalar field theory with no derivative interactions, $\pi^\mu{}_a = \partial^\mu \phi$ so $T^{\mu\nu} = \sum_a \partial^\mu \phi^a \partial^\nu \phi^a - g^{\mu\nu} \mathcal{L}$. Note that $T^{\mu\nu}$ is symmetric, so you don't have to remember which index is which. $T^{\mu\nu}$ can be nonsymmetric, which can lead to problems, for example gravity and other theories of higher spin. Go to symmetric

OCT. 9

Lorentz transformations

Under a Lorentz transformation all vectors transform as

$$a^\mu \longrightarrow \Lambda^\mu{}_\nu a^\mu$$

where $\Lambda^\mu{}_\nu$ specifies the Lorentz transformation. $\Lambda^\mu{}_\nu$ must preserve the Minkowski space inner product, that is if

$$b^\mu \to \Lambda^\mu{}_\nu b^\nu \qquad \text{then} \qquad a_\mu b^\mu \longrightarrow a_\mu b^\mu.$$

This must be true for arbitrary a and b. (The equation this condition gives is $g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta = g_{\alpha\beta}$.) We'll be interested in one parameter subgroups of the group of Lorentz transformations parameterized by λ. This could be rotations about some specified axis by an angle λ or a boost in some specified direction by a rapidity λ. In any case, the Lorentz transformation is given by a family $\Lambda(\lambda)^\mu{}_\nu$.

$$a^\mu \longrightarrow a^\mu(\lambda) = \Lambda(\lambda)^\mu{}_\nu a^\nu$$

Under this (active) transformation (we are not thinking of this as a passive change of coordinates) the fields ϕ^a transform as (ϕ^a is a scalar),

$$\phi^a(x) \to \phi^a(x,\lambda) = \phi^a\left(\Lambda(\lambda)^{-1}x\right).$$

We are restricting ourselves to scalar fields. Even though we only used scalar fields in our examples of $T^{\mu\nu}$, the derivation of the conservation of $T^{\mu\nu}$ from space-time translation invariance applies to tensor, or vector fields. With Lorentz transformations, we only consider scalars because there are extra factors in the transformation law when the fields are tensorial. For example a vector field $A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$.

We need to get $D\phi = \frac{\partial \phi}{\partial \lambda}\Big|_{\lambda=0}$. We'll define

$$D\Lambda^\mu{}_\nu \equiv \epsilon^\mu{}_\nu \quad \text{(defines some matrix } \epsilon^\mu{}_\nu\text{)}.$$

From the invariance of $a^\mu b_\mu$, we'll derive a condition on $\epsilon^\mu{}_\nu$.

$$\begin{aligned} 0 = D(a^\mu b_\mu) &= (Da^\mu) b_\mu + a^\mu (Db_\mu) \\ &= \epsilon^\mu{}_\nu a^\nu b_\mu + a^\mu \epsilon_\mu{}^\nu b_\nu \\ &= \epsilon_{\mu\nu} a^\nu b^\mu + \epsilon_{\nu\mu} a^\nu b^\mu \\ &= (\epsilon_{\mu\nu} + \epsilon_{\nu\mu}) a^\nu b^\mu \Longrightarrow \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \end{aligned}$$

(relabel dummy indices in the second term, $\mu \to \nu$, $\nu \to \mu$.)

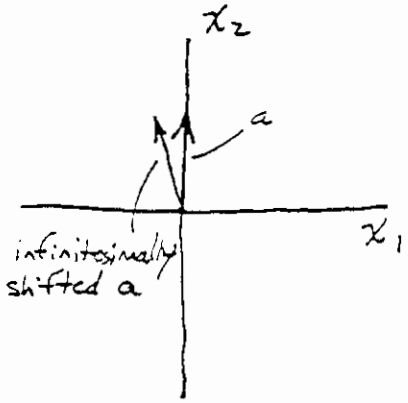
since this has to hold for arbitrary a and b. μ and ν range from 0 to 3, so there are $4 \cdot (4-1)/2 = 6$ independent ϵ, which is good since we have to generate 3 rotations (about each axis) and 3 boosts (in each direction).

As a second confidence-building check we'll do two examples.

Take $\epsilon_{12} = -\epsilon_{21} = 1$, all other components zero.

$$Da^1 = \epsilon^1{}_2 a^2 = -\epsilon_{12} a^2 = -a^2$$

$$Da^2 = \epsilon^2{}_1 a^1 = -\epsilon_{21} a^1 = +a^1$$



This says a^1 gets a little negative component proportional to a^2 and a^2 gets a little component proportional to a^1. This is a rotation, in the standard sense about the positive z axis.

Take $\epsilon_{01} = -\epsilon_{10} = +1$, all other components zero.

$$D a^0 = \epsilon^0{}_1 a^1 = \epsilon_{01} a^1 = a^1$$

$$D a^1 = \epsilon^1{}_0 a^0 = -\epsilon_{10} a^0 = a^0$$

This says x^1, which could be the first component of the position of a particle, gets a little contribution proportional to x^0 ↘ the time, which is definitely what a boost in the x^1 direction does. In fact, $Da^0 = a^1$ $Da^1 = a^0$ is just the infinitesimal version of

$$a^0 \to \cosh\lambda\, a^0 + \sinh\lambda\, a^1$$

$$a^1 \to \sinh\lambda\, a^0 + \cosh\lambda\, a^1$$

Without even thinking, the great index raising and lowering machine has given us all the right signs.

Now assuming $\mathcal{L}$ is a scalar, we are all set to get the 6 conserved currents.

From $\Lambda^{-1}(\lambda)\,\Lambda(\lambda) = 1$, $0 = D(\Lambda^{-1}\Lambda)$ and $D\Lambda^{-1\mu}{}_\nu\big|_{\lambda=1} = -\epsilon^\mu{}_\nu$

$$D\phi^a(x) = \frac{\partial}{\partial\lambda}\,\phi^a\left(\Lambda^{-1}(\lambda)^\mu{}_\nu x^\nu\right)\Big|_{\lambda=0}$$

There are extra terms in $D\phi^a$ if ϕ^a are not scalars

$$= \partial_\sigma \phi^a(x)\, D\left(\Lambda^{-1}(\lambda)^\sigma{}_\tau\, x^\tau\right)$$

$$= \partial_\sigma \phi^a(x)\left(-\epsilon^\sigma{}_\tau\right) x^\tau = -\epsilon_{\sigma\tau}\, x^\tau\, \partial^\sigma \phi^a(x)$$

Using the assumption that $\mathcal{L}$ is a scalar depending only on x through its dependence on ϕ^a and $\partial_\mu \phi^a$ we have

$$D\mathcal{L} = \epsilon_{\lambda\sigma} x^\lambda \partial^\sigma \mathcal{L}$$

$$= \partial_\mu\left[\epsilon_{\lambda\sigma}\, x^\lambda\, g^{\mu\sigma}\mathcal{L}\right]$$

The conserved current J^μ is

$$J^\mu = \sum_a \pi^\mu{}_a \epsilon_{\lambda\sigma} x^\lambda \partial^\sigma \phi^a - \epsilon_{\lambda\sigma} x^\lambda g^{\mu\sigma} \mathcal{L}$$
$$= \epsilon_{\lambda\sigma} \left(\sum_a \pi^\mu{}_a x^\lambda \partial^\sigma \phi^a - x^\lambda g^{\mu\sigma} \mathcal{L} \right)$$

This current must be conserved for all six independent antisymmetric matrices $\epsilon_{\lambda\sigma}$, so the part of the quantity in the parenthesis that is antisymmetric in λ and σ must be conserved. i.e. $\partial_\mu M^{\mu\lambda\sigma} = 0$ where

$$M^{\mu\lambda\sigma} = \sum_a \pi^\mu{}_a x^\lambda \partial^\sigma \phi^a - x^\lambda g^{\mu\sigma} \mathcal{L} - \lambda \leftrightarrow \sigma$$
$$= x^\lambda \left(\sum_a \pi^\mu{}_a \partial^\sigma \phi^a - g^{\mu\sigma} \mathcal{L} \right) - \lambda \leftrightarrow \sigma$$
$$= x^\lambda T^{\mu\sigma} - x^\sigma T^{\mu\lambda}$$

If the ϕ^a were not scalars, we would have additional terms, feeding in from the extra terms in $D\phi^a$. The 6 conserved charges are

$$J^{\lambda\sigma} = \int d^3x \, M^{0\lambda\sigma} = \int d^3x \left(x^\lambda T^{0\sigma} - x^\sigma T^{0\lambda} \right)$$

For example J^{12}, the conserved quantity coming from invariance under rotations about the 3 axis, often called, J^3 is $\left(J^i = \frac{1}{2} \epsilon_{ijk} J^{jk} \right)$,

$$J^3 = J^{12} = \int d^3x \left(x^1 T^{02} - x^2 T^{01} \right)$$

If we had point particles with

$$T^{0i}(\vec{x}, t) = \sum_a p_a^i \, \delta^{(3)} \left(\vec{x} - \vec{r}^{\,a}(t) \right)$$

J^3 would be

$$\sum_a \left(x^{a1} p_a^2 - x^{a2} p_a^1 \right) = \sum_a \left(\vec{r}^{\,a} \times \vec{p}_a \right)_3$$

We have found the field theory analog of the angular momentum. The particles themselves could have some intrinsic angular momentum. Those contributions to the angular momentum are not in the J^i. We only have the orbital contribution. Particles that have intrinsic angular momentum, spin, will be described by fields of tensorial character and that will be reflected in extra terms in the J^{ij}.

So far we have just found the continuum field theory generalization of three conserved quantities we learn about in freshman physics. But we have three other conserved quantities, the J^{0i}. What are they? Consider

$$J^{0i} = \int d^3x \left[x^0 T^{0i} - x^i T^{00} \right]$$

This has an explicit reference to x^0, the time, something we've not seen in a conservation law before, but there is nothing a priori wrong with that. We can pull the x^0 out of the integral over space. The conservation law is $\frac{dJ^{0i}}{dt} = 0$ so we have

$$0 = \frac{d}{dt} J^{0i} = \frac{d}{dt} \left[t \int d^3x\, T^{0i} - \int d^3x\, x^i T^{00} \right]$$

$$= t \frac{d}{dt} \underbrace{\int d^3x\, T^{0i}}_{P^i \to 0} + \underbrace{\int d^3x\, T^{0i}}_{P^i} - \frac{d}{dt} \int d^3x\, x^i T^{00}$$

Dividing through by P^0 gives

$$-\text{constant} = \frac{P^i}{P^0} = \frac{\frac{d}{dt} \int d^3x\, x^i T^{00}}{P^0} = \frac{\frac{d}{dt} (\text{"center of energy"}^{Hi})}{\text{total energy}}$$

This says that the "center" of energy" moves steadily. T^{00} is the relativistic generalization of mass. This is the relativistic generalization of the statement that the center of mass moves steadily. You aren't used to calling this a conservation law, but it is, and in fact it is the Lorentz partner of the angular

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Internal Symmetries

There are other conservation laws, like conservation of electric charge, conservation of baryon number, and conservation of lepton number that we have not found yet. We have already found all the conservation laws that are in a general Lorentz invariant theory. These additional conservation laws will only occur in specific theories whose Lagrange densities have special properties. Conservation laws are the best guide for looking for theories that actually describe the world, because the existence of a conservation law is a qualitative fact that greatly restricts the form of the Lagrange density. All these additional charges are scalars, and we expect the symmetries they come from to commute with Lorentz transformations. The transformations will turn fields at the same spacetime point into one another. They will not relate fields at different spacetime points. Internal symmetries are non-geometrical symmetries. Historically, the name comes from the idea that what an internal symmetry did was transform internal characteristics of a particle. For us internal just means non-geometrical. We'll study internal symmetries with two examples.

The first example is

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{2}\left(\partial_\mu \phi^a \partial^\mu \phi^a - \mu^2 \phi^a \phi^a\right) - g\left(\sum_a (\phi^a)^2\right)^2$$

This is a special case of a theory of two scalar fields. Both fields have the same mass, and the potential only depends on the combination $\phi^{1\,2}+\phi^{2\,2}$.

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We have left particle mechanics behind and I'll often use Lagrangian to mean Lagrange density.

This Lagrangian is invariant ($D\mathcal{L}=0$) under the transformation

$$\phi^1 \to \phi^1 \cos\lambda + \phi^2 \sin\lambda$$
$$\phi^2 \to -\phi^1 \sin\lambda + \phi^2 \cos\lambda$$

clockwise rotation makes the signs come out conventionally if a and c are b and c defined like a and b in Itzykson and Zuber p. 121

$SO(2)$ symmetry

The same transformation at every space time point. This is a rotation in the ϕ^1, ϕ^2 plane. The Lagrangian is invariant because it only depends on $\phi^{12}+\phi^{22}$ and that combination is unchanged by rotations.

$$D\phi^1 = \phi^2 \qquad D\phi^2 = -\phi^1 \qquad D\mathcal{L}=0 \qquad F^\mu = 0$$

$$J^\mu = \pi_1^\mu D\phi^1 + \pi_2^\mu D\phi^2 = (\partial^\mu\phi^1)\phi^2 - (\partial^\mu\phi^2)\phi^1$$

$$Q = \int d^3x J^0 = \int d^3x \left(\partial_0\phi^1\phi^2 - \partial_0\phi^2\phi^1 \right)$$

We can get some insight into this quantity by going to the case $g=0$ in which case ϕ^1 and ϕ^2 are both free fields and can be expanded in terms of creation and annihilation ops.

$$\phi^a(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(a^a_{\vec{k}} e^{-ik\cdot x} + a^{a\dagger}_{\vec{k}} e^{ik\cdot x} \right)$$

Now we'll compute Q. Let's have faith in our formalism and assume that the terms with two creation ops or two annihilation ops go away. If they didn't since they are multiplied by $e^{\pm 2i\omega_k t}$ Q wouldn't be time independent.

$$Q = \int \frac{d^3k}{2\omega_k} \left[a^1_{\vec{k}} a^{\dagger 2}_{\vec{k}} \left(-i\omega_{\vec{k}} - i\omega_{\vec{k}} \right) + 2i\omega_{\vec{k}} a^2_{\vec{k}} a^{\dagger 1}_{\vec{k}} \right]$$

We don't have to worry about the order because

$$\left[a^1_{\vec{k}}, a^{2\dagger}_{\vec{k}'} \right] = 0$$

$$Q = i\int d^3k \left[a^{\dagger 1}_{\vec{k}} a^2_{\vec{k}} - a^{\dagger 2}_{\vec{k}} a^1_{\vec{k}} \right]$$

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Exchanging the role of b_k and c_k is the way of making the signs come out conventionally with a counterclockwise rotation.

We are within reach of something intuitive. Define

$$b_{\vec{k}} = \frac{a^1_{\vec{k}} + i a^2_{\vec{k}}}{\sqrt{2}} \qquad b^+_{\vec{k}} = \frac{a^{1\dagger}_{\vec{k}} - i a^{2\dagger}_{\vec{k}}}{\sqrt{2}}$$

That's not the end of the definitions; we need the other combination to reconstruct $a^1_{\vec{k}}, a^2_{\vec{k}}, a^{1\dagger}_{\vec{k}}, a^{2\dagger}_{\vec{k}}$ So define,

$$c_{\vec{k}} = \frac{a^1_{\vec{k}} - i a^2_{\vec{k}}}{\sqrt{2}} \qquad c^+_{\vec{k}} = \frac{a^{1\dagger}_{\vec{k}} + i a^{2\dagger}_{\vec{k}}}{\sqrt{2}}$$

These linear combinations of operators are allowable. They are operators that create (or destroy) a superposition of states with particle 1 and particle 2. If one state is degenerate with another, and for a given $\vec{k}$ $|\vec{k},1\rangle$ is degenerate with $|\vec{k},2\rangle$, it is often convenient to work with linear combinations of these states as basis states. Because $b^+_{\vec{k}}$ and $c^+_{\vec{k}}$ create orthogonal states, $b_{\vec{k}}$ and $c^+_{\vec{k}}$ commute with each other, as is easily checked. The useful thing about these linear combinations is that Q has a simple expression.

$$i\left(a^{+1}_{\vec{k}} a^2_{\vec{k}} - a^{+2}_{\vec{k}} a^1_{\vec{k}}\right) = i\,\frac{b^+_{\vec{k}} + c^+_{\vec{k}}}{\sqrt{2}} \cdot \frac{b_{\vec{k}} - c_{\vec{k}}}{\sqrt{2}\,i} + i\,\frac{b^+_{\vec{k}} - c^+_{\vec{k}}}{\sqrt{2}\,i} \cdot \frac{b_{\vec{k}} + c_{\vec{k}}}{\sqrt{2}}$$

$$= \frac{1}{2}\left[2b^+_{\vec{k}} b_{\vec{k}} + 0 b^+_{\vec{k}} c_{\vec{k}} + 0 c^+_{\vec{k}} b_{\vec{k}} - 2 c^+_{\vec{k}} c_{\vec{k}}\right] = b^+_{\vec{k}} b_{\vec{k}} - c^+_{\vec{k}} c_{\vec{k}}$$

That is,

$$Q = \int d^3k \left(b^+_{\vec{k}} b_{\vec{k}} - c^+_{\vec{k}} c_{\vec{k}}\right) = N_b - N_c.$$

The b's carry Q charge +1, the c's carry Q charge -1. It's like particles and antiparticles; the b's and c's have the same mass and opposite charge. $Q = N_b - N_c$ is true as an operator equation. b and c type mesons are eigenstates of Q. 1 and 2 type mesons are not. An eigenstate of N_a and N_b is an eigenstate of Q, an eigenstate of N_1 and N_2 is not. By the way H and $\vec{P}$ have familiar forms in terms of the b's and c's. $H = \int d^3k\, \omega_{\vec{k}} \left(b^+_{\vec{k}} b_{\vec{k}} + c^+_{\vec{k}} c_{\vec{k}}\right)$

Taking all these linear combinations suggests that we could have changed bases earlier in the calculation and simplified things.

$$\psi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left[b_{\vec{k}} e^{-ik\cdot x} + c^{\dagger}_{\vec{k}} e^{ik\cdot x} \right]$$

ψ always diminishes Q by 1 either by annihilating a b type particle or by creating a c type particle

$$[Q, \psi] = -\psi \qquad \text{a } Q \text{ "eigenfield"}$$

There is also the hermitian conjugate of ψ, $\psi^\dagger$,

$$\psi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left[c_{\vec{k}} e^{-ik\cdot x} + b^{\dagger}_{\vec{k}} e^{ik\cdot x} \right]$$

($\psi^\dagger$ will be denoted ψ^* in the classical limit. a quantum field ψ^* will be understood to be $\psi^\dagger$.) $\psi^\dagger$ always increases Q by 1. $[Q, \psi^\dagger] = +\psi^\dagger$

ψ and $\psi^\dagger$ have neat commutation relations with Q.
ϕ^1 and ϕ^2 have commutation relations with Q.

$$[Q, \phi^1] = \frac{1}{\sqrt{2}} [Q, \psi + \psi^\dagger] = \frac{1}{\sqrt{2}} (-\psi + \psi^\dagger)$$

$$= -i\phi_2$$

In agreement with a formula which is usually true $[Q, \phi^a] = -i D\phi^a$

$$[Q, \phi_1] = i\phi_1$$

Under our transformation $\psi \to e^{-i\lambda}\psi$, $\psi^* \to e^{i\lambda}\psi^*$
This is called a $U(1)$ or phase transformation. It is equivalent to $SO(2)$.

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Digressions; we'll get back to symmetries.

As quantum fields, ψ and $\psi^\dagger$ are as nice as ϕ_1 and ϕ_2. They obey

$$(\Box+\mu^2)\psi = 0 \qquad (\Box+\mu^2)\psi^\dagger = 0$$

They obey simple equal time commutation relations:

$$[\psi(\vec{x},t),\psi(\vec{y},t)] = 0 = [\psi^\dagger(\vec{x},t),\psi^\dagger(\vec{y},t)] = [\psi(\vec{x},t),\psi^\dagger(\vec{x},t)]$$

$$[\psi(\vec{x},t),\dot{\psi}(\vec{y},t)] = 0 = [\psi^\dagger(\vec{x},t),\dot{\psi}^\dagger(\vec{y},t)]$$

$$[\psi(\vec{x},t),\dot{\psi}^\dagger(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) = [\psi^\dagger(\vec{x},t),\dot{\psi}(\vec{y},t)]$$

These equations can be obtained by doing something completely idiotic. Back to the classical Lagrangian which is

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - \mu^2\psi^*\psi \qquad (\pi_0 \tfrac{1}{2})$$

Imagine a person who once knew a lot of quantum field theory, but has suffered brain damage, is going to canonically quantize this theory. He has forgotten that $*$ stands for complex conjugate, and he is going to treat ψ and ψ^* as if they were independent fields. Here he goes

$$\pi^\mu{}_\psi = \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi} = \partial^\mu\psi^* \qquad \pi^\mu{}_{\psi^*} = \partial^\mu\psi$$

The Euler-Lagrange equations he gets are

$$\partial_\mu\pi^\mu{}_\psi = \frac{\partial\mathcal{L}}{\partial\psi} \qquad \text{i.e.} \qquad \Box\psi^* = -\mu^2\psi^*$$

The idiot got it right. He also gets $\Box\psi = -\mu^2\psi$. How about that!? Now he says "I'm going to deduce the canonical commutation relations":

$$i\delta^{(3)}(\vec{x}-\vec{y}) = [\psi(\vec{x},t),\pi^0_\psi(\vec{y},t)] = [\psi(\vec{x},t),\partial_0\psi^*(\vec{y},t)]$$

By God, he gets that right, too. How can you work with complex fields, which can't be varied independently, treat them as if they can be and nevertheless get the right answers.

We'll demonstrate that this works for the equations of motion. It works very generally. You can demonstrate that it works for the equal time commutation relations.

Suppose I have some action $S(\psi, \psi^*)$. The equations of motion come from

$$0 = \delta S = \int d^4x \left(A\, \delta\psi + A^* \delta\psi^* \right)$$

Treating the variations in $\delta\psi$ and $\delta\psi^*$ as independent we get } NAIVE APPROACH

$$A = 0 \quad \text{and} \quad A^* = 0$$

SHARP APPROACH:

However $\delta\psi$ is not independent of ψ^*. We can use the fact that ψ and ψ^* are allowed to be complex. We can make purely real variations in ψ so that $\delta\psi = \delta\psi^*$ and we can make purely imaginary variations in ψ so that $\delta\psi = -\delta\psi^*$. From the real variation we deduce

$$A + A^* = 0$$

and from the imaginary variation we deduce

$$A - A^* = 0$$

which implies $A = A^* = 0$.

For the ETCR write $\psi = \frac{\psi_r + i\psi_i}{\sqrt{2}}$ $\psi^\dagger = \frac{\psi_r - i\psi_i}{\sqrt{2}}$

Nothing tricky or slick.

Back to internal symmetries. For our second example, take

$$\mathcal{L} = \frac{1}{2}\sum_{a=1}^{n}\left(\partial_\mu \phi^a \partial^\mu \phi^a - \mu^2 \phi^a \phi^a\right) - g\left(\sum_{a=1}^{n} \phi^{a\,2}\right)^2$$

This is the same as our first example except we have n fields instead of 2. Just as in the first example the Lagrangian was invariant under rotations mixing up ϕ^1 and ϕ^2 this Lagrangian is invariant under rotations mixing up $\phi^1, \ldots, \phi^n$, because it only depends on $\phi^{1\,2} + \cdots + \phi^{n\,2}$.

The rotations are,

$$\phi^a \to \sum_b \underbrace{R^a{}_b}_{\text{n x n rotation matrix}} \phi^b$$

There are $\frac{n(n-1)}{2}$ independent planes in n dimensions and we can rotate in each of them, so there are $\frac{n(n-1)}{2}$ conserved currents and associated charges. This example is quite different from the first one because the various rotations don't in general commute. (They all commuted in the first one by virtue of the fact that there was only one.) We don't expect the various charges to commute. If they did they would generate symmetries that commute. Anyway, for any single rotation axis, the symmetry is just like the one we had in the first example, so we can read off the current,

$$J_\mu^{[a,b]} = \partial_\mu \phi^a \phi^b - \partial_\mu \phi^b \phi^a$$

You can't find combinations of the fields that have simple commutation relations with all the $Q^{[a,b]}$'s. For $n = 3$ you can choose the fields to have a simple commutation relation with one charge, say $Q^{[1,2]}$. This is just like isospin, which is a symmetry generated by I_1, I_2 and I_3. You can only choose the particle states, π^+, π^0 and $-$ to be eigenstates of one of them, I_3.

OCT. 14 Lorentz transformation properties of conserved quantities

We've worked with three currents, J^μ, the current of meson number, $T^{\mu\nu}$, the current of the νth component of momentum, and $M^{\mu\nu\lambda}$, the current of the $[\nu\lambda]$ component of angular momentum. We integrated the zeroth component of each of these currents to obtain Q, P^ν and $J^{\nu\lambda}$ respectively. These look like tensors with one less index, but do they have the right Lorentz transformation properties?

We'll prove that $P^\nu = \int d^3x T^{0\nu}$ is indeed a Lorentz vector given that $T^{\mu\nu}$ is a two index tensor and that $T^{\mu\nu}$ is conserved, $\partial_\mu T^{\mu\nu} = 0$. The generalization to currents with more indices or 1 index will be clear. We could do this proof in the classical theory or the quantum theory. we'll choose the latter because we need the practice. The assumption that $T^{\mu\nu}$ is an operator in quantum field theory that transforms as a two-index tensor is phrased mathematically as follows: Given $U(\Lambda)$ the unitary operator that effects Lorentz transformations in the theory

$$U(\Lambda) T^{\mu\nu}(x) U(\Lambda)^\dagger = \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}(\Lambda^{-1}x)$$

Now we'll try to show that

$$P^\nu = \int d^3x\, T^{0\nu}(\vec{x}, 0)$$

is a Lorentz vector. Introduce $n = (1,0,0,0)$, a unit vector pointing in the time direction. Then we can write P^ν in a way that makes its Lorentz transformation properties clearer.

$$P^\nu = \int d^4x\; n_\mu T^{\mu\nu}\, \delta(n\cdot x)$$

Perform a Lorentz transformation on P^ν.

$$U(\Lambda) P^\nu U(\Lambda)^\dagger = \int d^4x \; n_\mu \, U(\Lambda) \, T^{\mu\nu}(x) U(\Lambda)^\dagger \delta(n\cdot x)$$

$$= \int d^4x \, n_\mu \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}(\Lambda^{-1}x) \, \delta(n\cdot x)$$

Change integration variables $x' = \Lambda^{-1}x$ and define $n' = \Lambda^{-1}n$

$$U(\Lambda) P^\nu U(\Lambda)^\dagger = \int d^4x' \; \Lambda_{\mu\rho} n'^\rho \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau T^{\sigma\tau}(x') \delta(n'\cdot x')$$

$$= \int d^4x' \, n'^\rho \Lambda^\nu{}_\tau T^{\rho\tau}(x') \delta(n'\cdot x') \quad (\text{using } \Lambda_{\mu\rho}\Lambda^\mu{}_\sigma = g_{\rho\sigma})$$

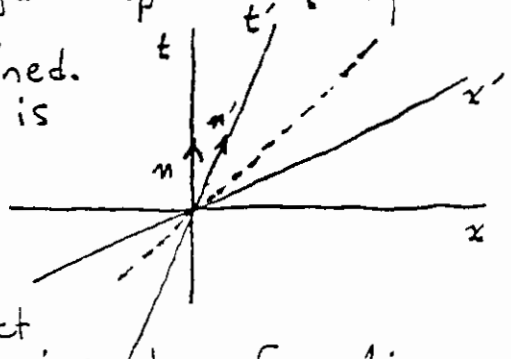
$$= \Lambda^\nu{}_\tau \int d^4x \, n'_\rho \, T^{\rho\tau}(x) \, \delta(n'\cdot x)$$

In the last expression we've dropped the prime on the variable integration. The only difference between what we have and what we would like to get

$$U(\Lambda) P^\nu U(\Lambda)^\dagger = \Lambda^\nu{}_\tau P^\tau = \Lambda^\nu{}_\tau \int d^4x \, n_\rho \, T^{\rho\tau} \delta(n\cdot x)$$

is that n has been redefined. The surface of integration is now $t' = 0$, and we take the component, $n'_\mu T^{\mu\nu}$, in the t' direction. Our active transformation has had the exact same effect as if we had made a passive transformation changing coordinates to $x' = \Lambda^{-1}x$. It's the same old story: alias (another name) versus alibi (another place)

passive active



You can think of this as Lorentz transforming the field or Lorentz transforming the surface.

This doesn't produce a vector P^ν for any tensor $T^{\mu\nu}$. For an arbitrary tensor, P^ν is not even independent of what time you computed, let alone changing the tilt of the surface. We need to use that $T^{\mu\nu}$ is conserved. More or less, that the current that flows through the surface $t'=0$ is the same as the current that flows through the surface $t=0$.

Note that $n_\mu \delta(n\cdot x) = \partial_\mu \theta(n\cdot x)$, so if I call the Lorentz transform of P^ν, P'^ν, what we are trying to show is

$$0 = P^\nu - \Lambda^{-1\,\nu}{}_\sigma P'^\nu$$

$$= \int d^4x \left[\partial_\mu \theta(n\cdot x) - \partial_\mu \theta(n'\cdot x)\right] T^{\mu\nu}(x)$$

$$= \int d^4x\, \partial_\mu \left[\left(\theta(n\cdot x) - \theta(n'\cdot x)\right) T^{\mu\nu}(x)\right]$$

This integral over all spacetime is a total divergence. In the far future or the far past $\theta(n\cdot x) = \theta(n'\cdot x)$ so there are no surface terms there, and as usual we won't worry about surface terms at spatial infinity.

DISCRETE SYMMETRIES

A discrete symmetry is a transformation $q^a(t) \to q^{a'}(t)$ that leaves the Lagrangian, L, unchanged $L \to L$. This could be parity, this could be rotation by π about the z axis. (It does not include time reversal, wait.)

See Dirac Principles of QM pp 103ff

Since all the properties of the theory are derived from the Lagrangian, (the canonical commutation relations, the inner product, the Hamiltonian all come from L) and since the Lagrangian is unchanged, we expect that there is a unitary operator effecting the transformation

$$U^\dagger q^a(t) U = q^{a'}(t) \qquad U^\dagger H U = H$$

The discrete symmetry could be an element of a continuous symmetry group. It could be rotation by 20°. There is no conserved quantity associated with a discrete symmetry however. What is special about the continuous symmetry is that there is a parameter, and that you have a unitary operator for each value of the parameter, θ, satisfying $U(\theta)^\dagger H U(\theta) = H$

You can differentiate this with respect to the parameter to find that $[I, H] = 0$ $I \equiv -i \frac{dU}{d\theta}\Big|_{\theta=0} \equiv -iD$ U. There is nothing analogous for discrete symmetries

Examples of internal symmetries (there is not a general theory, but we'll do prototypical examples)

Example (1). The transformation is

$$\phi(x) \to -\phi(x) \quad \text{at every space-time point}$$

This is a symmetry for any Lagrangian with only even powers of ϕ, in particular,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \lambda\phi^4 \to \mathcal{L}$$

There should be a unitary operator effecting the transformation

$$\phi \to U^\dagger \phi U = -\phi$$

For $\lambda = 0$, we will actually be able to construct U, that is give its action on the creation and annihilation operators, and thus its action on the basis states.

Since ϕ is a linear function of a_k and $a_k^\dagger$ it must be that

$$U^\dagger a_{\vec{k}} U = -a_k$$

and the h.c. equation

$$U^\dagger a_{\vec{k}}^\dagger U = -a_{\vec{k}}^\dagger$$

We'll also make a choice in the phase of ϕ U by specifying

$$U|0\rangle = |0\rangle$$

We can determine the action of U on the basis states

$$\begin{aligned} U|k_1,\ldots,k_n\rangle &= U\, a^\dagger_{\vec{k}_1} a^\dagger_{k_2} \cdots a^\dagger_{\vec{k}_n}|0\rangle \\ &= U a^\dagger_{\vec{k}_1} U^\dagger U a^\dagger_{k_2} U^\dagger \ldots U a^\dagger_{\vec{k}_n} U^\dagger U|0\rangle \\ &= (-1)^n a^\dagger_{\vec{k}_1}\cdots a^\dagger_{k_n}|0\rangle = (-1)^n |k_1,\ldots,k_n\rangle \end{aligned}$$

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As an operator statement we see that

$$U = (-1)^N \quad (= e^{i\pi N} \text{ if you prefer}) \qquad \left(N = \text{meson number} = \int d^3k\, a^\dagger_{\vec{k}} a_{\vec{k}}\right)$$

Suppose we could construct this operator when $\lambda \neq 0$. The existence of this unitary operator tells you that you'll never see 2 mesons scatter into 43 mesons or any odd number of mesons. Mesons are always produced in pairs. More formally, it must be that $0 = \langle n|S|m\rangle$ where $|n\rangle$ is a state with n mesons, $|m\rangle$ is state with m mesons and S is the scattering matrix made from the Hamiltonian, whenever $n+m$ is odd. For if $n+m$ is odd

$$\langle n|S|m\rangle = \langle n|\, U^\dagger U S U^\dagger U|m\rangle = \langle n|U^\dagger \underbrace{(USU^\dagger)}_{=S \text{ because } UHU^\dagger = H} U|m\rangle$$

$$= (-1)^{n+m}\langle n|S|m\rangle = -\langle n|S|m\rangle$$

Our second example of a discrete internal symmetry will be turn out to be

CHARGE CONJUGATION.

Recall that $\mathcal{L} = \sum_{a=1}^{2}\left(\frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{1}{2}\mu^2(\phi^a)^2\right) - \lambda\left[(\phi^1)^2 + (\phi^2)^2\right]^2$

had an $SO(2)$ symmetry. In fact it has an $O(2)$ symmetry, rotations and rotations with reflections in the $\phi^1\ \phi^2$ plane. It is invariant under proper and improper rotations. We'll take one standard improper rotation

$$\phi^1 \to \phi^1 \qquad \phi^2 \to -\phi^2$$

Any other one can be obtained by composing this one with an element of the internal symmetry group $SO(2)$.

It is easy to write down the unitary operator in the case $\lambda = 0$. Then we have two independent free scalar field theories

$$U_C = (-1)^{N_2}$$

As mathematicians are fond of saying, we have reduced it to the previous case.

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The action of U is especially nice if we put it in terms of the fields ψ and $\psi^\dagger$ already introduced

$$\psi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \longrightarrow U_c^\dagger \psi U_c = \frac{\phi_1 - i\phi_2}{\sqrt{2}} = \psi^\dagger$$

$$\psi^\dagger \longrightarrow \psi$$

the charge from the $SO(2)$ symmetry

For this reason, this is sometimes called a conjugation symmetry. Because $U_c^\dagger Q U_c = -Q$ it is also called charge conjugation or partical-anti-particle conjugation. From the action on ψ and $\psi^\dagger$ we see

$$U_c^\dagger b_k U_c = c_{\vec{k}} \quad \text{and} \quad U_c^\dagger c_k U_c = b_{\vec{k}}$$

and the equations obtained from these by hermitian conjugation. U_c is unitary and hermitian $U_c^2 = 1 = U_c U_c^\dagger \quad U_c = U_c^\dagger$

We could continue the discussion of discrete internal symmetries, but it is boring. You could write down a Lagranglan with four fields that is invariant under rotations in four-dimensional space and under permutations of any of the four fields. You could write down a theory with the icosahedral group.

PARITY TRANSFORMATIONS

Any transformation that takes

$$\phi^a(\vec{x}, t) \longrightarrow \sum_b M^a{}_b \, \phi^b(-\vec{x}, t)$$

we'll call a parity transformation. (We have used the fact that we live in an odd number of dimensions (3) and thus that $\vec{x} \to -\vec{x}$ is an improper rotation, in our definition. If we lived in two space dimensions we could do a similar thing with only $x^2 \to -x^2$.) A parity transformation transforms each fundamental observable at the point $\vec{x}$ into some linear combination of fundamental observables at the point $-\vec{x}$.

Usually parity takes $\mathcal{L}(\vec{x},t) \to \mathcal{L}(-\vec{x},t)$ (the usual confusing notation), when it is a symmetry, but all we really demand is that parity takes

EXAMPLE (1) $L(t) \to L(t)$ as usual.

Parity is a symmetry of

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 \qquad P: \phi(\vec{x},t) \to \phi(-\vec{x},t) \quad (M=1)$$

From

$$\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{x}} e^{-i\omega_{\vec{k}}t} + a^\dagger_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{x}} e^{+i\omega_{\vec{k}}t} \right]$$

and

$$U_P^\dagger \phi(\vec{x},t)\, U_P = \phi(-\vec{x},t)$$

we can see that the action on the creation and annihilation operators must be

$$U_P^\dagger \begin{Bmatrix} a_{\vec{k}} \\ a^\dagger_{\vec{k}} \end{Bmatrix} U_P = \begin{Bmatrix} a_{-\vec{k}} \\ a^\dagger_{-\vec{k}} \end{Bmatrix}$$

and on the basis states

$$U_P |\vec{k}_1, \ldots, \vec{k}_n\rangle = |-\vec{k}_1, \ldots, -\vec{k}_n\rangle$$

But there is a second possibility for parity

$$P': \phi(\vec{x},t) \to -\phi(-\vec{x},t) \qquad (M=-1)$$

Whenever there is an internal symmetry in a theory I can multiply one definition of parity by an element of that symmetry group (discrete or continuous) and get another definition of parity. In the case at hand the unitary operator $U_{P'}$ is given by

$$U_{P'} = (-1)^N U_P \qquad \text{or} \qquad U_{P'} |\vec{k}_1, \ldots, \vec{k}_n\rangle = (-1)^n |-\vec{k}_1, \ldots, -\vec{k}_n\rangle$$

Sometimes people distinguish between a theory with invariance under $P: \phi(\vec{x},t) \to \phi(-\vec{x},t)$ and a theory with invariance under $P': \phi(\vec{x},t) \to -\phi(-\vec{x},t)$, by calling the first the theory of a scalar meson and the second a theory of a pseudo scalar meson. Our theory is invariant under both; its the old plate of cookies problem again. The theory has a a set of invariances. As long as you are in agreement about the total set of invariances of the theory is, you shouldn't waste time arguing about what you'll call each one. This is why the conventions on the parity of some particles is arbitrary (relative parity). If $-\lambda\phi^4$ is added to $\mathcal{L}$ both P and P' are still invariances of $\mathcal{L}$. With a ϕ^3 interaction P is a symmetry, P' isn't. All the physicists in the world would agree that this is a scalar meson. One of the cookies has been poisoned.

EXAMPLE (2)

This awful Lagrangian has been cooked up to illustrate a point. After this lecture you won't see anything this bad again.

$$\mathcal{L} = \sum_{a=1}^{4} \left[\frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{\mu_a^2}{2}(\phi^a)^2\right] - g\,\epsilon^{\mu\nu\lambda\sigma}\,\partial_\mu\phi^1\partial_\nu\phi^2\partial_\lambda\phi^3\partial_\sigma\phi^4$$

All four meson masses are different. The new interaction involving the totally antisymmetric tensor in four indices is invariant for the same reason that $\vec{a}\cdot(\vec{b}\times\vec{c})$ is invariant under L.T. under proper rotations ($\vec{a}\cdot(\vec{b}\times\vec{c})$ is multiplied by $\det R$ under a rotation). Because $\epsilon^{\mu\nu\lambda\sigma}$ is nonzero only if one of its indices is timelike the other three spacelike three of the derivatives are on space coordinates. We get three minus signs under parity. An odd number of mesons are going to have to be pseudoscalar to get a net even number of minus signs. Because any $\phi^a \to -\phi^a$ is an internal symmetry of the free Lagrangian, it doesn't matter which one you choose or which three you choose to be pseudoscalar.

EXAMPLE (3)

Take the perverse Lagrangian of the last example by and make it worse by adding $\sum_{a=1}^{4}(\phi^a)^3$

Now there is no definition of parity that gives a symmetry. This theory violates parity.

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EXAMPLE 4

Sometimes people say that because the product of two reflections is 1, the square of parity is 1. This example is cooked up to show that a theory with parity can have indeed $U_P^2 \neq 1$ indeed U_P cannot be chosen to satisfy $U_P^2 = 1$.

$$\mathcal{L} = \sum_{a=1}^{4}\left[\frac{1}{2}(\partial_\mu \phi^a)^2 - \frac{\mu^2}{2}(\phi^a)^2\right] + \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi$$
$$- h\sum_{a=1}^{4}(\phi^a)^3 - g\,\epsilon_{\mu\nu\lambda\sigma}\,\partial^\mu\phi^1\,\partial^\nu\phi^2\,\partial^\lambda\phi^3\,\partial^\sigma\phi^4\left[\psi^2 + \psi^{*2}\right]$$

The transformation of the of the ϕ^a's must be $\phi^a(\vec{x},t) \to +\phi^a(\vec{x},t)$. The only way to make the last term parity invariant is for ψ to transform as

$$\psi \to \pm i\psi \qquad \psi^* \to \mp i\psi^*$$

In either case $U_P^2 \neq 1$, $U_P^\dagger U_P^\dagger \psi U_P U_P = -\psi$. Fortunately, nothing like this occurs in nature (as far as we know). If it did, and if parity were a symmetry (or an approximate symmetry) of the world we would have a name for fields transforming like ψ, a "seni-pseudo-scalar".

TIME REVERSAL

First a famous example from classical particle mechanics, a particle moving in a potential

$$L = \frac{1}{2}m\dot{q}^2 - V(q) \qquad T: q(t) \to q(-t)$$

T is not a discrete symmetry the way we have defined it

$$T: L(t) \to L(-t)$$

in the usual confusing notation where $L(t)$ refers to the time dependence through the coordinates

Nevertheless T does take one solution of the equations of motion into another. You might still hope there is a unitary operator that does the job in the quantum theory.

$$U_T^\dagger\, q(t)\, U_T = q(-t)$$

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There are two paradoxes I'll give to show this can't happen
e. 1ST PARADOX

Differentiate $U_T^\dagger q(t) U_T = q(-t)$ with respect to t. Because $p(t) \propto \dot{q}(t)$

$$U_T^\dagger p(t) U_T = -p(t)$$

Consider $U_T^\dagger [p,q] U_T = -i$. From the two relations we have just obtained we also have

$$U_T^\dagger [p(t), q(t)] U_T = -[p(-t), q(-t)] = -i$$

Particularly poignant at $t=0$

Looks like we would have to give up the canonical commutation relations to implement time reversal. If that isn't enough to make you abandon the idea of a unitary time reversal operator I'll continue to the

2ND PARADOX

Roughly, U_T should reverse time evolution, i.e. $U_T^\dagger e^{iHt} U_T = e^{iHt}$
I can prove this. For any operator $\mathcal{O}(t)$

$$\mathcal{O}(t) = e^{iHt} \mathcal{O} \cdot e^{-iHt}$$

Apply $U_T^\dagger$ the unitary transformation to both sides to obtain

$$\mathcal{O}(-t) = U_T^\dagger e^{+iHt} U_T \mathcal{O} U_T^\dagger e^{-iHt} U_T \qquad \mathcal{O} = \mathcal{O}(0)$$

but

$$\mathcal{O}(-t) = e^{-iHt} \mathcal{O} e^{iHt} \qquad \text{let } V \equiv U_T^\dagger e^{-iHt} U_T$$

I'd like to show $V = e^{iHt}$. what we have is $e^{-iHt} \mathcal{O} e^{iHt} = V^{-1} \mathcal{O} V$ which implies $V e^{-iHt} \mathcal{O} = \mathcal{O} V e^{-iHt}$. $V e^{-iHt}$ commutes with any operator $\mathcal{O}$. $V e^{-iHt} = 1$. Now that I've proved

$$U_T^\dagger e^{-iHt} U_T = e^{iHt},$$

take $\frac{d}{dt}\big|_{t=0}$ of this relation, $\quad U_T^\dagger (-iH) U_T = iH,$

cancelling the i's, we see that H is unitarily related to $-H$. The spectrum of H cannot be bounded below, because (the spectrum of unitarily related operators is the same and) the spectrum of H is not bounded above.
AUGGH!

A unitary time reversal operator is an object that makes no sense whatsoever. The answer is that time reversal is implemented by an antiunitary operator. Antiunitary operators is are antilinear. Dirac notation is designed to automate the handling of linear operators, so for a while we'll use some more cumbersome notation that does not automate the handling of linear operators.

Let a, b denote states
α, β complex numbers
A, B operators

(a,b) is the inner product of two states.

A <u>unitary operator</u> is an invertible operator, U, satisfying

$$(Ua, Ub) = (a,b) \quad \text{for all } a,b \quad \text{unitarity}$$

This is enough of an assumption to show U is linear (see related proof below). The simplest unitary operator is 1.

$$U(\alpha a + \beta b) = \alpha Ua + \beta Ub \quad \text{linearity}$$

The adjoint of a linear operator A is denoted $A^\dagger$ and is the operator defined by (this definition is not consistent if A is not linear)

$$(a, A^\dagger b) = (Aa, b) \quad \text{for all } a,b$$

I'll show that $U^\dagger = U^{-1}$ (which is sometimes given as the definition of unitarity).

$$(a, U^{-1}b) = (Ua, UU^{-1}b) = (Ua, b) \quad \checkmark$$

(unitarity of U)

A transformation of the states, $a \to Ua$, can also be thought of as a transformation of the operators in the theory

$(a, Ab) \to (Ua, AUb) = (a, U^\dagger AUb)$ can alternatively be thought of as $A \to U^\dagger AU$.

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An antiunitary operator is an invertible operator, $\mathcal{U}$, (this is a notational gem, an upside down U) satisfying

$$(\mathcal{U}a, \mathcal{U}b) = (b, a) \quad \text{for all } a, b \quad \text{antiunitarity}$$

We can immediately make a little table showing the result of taking products of unitary and antiunitary operators. (The product of a unitary operator with an antiunitary operator is an antiunitary operator, etc.)

	U	$\mathcal{U}$
U	U	$\mathcal{U}$
$\mathcal{U}$	$\mathcal{U}$	U

We can prove (this is the related proof referred to above) that any operator (not necessarily invertible) satisfying the antiunitarity condition is antilinear.

$$(\mathcal{U}a, \mathcal{U}b) = (b, a) \Longrightarrow \mathcal{U}(\alpha a + \beta b) = \alpha^* \mathcal{U}a + \beta^* \mathcal{U}b \quad \text{antilinearity}$$

Consider $(\mathcal{U}(\alpha a+\beta b) - \alpha^* \mathcal{U}a - \beta^* \mathcal{U}b, \mathcal{U}(\alpha a+\beta b) - \alpha^* \mathcal{U}a - \beta^* \mathcal{U}b)$. (You ask why?!) This is the inner product of $\mathcal{U}(\alpha a+\beta b) - \alpha^* \mathcal{U}a - \beta^* \mathcal{U}b$ with itself. If this is zero, the fact that the inner product is positive definite implies that $\mathcal{U}(\alpha a+\beta b) - \alpha^* \mathcal{U}a - \beta^* \mathcal{U}b = 0$. The result we want! Indeed, it is simply a matter of expanding this inner product out into its 9 terms, applying the antiunitarity condition to each term, and then expand the 5 terms containing $\alpha a+\beta b$ some more to show this is zero. (The analagous proof for operators satisfying the unitarity condition also only uses properties of the inner product and is even easier.)

The simplest antiunitary operator is complex conjugation, K. For the elements of some basis, b_i, $Kb_i = b_i$ and on any linear combination

$$K\left(\sum_i \alpha_i b_i\right) = \sum_i \alpha_i^* b_i$$

For consistency (b_i, b_j) must be real. This is the familiar complex conjugation of nonrelativistic quantum mechanics of position space wave functions. The basis is a complete set of real wave functions.

A useful fact (especially conceptually) is that any antiunitary operator, $\mathcal{U}$, is equal to UK for some unitary U. Proof by construction: take $U = \mathcal{U}K$.

In a more limited sense, the transformation of the states by an antiunitary operator $\mathcal{U}$, $a \to \mathcal{U}a$, can also be thought of as a transformation of the operators in the theory. Consider the expectation value of a Hermitian operator (observable) in the state a. It transforms as

$$(a, Aa) \to (\mathcal{U}a, A\mathcal{U}a) = (A\mathcal{U}a, \mathcal{U}a) \qquad \text{(hermiticity)}$$
$$= (\mathcal{U}\mathcal{U}^{-1}A\mathcal{U}a, \mathcal{U}a) \qquad \text{(invertibility)}$$
$$= (a, \mathcal{U}^{-1}A\mathcal{U}a) \qquad \text{(antiunitarity)}$$

This transformation can alternatively be thought of as

$$A \to \mathcal{U}^{-1}A\mathcal{U}$$

We don't write $\mathcal{U}^{\dagger}A\mathcal{U}$ because adjoint is not even defined for antilinear ops.

50 years ago, Eugene Wigner, proved a beautiful theorem telling us why unitary and antiunitary operators are important in QM. He showed that (up to phases) they are the only operators that preserve probabilities. It is not necessary to preserve inner products; they aren't measurable. It is the probabilities that are measurable. Look in the appendix of his book on group theory.

Given that $F(a)$ ($F: \mathcal{H} \to \mathcal{H}$) is continuous and for any a and b

$$|(F(a), F(b))|^2 = |(a,b)|^2$$

then $F(a) = e^{i\phi(a)} \times \begin{cases} \text{a unitary op} \\ \text{or an} \\ \text{antiunitary op} \end{cases}$ $\qquad (\phi: \mathcal{H} \to \mathbb{R})$

Time reversal is not a unitary operator. Time reversal is antiunitary.

It is easy to see now how the two paradoxes are avoided.

$$\mathcal{U}_T^{-1} i \mathcal{U}_T = -i$$

You can't cancel the i's as we did in paradox 2. $\mathcal{U}_T^{-1}(-iH)\mathcal{U}_T = iH$
$\Rightarrow \mathcal{U}_T^{-1} H \mathcal{U}_T = H$ ✓

We can explicitly construct the time reversal operator in free field theory. The simpler thing to look at in a relativistic theory is actually PT. Lets find

$$\mathcal{U}_{PT} \text{ such that } \mathcal{U}_{PT}^{-1}\,\phi(x)\,\mathcal{U}_{PT} = \phi(\tilde{x})$$

The simplest candidate is just complex conjugation, in the momentum state basis. That is $\mathcal{U}_{PT}$ does nothing, <u>absolutely nothing</u> to $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$

$$\mathcal{U}_{PT}^{-1}\, a_{\vec{k}}\, \mathcal{U}_{PT} = a_{\vec{k}} \quad \text{and} \quad \mathcal{U}_{PT}^{-1}\, a_{\vec{k}}^\dagger\, \mathcal{U}_{PT} = a_{\vec{k}}^\dagger$$

Furthermore we'll take $\mathcal{U}_{PT}|0\rangle = |0\rangle$ and it follows that

$$\mathcal{U}_{PT}\,|\vec{k}_1,\ldots,\vec{k}_n\rangle = |\vec{k}_1,\ldots,\vec{k}_n\rangle$$

again nothing, they just lie there. What does this operator do to $\phi(x)$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\left[a_{\vec{k}}\, e^{-ik\cdot x} + a_{\vec{k}}^\dagger\, e^{ik\cdot x}\right]$$

Apply $\mathcal{U}_{PT}$ to $\phi(x)$ it does nothing to $\frac{1}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}$, it does nothing to $a_{\vec{k}}$ and nothing to $a_{\vec{k}}^\dagger$.
But what about that i up in the exponential. It turns that into $-i$!

$$\mathcal{U}_{PT}^{-1}\,\phi(x)\,\mathcal{U}_{PT} = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\left[a_{\vec{k}}\, e^{ik\cdot x} + a_{\vec{k}}^\dagger\, e^{ik\cdot x}\right]$$

$$= \phi(-x)\,!$$

PT does nothing to momentum states. That is expected. Parity turns $\vec{k} \to -\vec{k}$ and time reversal changes it back again.

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OCT. 16 SCATTERING THEORY

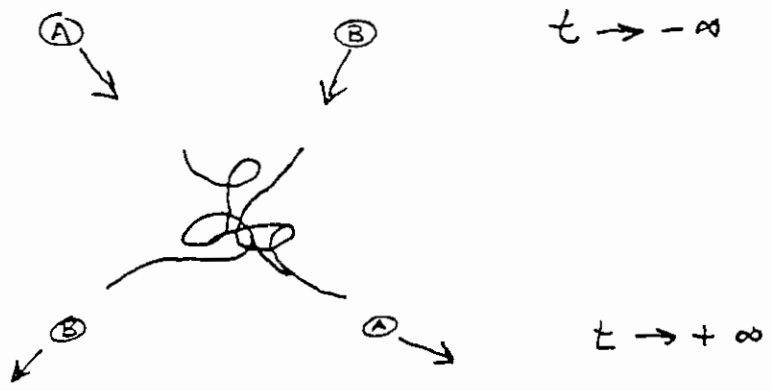
FOR A WIDE CLASS OF QUANTUM MECHANICAL SYSTEMS, THE DESCRIPTION OF ANY STATE IS SIMPLE IF YOU GO FAR ENOUGH INTO THE PAST OR FAR ENOUGH INTO THE FUTURE.

WE'LL ILLUSTRATE THIS BY A SEQUENCE OF THREE INCREASINGLY COMPLICATED SYSTEMS.

(1) NRQM, two particles interacting through a repulsive force, which dies off at large separation.

$$H = \frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B} + V(|\vec{r}_A - \vec{r}_B|) \qquad V \geq 0 \qquad V(\infty) = 0$$

Any state of the system, a real normalizable state, not a plane wave, looks in the far past/future like two particles far apart (actually a superposition of such states). The potential always pushes the particles far apart for times in the far future, and because the potential dies off they then look like noninteracting particles



There is no trace of the interaction in the far future or the far past. The states act like states with dynamics governed by a free Hamiltonian, H_0, which is simple,

$$H_0 = \frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B}$$

THE AIM OF SCATTERING THEORY IS TO TELL WHAT (SUPERPOSITION OF) SIMPLE STATE(S) IN THE FAR FUTURE A SIMPLE STATE IN THE FAR PAST EVOLVES INTO.

HOW FAR IN THE "FAR PAST" YOU HAVE TO GO DEPENDS ON THE INITIAL CONDITIONS AND THE INTERACTION. IF THE INTERACTION IS THE NUCLEAR FORCE AND WE COLLIDE TWO NEUTRONS AT LOW ENERGY WHICH ELASTICALLY SCATTER NEAR $t=0$, THEN $t=-7$ YEARS IS FAR ENOUGH IN THE FAR PAST SO THAT THE SYSTEM LOOKS LIKE TWO NON INTERACTING NUCLEONS. IF HOWEVER THE INITIAL CONDITIONS ARE SET UP SO THAT THE ELASTIC SCATTERING OCCURS AROUND $t=-1$ billion years then YOU MIGHT HAVE TO GO TO $t=-$(1billion and seven) years to make THE SYSTEM LOOK SIMPLE.

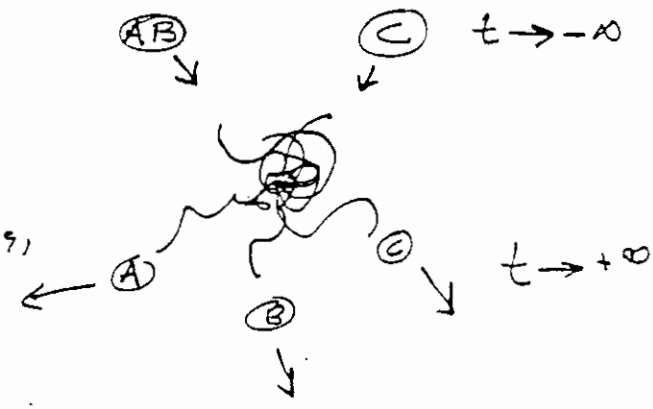
IT NEED NOT BE THAT THE SIMPLE HAMILTONIAN IN THE FAR FUTURE IS THE SAME AS THE SIMPLE HAMILTONIAN IN THE FAR PAST.

(2) NRQM, THREE Particles A, B, C which interact through attractive interactions which are strong enough to make an AB bound state. Start in the far past with C and the AB bound state. Everything still looks like free particles governed by a free Hamiltonian but the free Hamiltonian for the past is

$$H_0 = \frac{P_{AB}^2}{2m_{AB}} + \frac{P_C^2}{2m_C}$$

Now let scattering occur. In the far future, we can get states that look like three particles, A, B, C, non-interacting, governed by the free Hamiltonian

$$H_0 = \frac{P_A^2}{2m_A} + \frac{P_B^2}{2m_B} + \frac{P_C^2}{2m_C}$$



If you had a sufficiently advanced QM course, you may have studied such a system: $e^+ + H \to p + e^+ + e^-$

There is no way to truncate this system's full Hamiltonian into a free part and an interacting part, for which the free part describes the evolution of the system in the far future and the far past. If you use the far future H_0, you don't have an AB bound state.

(3) (A plausible picture of) the real world.

In the real world we have loads of (stable) bound states. If the real world has a laboratory bench as a stable bound state, then I need can do chalk-bench scattering, and I'll need a description of a freely flying piece of chalk and a freely flying laboratory bench. The description of states in the far past requires states with free electrons, hydrogen atoms, protons, Iron atoms, Iron nuclei, laboratory benches, chalk, and the associated free Hamiltonians.

Let's get some formalism up. (The first part of this lecture, with lots of words and few equations, is the part of a lecture that makes some people nervous and some people bored.)

Let H be the actual Hamiltonian of the world and $\mathcal{H}$ be the actual Hilbert space of the world. If you go sufficiently far in the past, every state in the actual Hilbert space looks simple. Let $\mathcal{H}_0$ be the Hilbert space of simple states and let $|\psi\rangle \in \mathcal{H}_0$. Somewhere in the real world Hilbert space there is a state that looks like $|\psi\rangle$ in the far past. We'll label that state $|\psi\rangle^{in}$, $|\psi\rangle^{in} \in \mathcal{H}$. Given another state $|\varphi\rangle \in \mathcal{H}_0$, there is another state in the real world Hilbert space that looks like $|\varphi\rangle$ in the far future. We'll label that state $|\varphi\rangle^{out}$, $|\varphi\rangle^{out} \in \mathcal{H}$. States in the complicated space are labelled by what they look like in the far past or the far future.

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What we are after in scattering theory is the probability, and hence the amplitude, that a given state looking like $|\psi\rangle$ in the far past, looks like $|\varphi\rangle$ in the far future. We are after ${}^{out}\langle\varphi|\psi\rangle^{in}$

The correspondence between $|\psi\rangle^{in}$ and $|\psi\rangle$ (for every state $|\psi\rangle \in \mathcal{H}_0$ there is a $|\psi\rangle^{in} \in \mathcal{H}$ that looks like $|\psi\rangle$ in the far past) and between $|\varphi\rangle^{in}$ and $|\varphi\rangle$ allows us to define an operator in the simple Hilbert space $\mathcal{H}_0$: S, the scattering matrix, is defined by

$$\langle\varphi|S|\psi\rangle \equiv {}^{out}\langle\varphi|\psi\rangle^{in}$$

An ideal scattering theory would have two parts

(1) A turn the crank method of obtaining the "descriptor" states $|\psi\rangle$, $|\varphi\rangle$, that is, generating $\mathcal{H}_0$, from the real world Hamiltonian. We also need H_0 which gives the evolution of the descriptor states. H_0 evolves the descriptor states without scattering.

(2) A turn the crank method of obtaining S.

90% of the rest of this course will be devoted to calculating the matrix elements of S perturbatively.

That's an ideal scattering theory. We want to get calculating so we'll start with a bargain basement, K-mart scattering theory.

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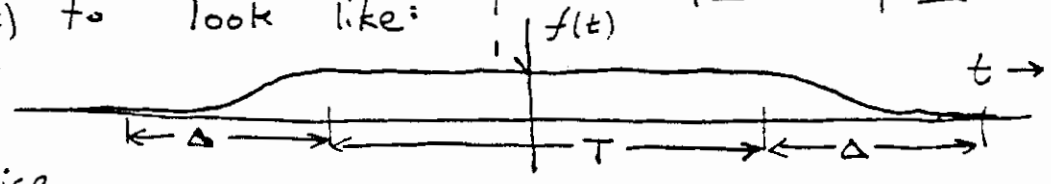
Low Budget Scattering Theory

Imagine that H can be written as $H = H_0 + f(t)H'$, $f(t) = 0$ for large $|t|$, and H_0 a free Hamiltonian that evolves states simply, without scattering. Unless the interaction is with some externally specified apparatus, interesting Hamiltonians and the real world Hamiltonian are not of this form. We want a simple description of states in the far past/future. Because the interaction is off in the far past/future, the simple descriptor states are simply the states in the full theory far enough in the past/future that $f(t) = 0$. ~~$\mathcal{H} = \mathcal{H}_0$~~.
Furthermore, the Hamiltonian that gives the evolution of the simple states, H_0, is just the full Hamiltonian H, far enough in the past/future that $f(t) = 0$.

Most Hamiltonians don't have an $f(t)$ in them that goes to zero as $|t| \to \infty$. However, many Hamiltonians are of the form $H = H_0 + H'$ where H_0 is a Hamiltonian we know the solution of. Maybe we could put an $f(t)$ into the Hamiltonian without changing scattering processes much. We know we can't do this in system (2). No matter how far you go into the far past/future it is the interaction that holds the stable AB bound state together, and you can't shut the interaction off without the bound state falling apart, totally changing scattering processes, no matter how long you wait to shut it off. The real world is like system (2) and we can only get a little ways studying the real world if we hack it up like this. We might get a ways studying system (1) like this. In the far past/future the particles in system (1) are far apart and noninteracting. If $f(t) \to 0$ in the far past/future, it should not affect their evolution since they are not interacting then anyway.

Suppose we wanted to insert an $f(t)$ to study a theory of electrons interacting through a repulsive Coulomb force. We can see a flaw developing. As a single electron goes off to ∞, away from all others, it still has a Coulomb field (cloud of photons) around it. If you weigh an electron, you get a contribution $\int \frac{E^2}{8\pi} d^3x$ to the mass-energy in addition to the contribution to the mass-energy at the heart of the electron. This is one and the same electric field that causes the scattering to take place, and you can't turn off scattering without turning off this cloud. Maybe, if we turn the interaction off sufficiently slowly the simple states in the real theory will turn into the states in the free theory with probability 1. We want $f(t)$ to look like:



$f(t)$ turns on and off adiabatically. A more precise way of stating the condition under which we hope inserting $f(t)$ into the theory won't change scattering processes much is there must be a 1-1 correspondence between the asymptotic (simple) states of the full Hamiltonian and the states of the free Hamiltonian. That means no bound states, no confinement. We hope scattering processes won't be changed at all under this assumption in the limit $\Delta \to \infty$, $T \to \infty$, $\frac{\Delta}{T} \to 0$. The last limit is needed so that edge effects are negligible. We want adiabatic turn on and turn off, but we also want the interaction to be on much longer than the amount of time we spend turning it on and off. Similar requirements must be imposed if you put a system in a spatial box, depending on what kind of quantities you want to know about. In slightly racy language, the electron without its cloud of photons is called a "bare" electron, and with its cloud of photons a "dressed" electron. The scattering process goes like this: In the far far past a bare electron moves freely along. A billion years before it is to interact it leisurely dresses itself. Then it moves along for a long time as a dressed electron, briefly interacts with another (dressed) electron and moves for a long time again, dressed. Then it leisurely undresses.

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We need to develop Time dependent Perturbation Theory for Hamiltonians of the form

$$H = H_0 + H'(t)$$

$H'(t)$ may depend on time because of externally varying interactions or because of the insertion of $f(t)$. We'll do the formalism in the interaction picture developed by Dirac. This is the best formalism for doing time-dependent perturbation theory. If you have laser light shining on an atom, and you know this formalism, it is the most efficient way of calculating what happens to the atom, although it is higher powered than the minimum formalism you need for that problem.

Schrödinger picture

We have states evolving in time according to

$$i \frac{d}{dt} |\psi(t)\rangle_S = H(p_S, q_S, t) |\psi(t)\rangle_S$$

↑ for Schrödinger, since we'll be working in several pictures

The fundamental operators, p_S and q_S are time independent

$$q_S = q_S(t) = q_S(0) \qquad p_S = p_S(t) = p_S(0)$$

The only operators that are not time independent are operators that explicitly depend on time. The time evolution operator $U(t,t')$ is given by

$$|\psi(t)\rangle_S = U(t,t') |\psi(t')\rangle_S$$

U is completely determined by the 1st order differential equation in t

$$i \frac{d}{dt} U(t,t') = H(p_S, q_S, t)\, U(t,t')$$

and the initial condition $U(t,t')|_{t=t'} = 1$. Think of t' as a parameter. U is unitary, (from the Hermiticity of H), i.e. $U(t,t')^\dagger = U(t,t')^{-1}$, which expresses the conservation of probability. U also obeys the composition law $U(t,t')\,U(t',t'') = U(t,t'')$ which implies $U(t,t') = U(t',t)^{-1}$

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Heisenberg picture

The states do not change with time

$$|\psi(t)\rangle_H = |\psi(0)\rangle_H = |\psi(0)\rangle_S$$

If $A_S(t)$ ← possible explicit time dependence is an operator in the Schrödinger picture and $A_H(t)$ its counterpart in the Heisenberg picture, demand that

$$\begin{aligned} {}_S\langle\varphi(t)|A_S(t)|\psi(t)\rangle_S &= {}_H\langle\varphi(t)|A_H(t)|\psi(t)\rangle_H \\ &= {}_S\langle\varphi(0)|A_H(t)|\psi(0)\rangle_S \\ &= {}_S\langle\varphi(t)|U(0,t)^\dagger A_H(t)\,U(0,t)|\psi(t)\rangle_S \end{aligned}$$

$$\therefore\quad A_H(t) = U(0,t)\,A_S(t)\,U(0,t)^\dagger = U(t,0)^\dagger A_S(t)\,U(t,0)$$

Suppose we have some function of operators and the time in the Schrodinger picture, itself an operator. For example H itself is a function of p_S, q_S and t. To get the operator in the Heisenberg picture, all you have to do is replace p_S and q_S by p_H and q_H.

$$H_H(t) = H(p_H(t), q_H(t), t)$$

Expand H as a power series and insert $U(t,0)U(t,0)^\dagger$ all over.

Interaction Picture

Assume H can be written as $H(p,q,t) = H_0(p,q) + H'(p,q,t)$
This defines the relation between the functions H, H_0 and H', the arguments of these functions will change.

The interaction picture is intermediate between the Schrödinger picture and the Heisenberg picture. You make the transformation you would make to get from the Schrödinger picture to the Heisenberg picture, but you do it using just the free part of the Hamiltonian only.

$$|\psi(t)\rangle_I = e^{iH_0(p_S,q_S)t}\,|\psi(t)\rangle_S$$

If there were no interaction H', there would be no time evolution of the states. That is what makes the interaction picture so useful.

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If H_0 depended explicitly on time we would have to define a $U_0(t,0)$ which would take the place of e^{-iH_0t}, but we will have no occasion to be that general. If $A_S(t)$ is an operator in the Schrödinger picture, and $A_I(t)$ its counterpart in the interaction picture, we get

$$A_I(t) = e^{iH_0(p_S,q_S)t} A_S(t) e^{-iH_0(p_S,q_S)t}$$

by demanding $\quad {}_S\langle\varphi(t)|A_S(t)|\psi(t)\rangle_S = {}_I\langle\varphi(t)|A_I(t)|\psi(t)\rangle_I$

We can find a differential equation for $|\psi(t)\rangle_I$

$$i\frac{d}{dt}|\psi(t)\rangle_I = i\frac{d}{dt}\left(e^{iH_0(p_S,q_S)t}|\psi(t)\rangle_S\right)$$

$$= e^{iH_0(p_S,q_S)t}\left[-H_0(p_S,q_S) + H(p_S,q_S,t)\right]|\psi(t)\rangle_S$$

$$= e^{iH_0(p_S,q_S)t}\left[H'(p_S,q_S,t)\right]e^{-iH_0(p_S,q_S)t}|\psi(t)\rangle_I$$

$$= H'(p_I,q_I,t)|\psi(t)\rangle_I$$ Expand $H'(p_S,q_S,t)$ in a power series and insert $e^{-iH_0t}e^{iH_0t}$ all over.

$$\equiv H_I(t)|\psi(t)\rangle_I$$ As promised, if H' is zero, no time evolution.

In field theory, $H_I(t)$ will contain the free fields

$$\phi(\vec{x},t) = e^{iH_0t}\phi_S(\vec{x})e^{-iH_0t}$$

That is why all our results about free fields are still going to be useful.

We can define $U_I(t,t')$ by $\quad |\psi(t)\rangle_I = U_I(t,t')|\psi(t')\rangle_I$

$$U_I^+(t,t') = U_I(t,t')^{-1} \qquad U_I(t,t')U_I(t',t'') = U_I(t,t'')$$

$$U_I(t,t') = U_I(t',t)^{-1} \qquad U_I(t,0) = e^{iH_0(p_S,q_S)t}U(t,0)$$

In a field theory, $q_S(x)$ obey free eq. of motion + commutation relations

$U_I(t,t')$ can be determined from the first order differential equation in t it satisfies

$$i\frac{d}{dt}U_I(t,t') = H_I(t)\,U(t,t') \quad \text{and the initial condition } U_I(t,t')\Big|_{t=t'} = 1$$

Now we'll apply interaction picture perturbation theory to scattering theory. In the interaction picture the scattering process looks like this: In the far past the interaction is not felt, both because of $f(t)$ and the fact that the particles are far apart. The states just lie there, although the p^{is} and q^{is} are changing. The time of scattering approaches and the state starts changing. After scattering they stop scattering, like a game of musical chairs, everything freezes again.

You want to connect the simple description in the far past to the simple description in the far future. Because of $f(t)$ the simple description in the far past/future is the actual description in the far past/future. The states in the far past and future are their own descriptors (used in arrowed step).

$$\langle\varphi|S|\psi\rangle \equiv {}^{out}\langle\varphi|\psi\rangle^{in} = {}_I\langle\varphi(0)|\psi(0)\rangle_I$$
$$= {}_I\langle\varphi(\infty)|U_I(\infty,-\infty)|\psi(-\infty)\rangle_I$$
$$= \langle\varphi|U_I(\infty,-\infty)|\psi\rangle$$

$$\therefore \quad S = U_I(\infty,-\infty)$$

Our number one priority then is to evaluate $U_I(\infty,-\infty)$. That will cause us to derelop Dyson's formula and wicks theorem. Then we'll apply this formalism to three models.

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Proof that $S = U_I(\infty,-\infty)$ in the Schrödinger picture.

$$i\frac{d}{dt}|\psi\rangle^{in} = H|\psi\rangle^{in}$$

$$|\psi(-\infty)\rangle = |\psi(-\infty)\rangle^{in}$$

$$i\frac{d}{dt}|\psi\rangle = H_0|\psi\rangle$$

$${}^{out}\langle\phi|\psi\rangle^{in} \equiv \langle\phi|S|\psi\rangle$$

$$\therefore \langle\phi|\psi\rangle^{in} = {}^{out}\langle\phi(t)|\psi(t)\rangle^{in}$$

← any time will do

$$= {}^{out}\langle\phi(\infty)|U(\infty,0)U(0,-\infty)|\psi(-\infty)\rangle^{in}$$

$$= \langle\phi(\infty)|U(\infty,-\infty)|\psi(-\infty)\rangle$$

$$= \langle\phi(0)|U_0(\infty,0)^{\dagger -1}U(\infty,-\infty)U_0(0,-\infty)^{-1}|\psi\rangle$$

$$= \langle\phi|U_I(\infty,-\infty)|\psi\rangle$$

$[S, H_0] = 0$ because S turns free states of some given energy into other free states of the same energy.

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We would like to find the solution of the equation

$$i\frac{d}{dt}U_I(t,t') = H_I(t)\,U_I(t,t') \qquad U_I(t,t') = 1$$

Imagine that $[H_I(t), H_I(t')] = 0$, which is not true. Then the solution would be

$$U_I(t,t') = e^{-i\int_{t'}^{t} dt''\, H_I(t'')}$$

Let's define a new exponential so this equation is right. Given a string of operators define the time-ordered product

$$T\left[A_1(t_1)\cdots A_n(t_n)\right]$$

to be the string rearranged so that later operators are to the left of earlier operators, with the operator with to the latest time on the leftist. The ambiguity of what to do at equal times does not bother us when the operators commute at equal times. This is certainly the case when all the operators are the same, $H_I(t)$, evaluated at various t. T, the symbol for the time ordering operation, is not an operator in Hilbert space. Time ordering is a notation.

Now we'll show that the differential equation for $U_I(t,t')$ is satisfied by

$$T\, e^{-i\int_{t'}^{t} dt''\, H_I(t'')} \qquad t > t'$$

Under the time ordering symbol everything commutes, so we can naively take a time derivative to get

$$i\frac{d}{dt}T e^{-i\int_{t'}^{t} dt''\, H_I(t'')} = T\left(H_I(t)\, e^{-i\int_{t'}^{t} dt''\, H_I(t'')}\right)$$

Now t is a special time. It is the latest time, so the time ordering puts $H_I(t)$ on the leftist, and we can pull it out on the left to get

$$i\frac{d}{dt}T e^{-i\int_{t'}^{t} dt''\, H_I(t'')} = H_I(t)\, T e^{-i\int_{t'}^{t} dt''\, H_I(t'')}$$

This solution of the differential equation also obeys the boundary condition (any old ordering does that),

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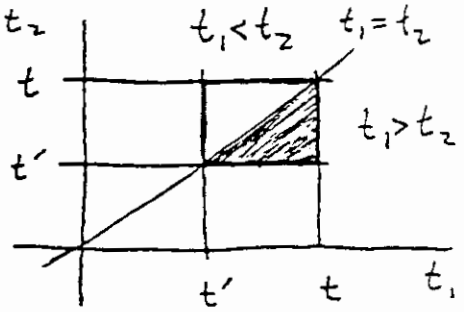
The solution of a first order differential equation with given initial value is unique. Therefore

$$U_I(t,t') = T\, e^{-i\int_{t'}^{t} dt''\, H_I(t'')} \qquad t>t'$$

DYSON'S FORMULA

To illustrate what Dyson's formula means, we'll look at the second order term in the power series expansion for the exponential.

$$\frac{(-i)^2}{2!}\int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, T\left(H_I(t_1)\, H_I(t_2)\right)$$



Think of this as an integration over the square. If the time ordering symbol were not there, you would be integrating $H(t_1)H(t_2)$ over the whole square. Because of the time ordering symbol, you get instead twice the integral of $H(t_1)H(t_2)$ over the lower half of the square, the shaded triangle.

Our formula is only valid for $t>t'$. You can easily get the formula for $t<t'$ by taking the adjoint.

We are going to apply Dyson's formula to three model theories in order of increasing complexity.

MODEL 1 $\qquad \mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - g\rho(x)\,\phi(x)$

$\rho(x)\to 0$ as $x\to\infty$ in space or time. $\rho(x)$ is a prescribed c-number function of space-time, a source which will create mesons.

The equation of motion is

$$(\Box+\mu^2)\,\phi(x) = -g\,\rho(x)$$

which generates the EM field

Electromagnetism with an external source looks like $\Box A^\mu = -ej^\mu$. Except that our field is massive, has no vector index, and is a quantum field these two theories look similar. We'll call model 1 quantum meso-dynamics. We'll be able to solve it exactly, which is not a big surprise; in momentum space it is just a bunch of independent forced harmonic oscillators.

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MODEL 2 $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{\mu^2}{2}\phi^2 - g\rho(\vec{x})$

$\rho(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$. This is the same as model 1 except the source is static. You might think this time independent problem would be easier than model 1, but it isn't because the source does not turn off as $|t| \to \infty$. We will have to use (and thus gain experience with) our adiabatic turning on and off function, $f(t)$. This is the quantum scalar analog of electrostatics, so we'll call it "mesostatics."

MODEL 3 $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\psi^*\psi\phi$

The equation of motion for the ϕ field is

$$(\Box + \mu^2)\,\phi = -g\,\psi^*\psi$$

This is beginning to look like the real thing. In real electrodynamics, the current j^μ is not prescribed, it is the current of charged particles. Here we have the charged field ψ as a source for ϕ ($\psi^*\psi$ is like a current). The ϕ field in turn appears in the equation of motion for the ψ field.

$$(\Box + m^2)\,\psi = -g\,\psi\phi$$

This theory also looks a lot like Yukawa's theory of the interaction between mesons and nucleons, except our charged particles are spinless and we only have one meson. We'll call this "meson-"nucleon"" theory. Actually, we had better not push this theory too far (we'll be doing low orders in P.T. only). The classical Hamiltonian contains $g\psi^*\psi$ and that is not bounded below for either sign of g.

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Wick's Theorem

When doing perturbative calculations in g in any of these three models, we are going to have to evaluate time ordered products of strings of Hamiltonians between states. In model 1 $\mathcal{H}_I = g\rho(x)\phi(x)$. At fourth order in g for a meson scattered by the source we would have to evaluate

(these are free fields)

$$\langle \vec{k}' | T\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right) | \vec{k} \rangle$$

The time ordered product contains 16 arrangements of creation and annihilation operators, from $a_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3} a_{\vec{k}_4}$ and $a_{\vec{k}_1} a_{\vec{k}_2} a_{\vec{k}_3} a^\dagger_{\vec{k}_4}$ to $a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_2} a^\dagger_{\vec{k}_3} a^\dagger_{\vec{k}_4}$. If we could rearrange these into normally ordered products, the only normally ordered product that could contribute would be the one with one creation operator on the left and one annihilation operator on the right, a great simplification. In model 3 we will have to evaluate time ordered products like

$$T\left(\phi(x_1)\psi^*(x_1)\psi(x_1)\phi(x_2)\psi^*(x_2)\psi(x_2)\right).$$

If we had an algorithm for normal ordering the time ordered product, we would again have great simplifications when we sandwiched this between states of mesons and nucleons. Wick's theorem turns time ordered products of free fields into normal ordered products of free fields. To state Wicks theorem we'll define the contraction.

$$\overbrace{A(x)B(y)} \equiv T\left(A(x)B(y)\right) - :A(x)B(y):$$

Suppose, without loss in generality, really, that $x^0 > y^0$. Then $T(A(x)B(y)) = A(x)B(y) = (A^{(+)} + A^{(-)})(B^{(+)} + B^{(-)}) = :AB: + [A^{(+)}, B^{(-)}]$ and $\overbrace{A(x)B(y)} = [A^{(+)}, B^{(-)}]$ which is a c-number. This is also a c# when $x^0 < y^0$. So whether $x^0 < y^0$ or $y^0 < x^0$ $\overbrace{A(x)B(y)}$ is a c# and is thus equal to its vacuum expectation value. Using its definition

$$\overbrace{A(x)B(y)} = \langle 0 | \overbrace{A(x)B(y)} | 0 \rangle = \langle 0 | T(A(x)B(y)) | 0 \rangle - \langle 0 | :A(x)B(y): | 0 \rangle$$

That's why the calculation of $\langle 0|T(\phi(x)\phi(y)|0\rangle$ in the first problem set is going to be useful.

$$\overbrace{\phi(x)\phi(y)} = \langle 0|T(\phi(x)\phi(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{\pm ik\cdot(x-y)} \frac{i}{k^2-\mu^2+i\epsilon}$$

$\lim_{\epsilon\to 0+}$ is understood. Convince yourself the $\pm$ doesn't matter. You can also see that

$$\overbrace{\psi(x)\psi^*(y)} = \overbrace{\psi^*(x)\psi(y)} = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-y)} \frac{i}{k^2-m^2+i\epsilon}$$

A little more obvious notation: $:A(x)\overbrace{B(y)C(z)}D(w):\equiv :A(x)C(z):\overbrace{B(y)D(w)}$

and let $\phi_1 \equiv \phi^{a_1}(x_1)$, $\phi_2 \equiv \phi^{a_2}(x_2)$, etc., just for this proof.

Theorem (Gian-Carlo Wick)

$$T(\phi_1\cdots\phi_n) = :\phi_1\cdots\phi_n: + :\overbrace{\phi_1\phi_2}\cdots\phi_n: + \text{all the other } \tfrac{n(n-1)}{2}-1 \text{ possible terms with one contraction}$$

(term with no contractions)

$$+ :\overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4}\cdots\phi_n: + \text{all the other } \tfrac{1}{2!}\tfrac{n(n-1)}{2}\tfrac{(n-2)(n-3)}{2}-1 \text{ possible terms with two contractions}$$

$$\vdots$$

$$+ \begin{matrix}\text{either terms like}\\ \text{or terms like}\end{matrix} \begin{matrix}:\overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4}\cdots\overbrace{\phi_{n-1}\phi_n}: & \text{if } n \text{ is even}\\ :\overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4}\cdots\overbrace{\phi_{n-2}\phi_{n-1}}\phi_n: & \text{if } n \text{ is odd}\end{matrix}$$

You draw all possible terms with all possible contractions. That you get all that is no surprise. The remarkable and graceful thing about this theorem is that each term occurs with coefficient $+1$.

Proof (By induction) Define the RHS of the expression to be $W(\phi_1\cdots\phi_n)$. We want to show $W=T$. Trivial for $n=1,2$. Choose without loss of generality $x_{10}^0 > x_{20}^0 > \cdots > x_{n0}^0$. Then

$$T(\phi_1\cdots\phi_n) = \phi_1 T(\phi_2\cdots\phi_n) \overset{\text{induction}}{=} \overset{\text{step}}{\phi_1} W(\phi_2\cdots\phi_n) = \phi_1^{(-)}W + W\phi_1^{(+)} + [\phi_1^{(+)}, W]$$

This expression is normal ordered. The first two terms contain all possible contractions that do not include ϕ_1. The second term contains all possible contractions that do include ϕ. Together they contain all possible contractions. Either a contraction includes ϕ_1 or it doesn't. The right hand side is thus $W(\phi_1\cdots\phi_n)$.

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DIAGRAMMATIC PERTURBATION THEORY

Dyson's formula applied to $S = U_I(\infty, -\infty)$ is

$$U_I(\infty,-\infty) = T\, e^{-i\int dt\, H_I(t)}$$

(Without the use of the time ordering notation, this the formula for $U_I(t,t')$ was written down by Dirac 15 years before Dyson wrote it this way, and Dyson says he should not have credit for little more than a change in notation.)

From this and Wick's theorem, which for those of you who really love combinatorics can be written

$$T(\phi_1 \cdots \phi_n) = :e^{\frac{1}{2}\sum_{i,j=1}^{n} \overbrace{\phi_i \phi_j} \frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}} \phi_1 \cdots \phi_n:,$$

we have enough work done to write down diagrammatic perturbation theory for $S = U_I(\infty,-\infty)$. The easiest way to see this is to look at a specific model and a contribution to $U_I(\infty,-\infty)$ at a specific order in g.

In model 3, $\mathcal{H}_I = g f(t) \psi^* \psi \phi$, $H_I = \int d^3x\, \mathcal{H}_I$,

$$U_I(\infty,-\infty) = T\, e^{-i\int d^4x\, \mathcal{H}_I} = T\, e^{-i\int d^4x\, g f(t) \psi^* \psi \phi},$$

the contribution at second order in g is

$$\frac{(-ig)^2}{2!} \int d^4x_1\, d^4x_2\, f(t_1) f(t_2)\, T\left(\psi^*\psi\,\phi(x_1)\,\psi^*\psi\,\phi(x_2)\right).$$

One of the terms in the expansion of the time ordered product into normal ordered products by Wick's theorem is

$$\frac{(-ig)^2}{2!} \int d^4x_1\, d^4x_2\, f(t_1) f(t_2) :\psi^*\psi\, \overbrace{\phi(x_1)\, \psi^*\psi}\, \phi(x_2):$$

This term can contribute to a variety of physical processes. The ψ field contains operators that annihilate a "nucleon" and operators that create an anti-"nucleon". The ψ^* field contains operators that annihilate an anti-nucleon and create a nucleon. The operator

$$:\psi^*\psi\,\overbrace{\phi(x_1)\,\psi^*\psi}\,\phi(x_2):\equiv:\psi^*\psi(x_1)\,\psi^*\psi(x_2):\,\overbrace{\phi(x_1)\,\phi(x_2)}$$

can contribute to $N+N\to N+N$. That is to say

$$\langle 2\text{ nucleon state}^{\text{final}}\,|:\psi^*\psi(x_1)\,\psi^*\psi(x_2):|\,2\text{ nucleon state}^{\text{initial}}\rangle$$

is nonzero because there are terms in the two ψ fields that can annihilate the two nucleons in the initial state and terms in the 2 ψ^* fields that can then create two nucleons, to give a nonzero matrix element. It can also contribute to $\bar{N}+\bar{N}\to\bar{N}+\bar{N}$ and $N+\bar{N}\to N+\bar{N}$. You can see that there is no combination of creation and annihilation operators in this operator that can contribute to $N+N\to\bar{N}+\bar{N}$. The ψ fields would have to annihilate the nucleons and the ψ^* fields cannot create antinucleons. This is good because this process does not conserve the $U(1)$ symmetry charge. However it looks like our operator can contribute to vacuum $\to N+N+\bar{N}+\bar{N}$, which would be a disaster. The coefficient of that term after integrating over x_1 and x_2 had better turn out to be zero.

Another term in the expansion of the time ordered product into normal ordered products is

$$\frac{(-ig)^2}{2!}\int d^4x_1\,d^4x_2\,f(t_1)f(t_2):\psi^*\overbrace{\psi\,\phi(x_1)\,\psi^*}\psi\,\phi(x_2):$$

This term can contribute to the following $2\to 2$ scattering processes: $N+\phi\to N+\phi$, $\bar{N}+\phi\to\bar{N}+\phi$, $N+\bar{N}\to 2\phi$, $2\phi\to N+\bar{N}$.

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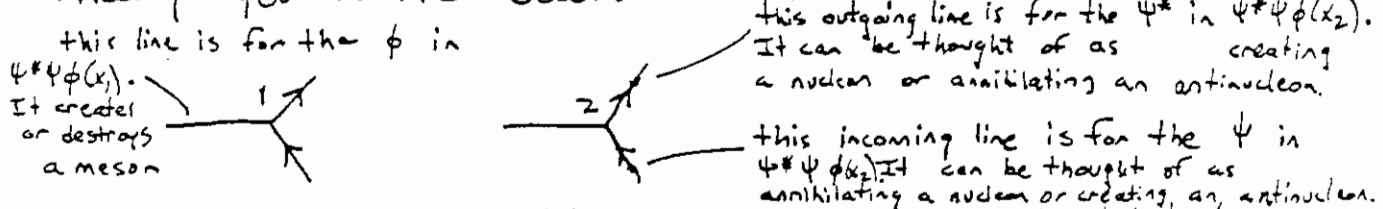
A single term is capable of contributing to a variety of processes because a single ψ field is capable of creating or destroying a particle.

The terms in the Wick expansion can be written down in a diagrammatic shorthand according to the following rules. At Nth order in perturbation theory, you start by writing down N interaction vertices and numbering them 1 to N. For model 3 at second order in perturbation theory you write down



The vertex represents the factor of $\int \psi^*\psi\phi$. From the fact that there are two in this diagram you know to include $\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2$

Contractions are represented by connecting the lines. Any time there is a contraction, join the lines of the contracted fields. The arrows will always line up, because the contractions for which they don't are zero. An unarrowed line will never be connected to an arrowed line because that contraction is also zero.

Our first term in the expansion of the time ordered product corresponds to the diagram

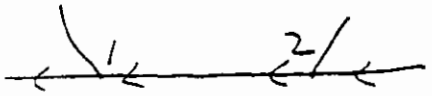
The second term corresponds to

The term in the Wick expansion

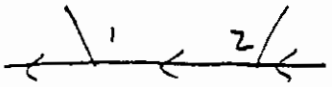
$$\frac{(-ig)^2}{2!}\int d^4x_1\, d^4x_2\, f(t_1)f(t_2)\, :\overbrace{\psi^*\psi\,\phi(x_1)}\,\overbrace{\psi^*\psi\,\phi(x_2)}:$$

is zero because $\overbrace{\psi^*\,\psi^*} = 0$, so we never write down

Because the arrows always line up we can shorten

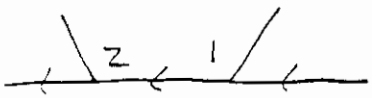


to

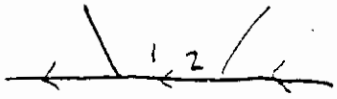


These diagrams are in one-to-one correspondence with terms in the Wick expansion of Dyson's formula.

We'll call them Wick diagrams. They stand for operators and the vertices are numbered. We are most of the way to Feynmann diagrams which stand for matrix elements, but these aren't them yet. The vertices are numbered in Wick diagrams and



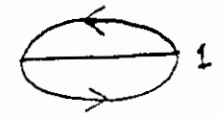
is distinct from



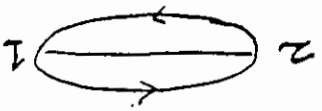
(there are two distinct terms in the Wick expansion) even though after integrating over x_1 and x_2 these are identical operators. In Feynmann diagrams the lines will be labelled by momenta.

On the other hand

is identical to



(there is only one way of contracting all three fields at one vertex with all three at the other). Although these two have been written down to look different they aren't. Rotate the right one by 180° and you see they are the same.



The contraction this diagram corresponds to is

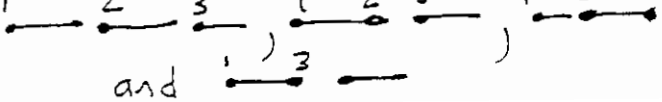
$$\frac{(-ig)^2}{2!}\int d^4x_1 d^4x_2 f(t_1) f(t_2) : \psi^*\psi\,\phi(x_1)\,\psi^*\psi\,\phi(x_2) :$$

In model 1 $\mathcal{H}_I = g\rho(x)\phi(x)$ (we don't have to insert a turning on and off function because the interaction goes to zero as $x \to \infty$ in any direction and in particular in the time direction in the far past/future. $\rho(x)$ is a prescribed c number source, so strongly we don't have to worry about the back reaction of the field ϕ on the source. The vertex in this model is made

That represents $\rho\,\phi(x)$. At $O(g)$ in U_I we have $(-ig)\int d^4x_1\, \rho\, \phi(x_1)$ which is represented

At $O(g^2)$ we have

and

At $O(g^3)$ we have 

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"Connected" means (in any theory) that the diagram is in one connected piece. It doesn't mean fully contracted.  is a fully contracted diagram that is not connected  is not contracted, but is connected.

A diagram at $O(g^4)$ is

We have been putting the normal ordering inside the integrand. Of course we could put it around the whole integral in which case we see that this $O(g^4)$ diagram corresponds to

$$\frac{(-ig)^4}{4!} : \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \, \rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4) \cdot \overbrace{\phi(x_1)\,\phi(x_2)}\,\phi(x_3)\phi(x_4) :$$

$$= \frac{(-ig)^4}{4!} : \int d^4x_1 d^4x_2 \, \overbrace{\phi(x_1)\phi(x_2)}\,\rho(x_1)\rho(x_2) \cdot \int d^4x_3 \phi(x_3)\rho(x_3) \int d^4x_4 \phi(x_4)\rho(x_4) :$$

That is the integrands factor into products of terms corresponding to each connected part of the diagram. This suggests we can sum the series and then normal order in this simple theory, because at all orders in g the diagrams only contain  and  various numbers of times. We could do the sum in this theory, but instead we will prove a general theorem.

$$\sum \text{all Wick diagrams} = : e^{\sum \text{connected Wick diagrams}} :$$

In a theory with only two connected diagrams this theorem is powerful enough to solve the theory exactly in a couple of lines. It will help a lot in model 3, but we won't since there are still an infinite number of connected diagrams in model 3 solve it. This formula is also useful in condensed matter physics where you develop a perturbation theory for $\mathrm{Tr}\, e^{-\beta H}$. The free energy which is the logarithm of the partition function is what is actually of interest. This theorem's analog tells you that you don't have to calculate a huge series for $\mathrm{Tr}\, e^{-\beta H}$ and then try to take its logarithm. The free energy is just the sum of the connected diagrams.

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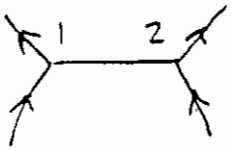
Let D be a general diagram with $n(D)$ vertices. associated with this diagram is an operator

$$\frac{:\mathcal{O}(D):}{n(D)!}$$

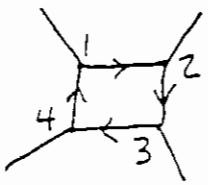
We have explicitly displayed the $n(D)!$ and we have pulled the normal ordering outside. For example for

$D =$ 

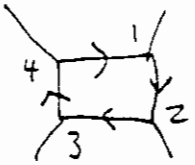
$$\mathcal{O}(D) = (-ig)^2 \int d^4x_1 d^4x_2 f(t_1) f(t_2) \overbrace{\phi(x_1)\phi(x_2)} \cdot \psi^*\psi(x_1)\,\psi^*\psi(x_2)$$

I will define two diagrams to be of the same "pattern" if they differ just by permuting the labels at the vertices, $1, 2, \ldots, N(D)$.

Since after integration over $x_1, \ldots, x_{N(D)}$ two different diagrams of the same pattern give identical contributions to U_I and since there are $N(D)!$ permutations of the numbers $1, \ldots, N(D)$, you might expect the sum over all diagrams of a given pattern to exactly cancel the $N(D)!$. This is not quite right however. For some diagrams there are permutations of the vertices that have no effect, for example



is not distinct from



(and there are two more cyclic permutations) but it is distinct from the diagrams with noncyclic permutations. This is in exact correspondence with the question of whether or not there is a new term in the Wick expansion from permuting $x_1, \ldots, x_{n(D)}$.

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For any pattern, there will be some symmetry number, $S(D)$, which is the number of permutations that have no effect on the diagram D. (and of course there is the analagous statement, that there are $S(D)$ permutations of $x_1, \ldots, x_{N(D)}$ that do not give additional contributions in the Wick expansion. Summing over all distinct diagrams of the same pattern as D yields

$$\frac{:O(D):}{S(D)}$$

Let $D_1, D_2, \ldots, D_r, \ldots$ be a complete set of connected diagrams, with one diagram of each pattern. A general diagram, D, has n_r components of pattern D_r. Because of the factorization of the integrands and because we have explicitly pulled out the $n(D)!$

$$:O(D): = :\prod_{r=1}^{\infty} [O(D_r)]^{n_r}:$$

Summing over all diagrams with the same pattern as D gives $:O(D):/S(D)$. What is $S(D)$? $S(D)$ certainly contains $\prod_r [S(D_r)]^{n_r}$. If I have 2 identical factors, I can take all the indices on one of them and exchange them with the other. If I have n identical factors there are $n!$ whole exchanges. So $S(D)$ contains $\prod n_r!$. The sum over all diagrams with the same pattern as D gives

$$\frac{:O(D):}{S(D)} = \frac{:\prod_{r=1}^{\infty} [O(D_r)]^{n_r}:}{\prod_{r=1}^{\infty} [S(D_r)^{n_r} n_r!]}$$

Now that we have done the sum over all diagrams of a given pattern, we have to sum over all patterns. Notice that there is a 1-1 correspondence between patterns and sets $\{n_r\}$. Thus summing over all patterns is the same as summing over all sets $\{n_r\}$.

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So, $\sum$ all Wick diagrams

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \frac{: \prod_{r=1}^{\infty} [O(D_r)]^{n_r} :}{\prod_{r=1}^{\infty} [S(D_r)^{n_r} n_r!]}$$

$$= : \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \prod_{r=1}^{\infty} \frac{O(D_r)^{n_r}}{S(D_r)^{n_r} n_r!} :$$

$$= : \prod_{r=1}^{\infty} \left(\sum_{n_r=0}^{\infty} \frac{[O(D_r)/S(D_r)]^{n_r}}{n_r!} \right) :$$

$$= : \prod_{r=1}^{\infty} e^{O(D_r)/S(D_r)} :$$

$$= : e^{\sum_{r=1}^{\infty} O(D_r)/S(D_r)} :$$

$$= : e^{\sum \text{connected Wick Diagrams}} :$$

This is a neat theorem because it expresses a fact about diagrams, pictures, algebraically.

Now we'll apply this to model 1

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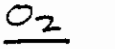
Model 1 Solved

$D_1 =$ $D_2 =$ $S(D_2) = 2$

$$U_I(\infty, -\infty) = :e^{O_1 + \frac{O_2}{2}}: = :e^{\text{  } + \text{  }}:$$

$$O_1 = -ig \int d^4x_1 \rho(x_1) \phi(x_1)$$

$$O_2 = (-ig)^2 \int d^4x_1 d^4x_2 \overbrace{\phi(x_1)\phi(x_2)}\, \rho(x_1)\rho(x_2) = \underset{\text{number}}{\text{some}} = \alpha + i\beta$$

You will compute α in the homework. We'll get it here by a consistency argument, demanding that U_I be unitary.

Lets rewrite O_1, using the expansion for $\phi(x)$.

$$O_1 = -ig \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \int d^4x\, \rho(x) \left(e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} a^\dagger_{\vec{k}} \right)$$

$$= -ig \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(\underbrace{\tilde{\rho}(-k)}_{\tilde{\rho}(k)^*} a^\dagger_{\vec{k}} + \tilde{\rho}(k) a_{\vec{k}} \right)$$

(Using the Fourier transform convention (this convention will be adhered to, see Nov. 6)

$$\tilde{f}(k) = \int d^4x\, e^{ik\cdot x} f(x) \qquad f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(k)$$

also in three space dimensions

$$\tilde{f}(\vec{k}) = \int d^4x\, e^{-i\vec{k}\cdot\vec{x}} f(\vec{x}) \qquad f(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{f}(\vec{k})$$

So as not to carry around so many factors, define

$$f(\vec{k}) \equiv \frac{-ig}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \tilde{\rho}(\vec{k}, \omega_{\vec{k}})$$

then

$$U_I(\infty, -\infty) = e^{\frac{1}{2}(\alpha + i\beta)} e^{\int d^3k\, f(\vec{k}) a^\dagger_{\vec{k}}} e^{-\int d^3k\, f(\vec{k})^* a_{\vec{k}}}$$

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Now that we have solved the model we can answer the usual questions you ask about when a field is driven by an external source.

Given that you start with nothing in the far past, $|0\rangle$,

What is the probability of finding n mesons in the far future?

The state in the far future is

$$U_I(\infty,-\infty)|0\rangle = e^{\frac{1}{2}(\alpha+i\beta)} e^{\int d^3k\, f(\vec{k})\, a^\dagger_{\vec{k}}} |0\rangle$$

$$= e^{\frac{1}{2}(\alpha+i\beta)} \sum_{n=0}^{\infty} \frac{1}{n!} \int d^3k_1 \cdots d^3k_n f(\vec{k}_1)\cdots f(\vec{k}_n) |\vec{k}_1,\ldots,\vec{k}_n\rangle$$

The probability, P_n, of finding n mesons is thus

$$P_n = \left| e^{\frac{1}{2}(\alpha+i\beta)} \frac{1}{n!} \int d^3k_1 \cdots d^3k_n f(\vec{k}_1)\cdots f(\vec{k}_n) |k_1,\ldots,k_n\rangle \right|^2$$

$$= e^{\alpha} \frac{1}{(n!)^2} \int d^3k_1 \cdots d^3k_n \, |f(\vec{k}_1)|^2 \cdots |f(\vec{k}_n)|^2 \, n!$$

$$= e^{\alpha} \frac{1}{n!} \left(\int d^3k_1 \, |f(\vec{k}_1)|^2 \right)^n$$

Now is where we demand unitarity of U_I to get α.

$$1 \overset{!}{=} \sum_n P_n = e^{\alpha} \sum_n \frac{1}{n!} \left(\int d^3k_1 |f(\vec{k}_1)|^2 \right)^n = e^{\alpha} e^{\int d^3k_1 |f(\vec{k}_1)|^2}$$

$$\alpha = -\int d^3k \, |f(\vec{k})|^2$$

So $P_n = e^{-|\alpha|} \frac{|\alpha|^n}{n!}$ Poisson distribution.

This state, created by a classical source, is called a coherent state. Coherent states of the harmonic oscillator are

$$|\lambda\rangle \equiv e^{\lambda a^\dagger}|0\rangle$$

They are special because they diagonalize a

$$a|\lambda\rangle = a e^{\lambda a^\dagger}|0\rangle = [a, e^{\lambda a^\dagger}]|0\rangle = \lambda e^{\lambda a^\dagger}|0\rangle = \lambda|\lambda\rangle$$

$\langle\lambda|x(t)|\lambda\rangle$ and $\langle\lambda|p(t)|\lambda\rangle$ oscillate sinusoidally like the classical variables.

The coherent states we have constructed are eigenvectors of $\phi^{(+)}(x)$ with eigenvalue

$$\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} e^{-ik\cdot x} f(\vec{k})$$

Except for the $\frac{1}{n!}$ this state's n particle part is just the product of n 1 particle states. It is about as uncorrelated as a state of mesons can be. If you remove a particle with $\phi^{(+)}(x)$, you get the same state back. Expectations of normal ordered products factorize.

What is the average number of mesons created?

$$\langle N\rangle = \sum_{n=0}^{\infty} n P_n = \sum_{n=1}^{\infty} \frac{e^{-|\alpha|}|\alpha|^n}{(n-1)!} = |\alpha| \quad \text{(pull out an } |\alpha| \text{ and reindex the sum)}$$

What is the average energy of the final state, i.e. the total energy of all the mesons created?

$$\langle H\rangle = \sum_{n=0}^{\infty} \frac{e^{-|\alpha|}}{(n!)^2} \int d^3k_1 \cdots d^3k_n \, |f(\vec{k}_1)|^2 \cdots |f(k_n)|^2 \underbrace{(\omega_{\vec{k}_1} + \cdots + \omega_{\vec{k}_n})}_{n\,\omega_{\vec{k}_1}} n!$$

$$= \sum_{n=1}^{\infty} \frac{e^{-|\alpha|}}{(n-1)!} |\alpha|^{n-1} \int d^3k \, |f(\vec{k})|^2 \omega_{\vec{k}} = \int d^3k \, |f(\vec{k})|^2 \omega_{\vec{k}}$$

Average momentum? $\langle \vec{P}\rangle = \int d^3k \, |f(\vec{k})|^2 \vec{k}$

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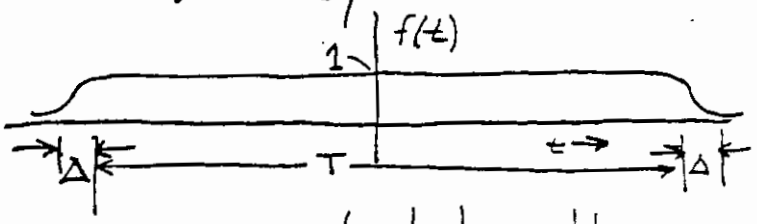
Model 2 solved (beginning)

Combinatorically, model 2 is identical to model 1, but physically the content is different. The interaction doesn't actually turn off in the far past/future. We put that in by hand.

$$\mathcal{H}_I = g\phi(x)\rho(\vec{x})f(t)$$



Assumming the theory has a ground state, the vacuum-to-vacuum scattering matrix element ought to be easy to calculate. ex nihil nihil. You start out with nothing you end up with nothing.

If you calculate $\langle 0|S|0\rangle$ however, you will not get one.

Let

$|0\rangle_p =$ ground state of the whole Hamiltonian, with energy E_0

$|0\rangle =$ ground state of H_0 as usual.

Let's look at the scattering process in the Schrodinger picture.

for $t < -\frac{T}{2}$ we have $|0\rangle$. At $t \approx -\frac{T}{2}$, in time Δ, the interaction turns on adiabatically. The adiabatic hypothesis says that $|0\rangle$ turns into $|0\rangle_p$ with probability 1. It can pick up a phase, $e^{-i\gamma_-}$. from $t=-\frac{T}{2}$ to $t=+\frac{T}{2}$ the state evolves with the full Hamiltonian, it rotates as e^{-iE_0t} and picks up a total phase e^{-iE_0T}. At $t \approx \frac{T}{2}$, as the interaction turns off adiabatically, $|0\rangle_p$ turns back into $|0\rangle$, getting one more phase $e^{-i\gamma_+}$. The state we have for $t > \frac{T}{2}$ is

$$e^{-i(\gamma_- + \gamma_+ + E_0T)}|0\rangle$$

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We can transfer this to the interaction picture, to get

$$\langle 0| U_I(\infty,-\infty)|0\rangle = e^{-i(\gamma_+ + \gamma_- + \mathcal{E}_0 T)}$$

this is disgusting. A divergent phase. How will we get rid of it?

Well change the the theory. The problem is that there is a mismatch between the ground state energy of the full Hamiltonian and the ground state energy of the free Hamiltonian. Subtract the mismatch and well eliminate the problem.

$$H_I \longrightarrow \left[g\int d^3x\, \phi(\vec{x},t)\rho(\vec{x}) - a\right] f(t) \qquad a = \mathcal{E}_0$$

Its obvious what will happen. The number, a just exponentiates while the interaction is on

$$\langle 0|S|0\rangle = e^{-i\left[(\gamma_+ + \gamma_- + \mathcal{E}_0 T) - aT(1+O(\frac{1}{T}))\right]} \doteq 1$$

take $\quad a = \mathcal{E}_0 + O\left(\frac{1}{T}\right)$

This is the first example of what is called a counterterm. It counters a problem we ran into in scattering theory. It doesn't change the physics, but it fixes up the problem.

You might worry that there will be energy mismatches in the one or many particle states even after we get the energy mismatch in the ground states fixed up. There shouldn't be though. Because the physical states get far away from the potential at large times, we expect the energy difference between a state with one physical particle and the physical vacuum to be the same as the energy difference between the bare particle and the bare vacuum. If the vacuum energies are lined up, the one particle state energies should be lined up. We don't expect this to be true in model 3. The particles interact with themselves and they can never get away from that as a particle can get away from an external potential.

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Model 2 solved (conclusion)

$$H_I = f(t)\left[g\int d^3x\, \rho(\vec{x})\,\phi(\vec{x},t) - a\right]$$

a is the vacuum energy counterterm chosen so that

$$\langle 0|S|0\rangle = 1$$

We have already argued that as $T \to \infty$, $a \to E_0$, where E_0 is the vacuum energy of the interacting theory (without the counterterm a). Finding a is going to give us E_0. Now it is clear what the addition of a constant to the Hamiltonian does to $U_I(\infty, -\infty)$. The constant just exponentiates. Let's see this come out of our diagrammatic perturbation theory.

There are now three connected diagrams

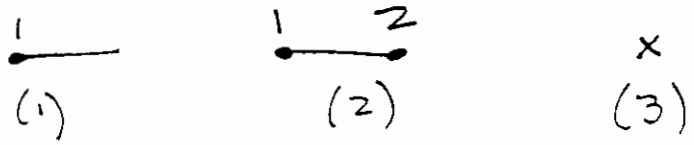


Diagram (3), which has no lines coming out, is for the counterterm.

$$S = U_I(\infty, -\infty) = :e^{(1)+(2)+(3)}: = e^{(2)+(3)}:e^{(1)}:$$

(Since (2) and (3) are just numbers). To set $\langle 0|S|0\rangle = 1$ is to set

$$e^{(2)+(3)} = 1 \quad \text{i.e.} \quad (3) = -(2).$$

Since a is an addition to the Hamiltonian, it had better be purely real. Diagram (3) is then pure imaginary, and diagram (2) in order to be cancellable had better come out pure imaginary. It didn't come out pure imaginary in Model (1), but there the source was time dependent!

Photons don't scatter off nailed down charges. OCT. 23
Mesons don't scatter off nailed down nucleons. 2
They only scatter off real nucleons (or off nailed down nucleons if there is some dynamical charged field in the theory).

$$(1) = -ig \int f(t)\rho(x)\phi(x)\, d^3x\, dt$$

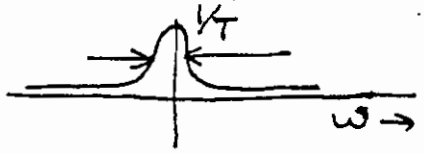
and

$$\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left(e^{-ik\cdot x} a_{\vec{k}} + e^{ik\cdot x} a^{\dagger}_{\vec{k}} \right) \quad \text{so}$$

$$(1) = -ig \left(\frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}} \left(\tilde{f}(\omega_{\vec{k}}) a^{\dagger}_{\vec{k}} \tilde{\rho}(k) + \underbrace{\tilde{f}(-\omega_{\vec{k}})^*}_{\tilde{f}(\omega_{\vec{k}})} a_{\vec{k}} \underbrace{\tilde{\rho}(-k)}_{\tilde{\rho}(\vec{k})^*} \right) \right)$$

(hermitian conjugates)

$f(t)$, our turning on and off function, has a fourier transform that looks like



As $T \to \infty$ $\tilde{f}(\omega_{\vec{k}})$ goes to zero for every $\omega_{\vec{k}} \neq 0$ and since $\omega_{\vec{k}} \geq \mu$ $\tilde{f}(\omega_{\vec{k}}) \to 0$ for all $\vec{k}$. That is $(1) \to 0$ as $T \to \infty$. (as long as we can set $(3) = -(2)$) we have found

$$\boxed{S = 1}$$

This theory is a complete washout as far as scattering is concerned

While this was easy to see in the formalism we have built up, it was not easy when they were evaluating the theory in the Born approximation. Not until miraculous cancelations of all the terms at 4th order in the Born series ~~record~~ did people realize that they should try to prove $S=1$ to all orders.

Why is $S=1$? A time independent source can impart no energy. Since it can only create mesons one at a time and since $\omega = 0$ is not on the mass shell, it cannot create mesons.
This result holds in the massless theory too. Since there is clearly no scattering for all $\vec{k} \neq 0$, you only have to prove that for wave packets centered about $\vec{k}=0$ the failure at $k=0$, a set of measure 0, does not screw up the wave packet.

OCT. 23 3

Ground State Energy, Ground State Wavefunction

In most QM courses these are discussed in a model long before scattering. You usually use time independent perturbation theory. I'll show you how to get these quantities out of the time dependent perturbation theory we have already developed.

Why is the ground state energy interesting? We have been studying the response of the meson field to a classical source. In meson-"nucleon" theory, the source will be $\psi^*\psi$. Our classical source theory is a lot like a meson-"nucleon" theory with the "nucleons" nailed down. Take

$$\rho = "\delta"(\vec{x}-\vec{y}_1) + "\delta"(\vec{x}-\vec{y}_2)$$

The quotes are around the δ functions because we might want to smear them out a little bit. This is the charge density of two nucleons at $\vec{y}_1$ and $\vec{y}_2$. By computing the ground state energy and then by varying the positions we can find the potential between two "nucleons".

This is the same thing we do in QM. We calculate the interaction between the two protons due to their interaction with the electron in H_2^+ by considering how the ground state energy of the electron varies with the separation of the protons. The protons are nailed down in that calculation, usually you say that the protons are so much heavier than the electrons and move so slowly that we can treat the response of the electron field to changes in positions of the protons as if the changes take place adiabatically. Of course in that calculation we also have a Coulomb potential between the protons. Here we are trying to get at the whole internuclear potential by saying it all comes from the interaction with the meson field. Of course the Coulomb potential V really comes from the interaction with the photons... in QM

Now to calculate $a = E_0$ by setting $(3) = -(2)$:

$$(3) = -i \int dt\, f(t)(-a) = iaT\left(1 + O\left(\frac{\Delta}{T}\right)\right) = iE_0 T\left(1 + O\left(\frac{\Delta}{T}\right)\right)$$

$$(2) = \frac{(-iq)^2}{2!} \int d^4x_1 d^4x_2\, f(t_1) f(t_2)\, \rho(\vec{x}_1)\rho(\vec{x}_2)\, \overbrace{\phi(x_1)\phi(x_2)}$$

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\epsilon} e^{ik\cdot(x_1 - x_2)}$$

$$= \frac{-iq^2}{2} \int \frac{d^3k}{(2\pi)^3} |\tilde{\rho}(\vec{k})|^2 \int \frac{dk^0}{2\pi} |\tilde{f}(\omega)|^2 \frac{1}{\omega^2 - \vec{k}^2 - \mu^2 + i\epsilon}$$

Now $|\tilde{f}(k^0)|^2$ is sharply concentrated at $k^0 = 0$ we can replace ω in $\frac{1}{\omega^2 - \vec{k}^2 - \mu^2 + i\epsilon}$ by 0 (and then the $i\epsilon$ is not needed any longer)

Also $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\tilde{f}(\omega)|^2 = \underbrace{\int_{-\infty}^{\infty} dt\, |f(t)|^2}_{T + O(\Delta)} = T\left(1 + O\left(\frac{\Delta}{T}\right)\right)$

famous theorem, Parseval's theorem

We could sum up these properties by saying something sloppy like

$$\lim_{T\to\infty} |\tilde{f}(\omega)|^2 = 2\pi T \delta(\omega)$$

but what I have just shown is all (no more, no less) than that sloppy statement means.

$$(2) = \frac{iq^2}{2} T\left(1 + O\left(\frac{\Delta}{T}\right)\right) \int \frac{d^3k}{(2\pi)^3} |\hat{\rho}(\vec{k})|^2 \frac{1}{|\vec{k}|^2 + \mu^2}$$

The moment of truth: Set $(3) = -(2)$, the T's and i's cancel.

$$E_0 \underset{T\to\infty}{=} \frac{-q^2}{2} \int \frac{d^3k}{(2\pi)^3} |\hat{\rho}(\vec{k})|^2 \frac{1}{\vec{k}^2 + \mu^2}$$

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The potential has come out in ~~position~~ momentum space. To convert it to position space,

Define $$V(\vec{x}) = -g^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2+\mu^2}$$

Then $$E_0 = \frac{1}{2} \int d^3x d^3y \; \rho(\vec{x})\rho(\vec{y}) \; V(\vec{x}-\vec{y})$$

The $\frac{1}{2}$ is the usual factor found even in electrostatics from overcounting the interaction when integrating over all space. For the two nucleon charge density, there will be four contributions. Two will be the interaction of the nucleons with themselves and two will be their interaction with each other (cancelled by the $\frac{1}{2}$).

$$\rho(\vec{x}) = \text{"}\delta\text{"}(\vec{x}-\vec{y}_1) + \text{"}\delta\text{"}(\vec{x}-\vec{y}_2)$$

$$E_0 = \text{something independent of } \vec{y}_1, \vec{y}_2 \; + \; V(\vec{y}_1 - \vec{y}_2)$$

If "δ" $\to \delta$ this part $\to -\infty$. Same problem as the self energy of a charged sphere in E.D.

The usual procedure for the integration of spherically symmetric fourier transforms, followed by a contour integration gives

$$V(r) = \frac{-g^2}{4\pi r} e^{-\mu r} \qquad r = |\vec{y}_1 - \vec{y}_2|$$

Yukawa potential

Looks like the coulomb potential for $r \ll \mu^{-1}$ the compton wavelength of the meson, and falls off rapidly for $r \gg \mu^{-1}$.

The force is attractive (between like charges) (because the particle mediating it has even integer spin) and short ranged because the mediating particle is massive. This potential has some of the essential features of the real nuclear force. Of course it doesn't include the effect of the whole family of mesons in the real world or of multi-meson processes, but with $\mu = m_\pi$ it is a start.

M_W is much larger and the weak force is thus much shorter-ranged.

The ground state wave function, is of course not a position space wave function (the expansion of $|\psi\rangle$ into $|x\rangle$'s), it is an expansion of $|0\rangle_p$ into the basis states $|\vec{k}_1, \ldots, \vec{k}_n\rangle$ of the noninteracting theory.

To get the ground state wave function of model 2 using time dependent perturbation theory, we'll use the results of model 1.

Consider $\rho(\vec{x}, t) = \rho(\vec{x}) e^{\epsilon t} \quad t<0 \quad \epsilon \to 0+$

$\rho(\vec{x}, t) = 0 \quad t>0$

That is we turn it on very slowly, arbitrarily slowly so that at $t=0$ we finally have the full interaction of model 2, then we turn the interaction off abruptly.

Consider the S matrix in this theory

$$\langle \vec{k}_1, \ldots, k_n | U_I(\infty, -\infty) | 0 \rangle$$

$$= \langle \vec{k}_1, \ldots, \vec{k}_n | U_I(\infty, 0) U_I(0, -\infty) | 0 \rangle$$

Since the interaction is turned on arbitrarily slowly

$U_I(0, -\infty)|0\rangle$ should turn the bare vacuum into $|0\rangle_p$

screw the phase factor

$U_I(\infty, 0)$, the evolution by the free Hamiltonian alone, which is in the interaction picture, does nothing on the left

so

$$\langle \vec{k}_1, \ldots, k_n | U_I(\infty, -\infty) | 0 \rangle = \langle \vec{k}_1, \ldots, \vec{k}_n | 0 \rangle_p$$

which is what we are after.

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Now we can apply the results of model 1 (Oct. 21, p. 11)

$$\langle \vec{k}_1, \dots, \vec{k}_n | 0 \rangle_p = \langle \vec{k}_1, \dots, \vec{k}_n | U_I(\infty, -\infty) | 0 \rangle$$

$$= e^{-\frac{|\alpha|}{2}} e^{\frac{i\beta}{2}} f(\vec{k}_1) \cdots f(\vec{k}_n)$$

$$f(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (-ig) \tilde{\rho}(\vec{k}, \omega_{\vec{k}}) \qquad |\alpha| = \int d^3k \, |f(\vec{k})|^2$$

$$\tilde{\rho}(k) = \int d^4x \, e^{ik\cdot x} \rho(x) = \int d^3x \, e^{-i\vec{k}\cdot\vec{x}} \rho(\vec{x}) \int_{-\infty}^{0} dt \, e^{ik^0 t} e^{\epsilon t}$$

$$\underset{\epsilon > 0}{=} \tilde{\rho}(\vec{k}) \frac{1}{ik^0 + \epsilon} \xrightarrow[\epsilon \to 0]{} -\frac{i}{k^0} \tilde{\rho}(\vec{k})$$

The probability for having n mesons is

$$P_n = e^{-|\alpha|} \frac{|\alpha|^n}{n!}$$

What is P_n for a point charge at the origin? That is

$$\rho(\vec{x}) \to \delta(\vec{x})$$ (not "δ" smeared, the limit of a real point charge)

well,

$$\tilde{\rho}(\vec{k}) \to 1 \quad \text{and}$$

this is bad news: at high k we have a UV divergence in the integral for $|\alpha|$.

$$|\alpha| = \int d^3k \, |f(\vec{k})|^2 \underset{\text{high } k}{\sim} \int d^3k \, \frac{1}{\omega_{\vec{k}}^3} \underset{\text{high } k}{\sim} \int \frac{k^2 dk}{k^3}$$

(↙ Oct. 21, p. 12)

The integral is log divergent. Since $\langle N \rangle = |\alpha|$ we see that not only the energy of the field becomes infinite in the limit of a point nucleon, but the ground state flees Fock space.

These infinities are scary but not harmful.

Physically observable quantities like the S matrix and the internucleon potential are heartening sensible.

Even if we don't go to the limit "$\delta \to \delta$", but instead take the limit of massless mesons $\mu \to 0$, $\langle N \rangle = |\alpha| \to \infty$, this time because of a small k divergence. What about that?

Answer:

So what if there are an infinite number of mesons in the ground state. An experimentalist will tell you he can only measure the existence of a meson down to some low energy, not arbitrarily low. If you say there are an in 1,000,000 mesons with a wavelength between ½ light year and 1 light year, so what. It might be a problem if there were an infinite amount of energy at small $\vec{k}$, but there isn't.

$$\langle H \rangle = \int d^3k \, |f(\vec{k})|^2 \omega_{\vec{k}}$$

This is manifestly positive. Seems to contradict Yukawa pot. result.

And the extra factor of $\omega_{\vec{k}}$ moderates the IR divergence. The integral is finite even as $\mu \to 0$. Same with

$$\langle \vec{P} \rangle = \int d^3k \, |f(\vec{k})|^2 \vec{k}$$

So it seems we have been lucky. In this simple theory, the divergences have restricted themselves to unobservable quantities. Maybe the divergences will break this quarantine in more complicated theories. In fact there is a surprisingly wide class of theories in which the divergences don't break the quarantine. They are called "renormalizable" theories.

Next: Mass renormalization.

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Model 3 and Mass Renormalization

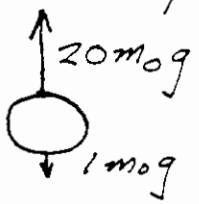
The ground state energy, in perturbation theory, of this system is not necessarily zero. We will need a vacuum energy counterterm to make it zero. In this theory, that is not enough to make the 1-particle states of the Hamiltonian equal in energy to the 1-particle states of the full Hamiltonian. Indeed, the energy of a "static nucleon" depended on it's interaction with the meson field. Not only did the vacuum energy of a state with one static nucleon depend on g, the coupling, it depended on how smeared the nucleon was.

The change in energy of a particle due to its interaction with another field is called "mass renormalization." The cure for this disease is also called "mass renormalization."

Mass renormalization goes all the way back to hydrodynamics.

Suppose I have a ping pong ball, with the mass equal to 1/20 of the water it displaces.

$$m_0 = \frac{1}{20}\rho V$$



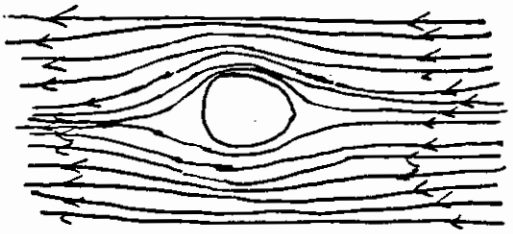
Elementary hydrostatics tells you that there is an upward force on the ping pong ball equal to g times the mass of the water it displaces. There is also the downward force of gravity on the ping pong ball itself. The net force on the ball is thus $19m_0g$ upward. Putting this in Newton's equation we have

$$m_0 a = 19 m_0 g \qquad a = 19g$$

The ball accelerates upwards at $19g$.

This is nonsense, as anyone who has ever held a ping pong ball underwater knows. The ping pong ball may accelerate up fast, but not at 19g. The answer is not friction. You can see that the ping pong ball is not accelerating with 19g even when its velocity is low and friction or viscosity is negligible.

The answer is that in order to move the ping pong ball, you have to move some fluid. In order to accelerate the ping pong ball, you have to accelerate some fluid. Stokes solved the fluid motion around a sphere.



Flow lines in rest frame of sphere, moving with velocity $\vec{V}$ through fluid.

If a ball that displaces volume V is moving with velocity $\vec{V}$ through the fluid, the fluid flow has a momentum, in the same direction as the ball, of $\frac{1}{2}\rho V\vec{V}$. The total momentum of the system, ball and fluid, is thus

$$\tfrac{1}{2}\rho V\vec{V} + m_0\vec{V} = 11\, m_0\vec{V}$$

Which we set equal to the force after taking $\frac{d}{dt}$

$$\frac{dp}{dt} = 11 m_0 a = 19 m_0 g = F$$

$$a = \frac{19}{11} g$$

See the derivation in Landau and Lifshitz, Fluid Mechanics, leading up to the problem on p.36, for a more detailed understanding of the problem.

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More motivation for mass renormalization:

Two limits of classical field theory:

Point Particle limit

Mass renormalization occurs

Example: G.R

Point particle of mass m creates a gravitational field which itself has energy density and creates further gravitational field.

Classical Field limit

Can read dispersion relation for low amplitude plane waves off of the quadratic part of the Lagrangian

Since the quantum field theory will probably exhibit all the behavior of the worst/classical limit, we better be prepared for mass renormalization

Another example of renormalized perturbation theory.

You could try, in the statistical-mechanical theory of critical phenomena to calculate the critical temperature, as well as other properties of the system in terms of the microphysical parameters. However you may be able to do computations much more easily if you trade in one of the microphysical parameters for the critical temperature.

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The classical theory of the electron also suffers mass renormalization. Imagine the electron as a charged shell. The bare mass is m_0, the charge e, the radius r. There is a contribution to the measured mass of the electron other than m_0. There is the electrostatic energy (divided by c^2).

$$m = m_0 + \frac{e^2}{2rc^2}$$

↑ measured, physical mass

Model 3 is going to suffer mass renormalization. The energy of a single meson state or a single nucleon state is going to depend on its interaction. We looked at static "nucleons" interacting with the meson field in model 2. Recall that even for a single nucleon nailed down at $\vec{y}$

$$\rho(\vec{x}) = \text{"}\delta\text{"}(\vec{x}-\vec{y})$$

the energy of the system depends in detail on how we smear out the δ function. In fact if we don't smear it out at all

$$\rho(\vec{x}) = \delta(\vec{x}-\vec{y})$$

the energy of the meson ground state $\to -\infty$. Now the energy of a one nucleon state includes the change in energy of the meson field its presence causes, and although the features of this effect may change when the coupling to the ϕ field goes from $\rho(\vec{x})\phi(x)$ (nailed down nucleons) to $\psi^*\psi\phi(x)$ (dynamical nucleons) there is no reason to expect it to go away!

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This is going to be bad news for scattering theory. Just as the failure to match up the ground state energy for the noninteracting and full Hamiltonians in model 2 produced f, T and Δ dependent phases in $\langle 0|S|0\rangle$, the failure to match up one particle state energies in model 3 will yield T and Δ dependent phases in $\langle \vec{k}|S|\vec{k}'\rangle$. Worse than that, it can even cause two wave packets that were arranged to collide, not to collide. I'll show how:

In our relativistic interacting theory, for sufficiently weak coupling, we expect that there will be one nucleon states, $|\vec{p}\rangle_p$, which are eigenstates of the full Hamiltonian, H:

$$H|\vec{p}\rangle_p = \sqrt{\vec{p}^2+m^2}\,|\vec{p}\rangle_p$$

There are also eigenstates of the free Hamiltonian, which because of this mass renormalization mess, can have a different mass. If I prepare a packet of these free Hamiltonian eigenstates they propogate along with group velocity

$$\vec{V} = \frac{\partial E}{\partial \vec{p}} \left(= \frac{\vec{p}}{E} \quad \text{when } E^2 = \vec{p}^2+m^2\right)$$

When I turn the interaction on (slowly, so that the free Hamiltonian eigenstate (bare nucleon) turns into the dressed nucleon (full Hamiltonian eigenstate)) the group velocity changes because the mass in $E=\sqrt{\vec{p}^2+m^2}$ changes. I could set up a nucleon and meson to scatter, and if I turn on the interaction too early or too late, they might not even come close!

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To fix up this problem we are going to introduce new counterterms into our theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 + \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi$$

$$+ f(t)\left[-g\,\psi^*\psi\phi + a + \frac{b}{2}\phi^2 + c\,\psi^*\psi\right]$$

- a: vacuum energy density counterterm
- $\frac{b}{2}\phi^2$: meson mass counterterm
- $c\,\psi^*\psi$: "nucleon" mass counterterm

μ is the measured mass of the meson.
m is the measured mass of the nucleon.

When the interaction is off ($f(t)=0$) this theory is a free theory with mesons of mass μ and nucleons of mass m. ✓

When the interaction is turned on ($f(t)=1$) we arrange, by adjusting b and c, that the one meson state has mass μ and the one nucleon state has mass m. This eliminates the phases in the one particle matrix elements, by matching the energy of the one particle states (the vacuum energy, which is an infinite volume system may be infinite, being proportional to the volume is adjusted to zero with the help of a).

To summarize, the conditions determining a, b, and c are

$$\langle 0|S|0\rangle = 1 \Longrightarrow a$$

one meson states:
$$\langle \vec{q}|S|\vec{q}\,'\rangle = \delta^{(3)}(\vec{q}-\vec{q}\,') \Longrightarrow b$$

one nucleon states:
$$\langle \vec{p}|S|\vec{p}\,'\rangle = \delta^{(3)}(\vec{p}-\vec{p}\,') \Longrightarrow c$$

The one meson and one nucleon states shouldn't do anything; they have got nothing to scatter on.

This procedure should match up the energies of states of widely separated nucleons and mesons, without any additional twiddling. The energy of two widely separated mesons, even in model 3 when they are affected by self-interaction, should be the sum of their respective energies. Matching the energies of the vacuum and the one particle states matches the energy of states of widely separated mesons too. Although the particles in model 3 never become separated from their own fields in the far past/future, they always become widely separated from each other.

Sometimes it is useful to think about

$$\mu_0^2 \equiv \mu^2 - b \quad \text{and} \quad m_0^2 \equiv m^2 - c$$

the coefficients of $\frac{1}{2}\phi^2$ and $\psi^*\psi$ in the full Lagrangian, respectively, although they have very little physical significance. What our procedure amounts to is breaking up the free and interacting parts of the Hamiltonian in a less naive way. You are always free to break up the free and interacting parts of the Hamiltonian any way you like, although you won't get anywhere unless you can solve the free Hamiltonian.

We have put $\frac{b}{2}\phi^2$ and $c\psi^*\psi$ in with the interaction $-g\psi^*\psi\phi$ because that way the mass of the meson (nucleon) is μ (m) when the interaction is off (manifestly) and the mass of the meson (nucleon) is μ (m) when the interaction is on (by our careful choice of b and c).

This procedure gives us a BONUS.

By making b and c (hence μ_0^2 and m_0^2) quantities you compute, our perturbation theory is expressed in terms of the actual physical masses, not the dumb quantities, μ_0 and m_0.

If you treated m_0 and μ_0 as fundamental, you would calculate all your cross sections, bound state energy levels, all quantities of interest, in terms of them, and then to make contact with reality, you would have to calculate μ and m, the physical masses, in terms of m_0 and μ_0 too. Since no one is interested in your μ_0 and m_0, to present your results, you would have to express all your cross sections in terms of μ and m.

We have bypassed that mess by turning perturbation theory on its head. Instead of a perturbation theory for m^2, μ^2 and all other physical quantities in terms of m_0^2 and μ_0^2, we have a perturbation theory for m_0^2 and μ_0^2 and all physical quantities in terms of the observed masses, m and μ.

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FEYNMAN DIAGRAMS in Model 3

$\langle 0|(S-1)|0\rangle = \langle \vec{p}\,|(S-1)|\vec{p}\,'\rangle = \langle \vec{x}\,|(S-1)|\vec{c}\,'\rangle$
one-nucleon, one-meson

Let's look at nucleon-nucleon scattering at $O(g^2)$. That is the first order at which there is a contribution to

$\langle p_1' p_2' | (S-1) | p_1 p_2 \rangle$ ← two-nucleon states

The -1 in $S-1$ is there because we aren't real interested in the no scattering process, which comes from the 1, the $O(g^0)$ term in

$$S = T\, e^{-ig\int d^4x \left(\psi^*\psi\phi - \frac{b}{2g}\phi^2 - \frac{c}{g}\psi^*\psi\right)}$$

(Since the power series for b and c begin at order g^2 at the earliest (they are zero if $g=0$), it is not misleading to pull the g out in front of the whole interaction and talk about the $O(g^0)$ contribution to S.)

There are no arrows over p_1, p_2, p_1' and p_2' because we are going to use the states that transform nicely under Lorentz transformations

$$U(\Lambda)\,|p_1, p_2\rangle = |\Lambda p_1, \Lambda p_2\rangle$$

$$|p_1, p_2\rangle = b^+(p_1)\,b^+(p_2)\,|0\rangle$$

$$b^+(p) = (2\pi)^{3/2}\sqrt{2\omega_{\vec{p}}}\; b_{\vec{p}}$$

So we don't have to worry about Bose statistics forward $p_1 \neq p_2$ and $p_1' \neq p_2'$. We can recover what we've lost by building wave packets concentrated around $p_1' = p_2$.

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The term in S with two factors of the interaction is

$$\frac{(-ig)^2}{2!}\int d^4x_1\, d^4x_2\, T\left[\psi^*\psi\phi(x_1)\,\psi^*\psi\phi(x_2)\right]$$

After all the hoopla about the turning on and off function, we are abandoning it, being sloppy. The only term in the Wick expansion of this term in S that can contribute to two nucleons goes to two nucleons is

$$\frac{(-ig)^2}{2!}\int d^4x_1\, d^4x_2 : \psi^*\psi\,\overbrace{\phi(x_1)\,\psi^*\psi}\,\phi(x_2) :$$

$\overbrace{\phi(x_1)\,\phi(x_2)}$ is some number. Let's look at

$$\langle p_1' p_2' | : \psi^*(x_1)\psi(x_1)\,\psi^*(x_2)\psi(x_2) : | p_1 p_2\rangle$$

The nucleon annihilation terms in $\psi(x_1)$ and $\psi(x_2)$ have to be used to annihilate the two incoming nucleons. The nucleon creation terms in $\psi^*(x_1)$ and $\psi^*(x_2)$ have to be used to create two nucleons, so as not to get zero inner product (alternatively I could say they have to be used to "annihilate" two nucleons on the left"). In equations

$$\langle p_1' p_2' | : \psi^*\psi(x_1)\,\psi^*\psi(x_2) : | p_1 p_2\rangle$$

$$= \langle p_1' p_2' | \psi^*(x_1)\psi^*(x_2) | 0\rangle\langle 0 | \psi(x_1)\psi(x_2) | p_1 p_2\rangle$$

You can easily show that the two contributions to the second matrix element are (the c.c. equation is also used)

$$\langle 0 | \psi(x_1)\psi(x_2) | p_1 p_2\rangle = e^{-ip_1\cdot x_1 - ip_2\cdot x_2} + e^{-ip_1\cdot x_2 - ip_2\cdot x_1}$$

(p_1 absorbed at x_1)

So there are four contributions to our matrix element

$$\langle p_1' p_2' | :\psi^*\psi(x_1)\psi^*\psi(x_2): | p_1 p_2\rangle$$

$$= \left(e^{ip_1'\cdot x_1 + ip_2'\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1}\right) \cdot \left(e^{-ip_1\cdot x_1 - ip_2\cdot x_2} + e^{-ip_1\cdot x_2 - ip_2\cdot x_1}\right)$$

$$= e^{ip_1'\cdot x_1 + ip_2'\cdot x_2 - ip_1\cdot x_1 - ip_2\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1 - ip_1\cdot x_2 - ip_2\cdot x_1}$$
$$+ e^{ip_1'\cdot x_2 + ip_2'\cdot x_1 - ip_1\cdot x_1 - ip_2\cdot x_2} + e^{ip_1'\cdot x_1 + ip_2'\cdot x_2 - ip_1\cdot x_2 - ip_2\cdot x_1}$$

I (pair of) O

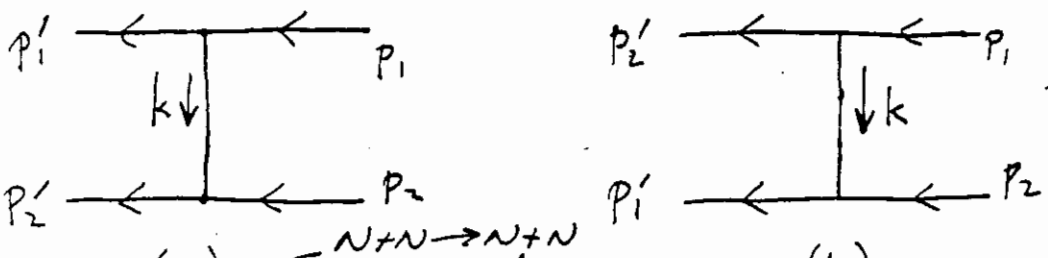
Notice that the terms on the first line differ only by $x_1 \leftrightarrow x_2$ and that the (pair of) terms on the second line only differ by $x_1 \leftrightarrow x_2$. Since x_1 and x_2 are to be integrated over and since $\overline{\phi(x_1)\phi(x_2)}$ is symmetric under $x_1 \leftrightarrow x_2$ these pairs give identical contributions to the matrix element. We will just write one of (each of) the pairs, which cancels the $\frac{1}{2!}$. We have,

$$(-ig)^2 \int d^4x_1 d^4x_2 \; \overline{\phi(x_1)\phi(x_2)} \cdot \left(e^{ip_1'\cdot x_1 + ip_2'\cdot x_2 - ip_1\cdot x_1 - ip_2\cdot x_2} + e^{ip_1'\cdot x_2 + ip_2'\cdot x_1 - ip_1\cdot x_1 - ip_2\cdot x_2}\right)$$

$$= (-ig)^2 \int d^4x_1 d^4x_2 \int \frac{d^4k}{(2\pi)^4} \left[e^{ix_1\cdot(p_1'-p_1+k)} e^{ix_2\cdot(p_2'-p_2-k)} + e^{ix_1\cdot(p_2'-p_1+k)} e^{ix_2\cdot(p_1'-p_2-k)} \right] \cdot \frac{i}{k^2-\mu^2+i\epsilon}$$

I have used the expression for $\overline{\phi(x_1)\phi(x_2)}$ and grouped all the exponential factors by spacetime point.

With these two integrals (there are two terms in the integrand) go two pictures



$N+N \to N+N$

(a) Feynmann diagrams @ $O(g^2)$ (b)

and two stories. The story that goes with picture (a) is this: a nucleon with momentum p_1 comes in and interacts. Out of the interaction point comes a nucleon with momentum p_1' and a "virtual meson" This "virtual meson" then interacts with a nucleon with momentum p_2 and out of the interaction point comes a nucleon with momentum p_2'. The interaction points x_1 and x_2 can occur anywhere, and so they are integrated over. Furthermore, this "virtual meson" can have any momentum k and this is integrated over, although you can see from the factor $\frac{i}{k^2-\mu^2+i\epsilon}$, "Feynman's propogator" that it likes to be on the meson mass shell, although with $k^0 = \pm\sqrt{k^2+\mu^2}$.

Fairy tales like this helped Feynmann discover and think about quantum electrodynamics. In our formalism, they are little more than fairy tales, but in a formulation of quantum particle mechanics like the path integral formulation, they gain some justification. The words not only match the pictures, they parallel the mathematics.

convention: in on right
out on left

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5

The x_1 and x_2 integrations are easy to do. We get

$$(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\epsilon} \left[(2\pi)^4\delta^{(4)}(p_1'-p_1+k)(2\pi)^4\delta^{(4)}(p_2'-p_2-k) + (2\pi)^4\delta^{(4)}(p_2'-p_1+k)(2\pi)^4\delta^{(4)}(p_1'-p_2-k)\right]$$

Because the interaction spacetime is translationally invariant, after integrating V over all space-time the interaction point we get delta functions which enforce energy-momentum conservation at every vertex.

all the features of this computation generalize to more complicated S matrix element contributions. I'll give a set of rules for writing down these integral expressions for contributions to S matrix elements. First, I'll explain in general why there are no combinatoric factors in model 3 to worry about, no symmetry numbers in the integral expressions.

Take a given operator in the Wick expansion, which has an associated Wick diagram

$$D, \text{ a diagram} \iff \frac{:O(0):}{n(D)!}$$

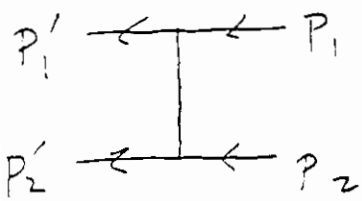
Designate which of the lines leading out of the diagram annihilates each incoming particle, and which of the lines creates each outgoing particle.

This is one contribution to the S matrix element.

Now consider summing over the permutations of the numbered points in the Wick diagram. While only $n(D)!/S(D)$ of these permutations actually correspond to different terms in the Wick expansion, in model 3 all $n(D)!$ of these permutations correspond to different contributions to the S matrix element. This cancels the $\frac{1}{n(D)!}$ exactly.

There are other possible designations (in general) for the way the external lines connect to the vertices of the Wick diagram. If they differ just by a permutation of the vertices then we have already counted them (by cancelling the $1/n(0)!$). If they don't differ by just a permutation of the vertices then they correspond to a different Feynman diagram (the difference between (a) and (b) on page 4).

Only in certain theories, like Model 3, do the $n(0)!$ permutations of the vertices all make different contributions to the S matrix, and in fact this is true in model 3 only when a diagram has at least one external line. In that case there is an unambiguous way of identifying each vertex in the diagram. In fact, each connected part has to have at least one external line. Contributions to $\langle 0|S|0\rangle$, which have no external lines, can have symmetry factors. The unambiguous labelling statement for diagram (a) is



The upper vertex is uniquely labelled as the one where p_1 is absorbed.
The lower vertex is the one connected to the upper vertex by a meson line.

In this theory, as soon as one vertex is labelled (by an external line) they are all uniquely labelled.

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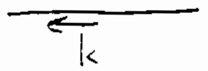
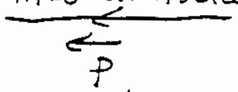
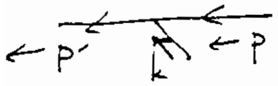
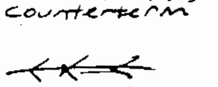
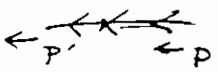
Feynman Rules for Model 3

You should convince yourself by taking some other low order terms in the Wick expansion of S and looking at some simple matrix elements they contribute to that the following set of rules applied to the diagram always gives you the correct contribution to the S matrix element.

For external lines $\begin{Bmatrix}\text{incoming}\\ \text{outgoing}\end{Bmatrix}$ momenta are directed $\begin{Bmatrix}\text{in}\\ \text{out}\end{Bmatrix}$.

Assign a directed momentum to every internal line.

For every	Write
internal meson line 	$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-\mu^2+i\epsilon}$
internal nucleon line 	$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon}$
vertex 	$(-ig)(2\pi)^4\delta^4(p'-p-k)$
meson vacuum counterterm •	$ia(2\pi)^4\delta^{(4)}(0)$
meson mass counterterm 	$ib(2\pi)^4\delta^{(4)}(k-k')$
nucleon mass counterterm 	$ic(2\pi)^4\delta^{(4)}(p-p')$

$(2\pi)^4\delta^{(4)}(0)$ would turn into the volume of all spacetime if the system were in a box. This c.t. diagram is designed to cancel diagrams with no external lines which you will see also have a factor of $\delta^{(4)}(0)$.

Although the meson mass counterterm had a $\frac{1}{2}$ in the Lagrangian, there is no $\frac{1}{2}$ here because there are two ways to do the contraction.

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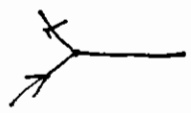
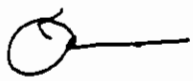
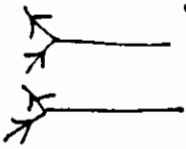
A catalog of all Feynmann diagrams in model 3 up to $O(g^2)$ (except those related by C or T will not be written down twice)

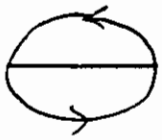
Order g

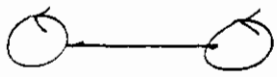
(1)  $= 0$ if $\mu < 2m$ by energy momentum conservation

(2)  $= 0$

Order g^2

(3)  $= 0^2$ $\mu < 2m$

(4) (a) 

(b) 

(c)

Because we demand ~~there should~~ there be no corrections to $\langle 0|S|0\rangle$ these sum to zero. This fixes the vacuum energy c.t. to $O(g^2)$

vacuum energy c.t. to $O(g^2)$,

In 4 (c) think of the c.t. as $O(g^2)$. Its value is determined by the fact that it has to cancel some $O(g^2)$ contributions to the vacuum-to-vacuum S matrix element.

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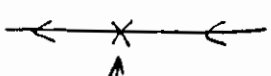
(c)  this diagram comes out zero for the same reason as (2) does

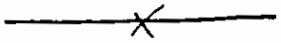
(5) (a) 

(b) 

meson mass c.t. to $O(g^2)$

} Sum to zero because we demand that there are no corrections to $\langle \vec{p} | S | \vec{p}' \rangle$ (one nucleon)

(6) (a) 

(b) 

} sum to zero because we demand that there are no corrections to $\langle \vec{k} | S | \vec{k}' \rangle$ (one meson)

The remaining order g^2 diagrams are more interesting. They contribute to the following processes

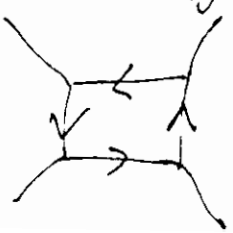
(7) $N + N \longrightarrow N + N$ (connected by C to $\bar{N} + \bar{N} \to \bar{N} + \bar{N}$)

(8) $N + \bar{N} \longrightarrow N + \bar{N}$

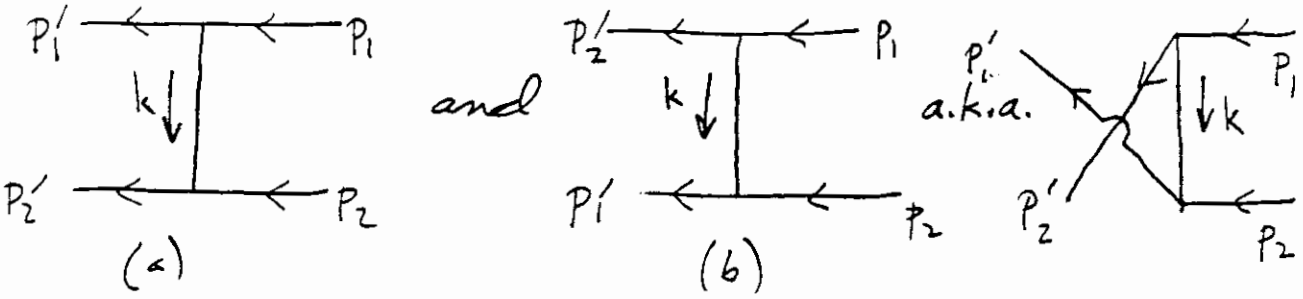
(9) $N + \varphi \longrightarrow N + \varphi$ (connected by C to $\bar{N} + \varphi \to \bar{N} + \varphi$)

(10) $N + \bar{N} \longrightarrow \varphi + \varphi$ (connected by T to $\varphi + \varphi \to N + \bar{N}$)

although $\phi + \phi \to \phi + \phi$ appears in this theory, indeed it must appear, it does not do so until $O(g^4)$. the diagram is



We have already written down the contributions to process (7). The diagrams are



They give (see page 5)

$$(-ig)^2 \frac{i}{(p_1-p_1')^2-\mu^2+i\epsilon} (2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2)$$

$$+ (-ig)^2 \frac{i}{(p_1-p_2')^2-\mu^2+i\epsilon} (2\pi)^4 \delta^{(4)}(p_1'+p_2'-p_1-p_2)$$

Note that the $\frac{1}{(2\pi)^4}$ associated with d^4k exactly cancels with the $(2\pi)^4$ associated with the δ functions used to do the integral. All our formulas have been arranged so that 2π's always go with δ's and $\frac{1}{2\pi}$'s always go with $\int dk$'s.

Note that you can shortcut these trivial integrations over δ functions by first assigning internal momenta so as to conserve momentum, whenever an internal momentum is determined by the other momenta at a vertex.

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finally, note that performing the trivial integrals over δ functions always gives you a factor

$$(2\pi)^4 \delta^{(4)}(P_f - P_i)$$

P_f: sum of all final momenta; P_i: sum of all initial momenta

at least when the diagram is of one connected piece

We define a_{fi}, the invariant Feynman amplitude by

$$\langle f | (S-1) | i \rangle = i a_{fi} \delta^{(4)}(P_f - P_i)$$

The factor of i is inserted to match the phase conventions of NRQM.

For $N+N \to N+N$

$$ia = (-ig)^2 \left[\frac{i}{(P_1 - P_1')^2 - \mu^2 + i\epsilon} + \frac{i}{(P_1 - P_2')^2 - \mu^2 + i\epsilon} \right]$$

Let's look at this in the COM frame.

$$P_1 = \left(\sqrt{p^2+m^2}, p\vec{e} \right)$$
$$P_2 = \left(\sqrt{p^2+m^2}, -p\vec{e} \right)$$
$$P_3 = \left(\sqrt{p^2+m^2}, p\vec{e}' \right)$$
$$P_4 = \left(\sqrt{p^2+m^2}, -p\vec{e}' \right)$$

$\vec{e} \cdot \vec{e}' = \cos\theta$

θ is the scattering angle in the COM frame

$E_T = 2\sqrt{p^2+m^2}$ is often used to characterize collisions. In the nonrelativistic limit, which we will be taking p is more useful.

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Define the momentum transfer, Δ, and the crossed momentum transfer, Δ_c, by

$$(P_1 - P_1')^2 = -\Delta^2$$

$$(P_1 - P_2')^2 = -\Delta_c^2$$

In our COM variables

$$\Delta^2 = 2p^2(1-\cos\theta) \qquad \Delta_c^2 = 2p^2(1+\cos\theta)$$

The invariant Feynmann amplitude is

$$a = g^2\left[\frac{1}{\Delta^2+\mu^2} + \frac{1}{\Delta_c^2+\mu^2}\right]$$

We have dropped the $i\epsilon$ because it is unnecessary. For physically accessible values of Δ^2 and Δ_c^2 the denominators are never less than μ^2.

The first term is peaked in the forward ($\theta \approx 0$) direction. The second term produces an identical peak in the backward direction. (peaked sharper at higher p) Of course when identical particles collide who is to say what is forward and what is backward. $\theta = 0$ is indistinguishable from a scattering angle of $\theta = \pi$. The amplitude had better have come out symmetrical. (probability)

People were scattering nucleons off nucleons long before quantum field theory was around, and at low energies they could describe scattering processes adequately with NRQM. Lets try to understand our amplitude in NRQM. First we'll find the NR analog of the first term.

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In the COM frame, two body scattering is simplified to the problem of scattering of a potential (classically and quantum-mechanically). P.T. at lowest order gives "Born" approximation

$$\langle \vec{k}' | S-1 | \vec{k} \rangle \propto \langle \vec{k}' | V | \vec{k} \rangle$$
$$= \int d^3r \; V(\vec{r}) e^{-i\vec{\Delta}\cdot\vec{r}}$$
$$= \tilde{V}(\vec{\Delta}) \qquad \vec{\Delta} = \vec{k}' - \vec{k}$$

To explain the first term in our scattering amplitude using NRQM we must have

$$\tilde{V}(\vec{\Delta}) \propto \frac{1}{\Delta^2+\mu^2} \Longrightarrow \tilde{V}(\vec{r}) \propto \frac{g^2 e^{-\mu r}}{r}$$

Our amplitude, which is characterized by having a simple pole in a physically unobservable region, at $\Delta^2 = -\mu^2$, corresponds to the Born approx. to the Yukawa interaction!

The second term also has an analog in NRQM. With two identical particles, the Hamiltonian should contain an exchange potential—

$$H = H_0 + V + VE \qquad E|\vec{r}_1, \vec{r}_2\rangle = |\vec{r}_2, \vec{r}_1\rangle$$

E is the exchange operator.

V: Yukawa potential; VE: exchange Yukawa potential

$$V|\vec{r}_1, \vec{r}_2\rangle \propto \frac{g^2 e^{-\mu r}}{r} |\vec{r}_1, \vec{r}_2\rangle \qquad r = |\vec{r}_1 - \vec{r}_2|$$

The exchange Yukawa potential is the source of a simple pole in the amplitude at $\Delta_c^2 = -\mu^2$ also in a physically unobservable region. The NRQM amplitude is proportional to $\tilde{V}(\vec{\Delta}) + \tilde{V}(\vec{\Delta}_{exch})$ $\vec{\Delta}_{exch} = \vec{k}' + \vec{k}$. In a partial wave expansion of the amplitude, the exchange potential gives a contribution {identical / opposite} to the direct potential if l is {even / odd}.

This is because in the COM an eigenstate of angular momentum is an eigenstate of the exchange operator E with eigenvalue $(-1)^\ell$ 1/17

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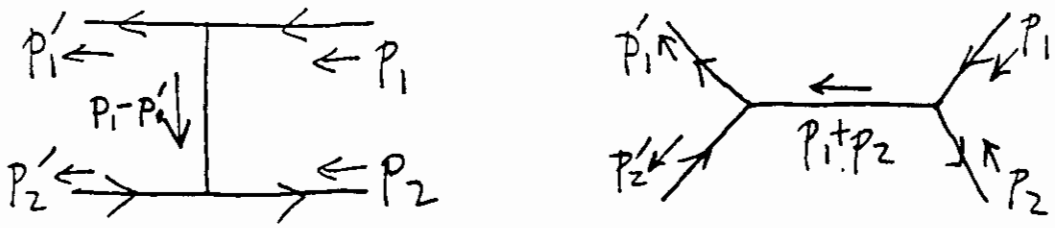
(8) $N + \bar{N} \to N + \bar{N}$ 'nucleon' antinucleon' scattering



Notice that my labeling of internal lines has been done so as to be consistent with energy-momentum conservation. This does away with the two steps of (1) labelling the momenta arbitrarily and (2) performing the trivial integration over the arbitrarily labelled momenta ~~with the~~ that are actually fixed by δ functions. From the diagrams I write down

$$iA = (-ig)^2 \left[\frac{i}{(p_1 - p_1')^2 - \mu^2} + \frac{i}{(p_1 + p_2)^2 - \mu^2} \right]$$

This has a less symmetric structure than the amplitude for $N+N \to N+N$ but that is not unexpected. The symmetry of the amplitude for $N+N \to N+N$ was forced upon us because of identical particles in the incoming and outgoing states. Bose statistics does not apply to the incoming and outgoing states of N and $\bar{N}$.

We've already found what the first term is by going to the COM frame and taking the NR limit. It is a Yukawa potential. What about the second term? In the COM frame

$$(p_1 + p_2)^2 = \left[2\sqrt{p^2 + m^2} \right]^2 = \left[2(p^2 + m^2) \right]^2 = E_T^2$$

(E_T = total energy in the COM frame)

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Now, $\frac{1}{(p_1+p_2)^2-\mu^2} = \frac{1}{E_T^2-\mu^2} = \frac{1}{E_T-\mu}\,\frac{1}{E_T+\mu}$

$\approx \frac{1}{2m+\mu}\,\frac{1}{E_T-\mu}$ in the NR limit

We have not replaced E_T by $2m$ in the second term because $2m$ could be very near μ. This can cause a rapid variation in this factor.
We'll see this is because the intermediate state is a pion.
Notice that this amplitude is independent of the scattering angle, θ. A partial wave decomposition would show a contribution only to the S wave.

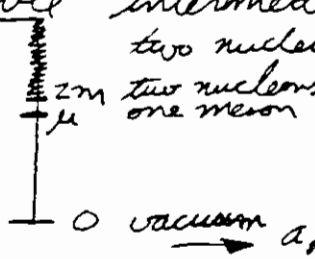
What is the explanation of this in terms of NRQM?

Let us suppose there is an energy eigenstate just below threshold, i.e. an energy eigenstate with an energy slightly less than $2m$. Then even in perturbation theory it may cause a significant contribution to the scattering amplitude in the second term of the Born expansion.

$$a \propto \langle f|V|i\rangle + \sum_n \frac{\langle f|V|n\rangle\langle n|V|i\rangle}{E_T - E_n \pm i\epsilon}$$

← possible source of an energy eigenstate pole.

In the COM frame the energy spectrum of possible intermediate states looks like:



A state with $E_n = \mu$ and non-zero matrix elements could produce a pole in the amplitude. (Continuum states produce a branch cut in the amplitude.) Angular momentum

→ a pole occurs in the partial wave that has the same quantum numbers as the intermediate state.

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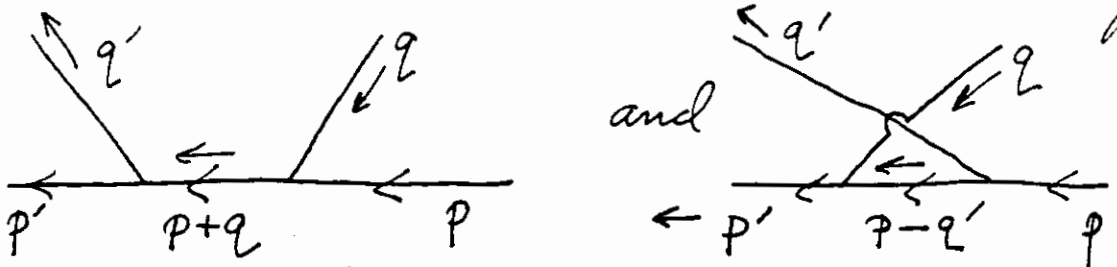
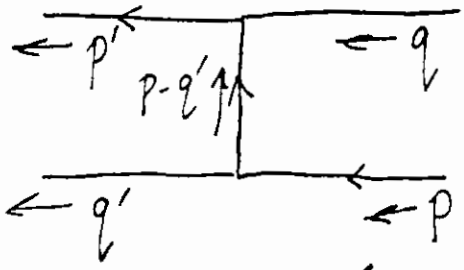
(9) $N + \phi \rightarrow N + \phi$ meson-nucleon scattering 3



The language that goes along with the second graph, the crossed graph, is that the outgoing meson is emitted first, then the incoming meson is absorbed. Of course this is no more kiematically possible for the intermediate nucleon than the first graph.

$$iA = (-ig)^2 \left[\frac{i}{(p+q)^2 - m^2 + i\epsilon} + \frac{i}{(p-q')^2 - m^2 + i\epsilon} \right]$$

This is an energy eigenstate pole and an exchange Yukawa potential. Perhaps it is clear to see that this is an exchange Yukawa potential if we redraw the second graph as



This is classical expectation because the angular momentum barrier is lowest.

Notice there is no direct Yukawa term. There is a resonance in the p wave of pion nucleon scattering called the $\Delta^*(1232)$. There is nothing in the s wave. Usually one thinks of a resonance as caused by an attractive force that nearly creates a bound state. Usually the s state is the most tightly bound. What kind of force could create a p wave state but no s wave state? a repulsive exchange force, which is attractive in odd partial waves

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4

Because the particles have different masses this potential is different from the ones we've had before. The second term in the amplitude

$$-g^2\frac{1}{(p-q')^2-m^2+i\epsilon} \overset{\text{unnecessary}}{=} g^2\frac{1}{-(p-q')^2+m^2}$$

has a denominator in the COM frame of

$$p=(\sqrt{p^2+m^2},\ \vec{p})$$
$$p'=(\sqrt{p^2+m^2},\ \vec{p}')$$
$$q=(\sqrt{p^2+\mu^2},\ -\vec{p})$$
$$q'=(\sqrt{p^2+\mu^2},\ -\vec{p}')$$

$$-(p-q')^2+m^2 = -\left(\sqrt{p^2+m^2}-\sqrt{p^2+\mu^2}\right)^2 + \overbrace{p^2(1+2\cos\theta)}^{\Delta_c^2} + m^2$$

This has the usual $p^2(1+2\cos\theta)$ exchange Yukawa forward peak, but what we would call the range is p dependent.

$$\frac{1}{(\text{range of pot})^2} = -\left(\sqrt{p^2+m^2}-\sqrt{p^2+\mu^2}\right)^2+m^2$$ i.e. energy dependent

✓ The energy dependent part vanishes when $\mu^2=m^2$

as $p\to\infty$ this $\to m^2$

as $p\to 0$ this $\to$

$$-(m-\mu)^2+m^2 = 2m\mu-\mu^2 = \mu(2m-\mu)$$

It can have a long range at low energies if the mass μ is small. If one is bold we can start applying these ideas to real pion nucleon interactions. However we still need to develop spin and isotopic spin to really get things right (the sign of the potential for one) and the pion-nucleon coupling is strong which means lowest order calculations can't be trusted (except at long range or high partial waves).

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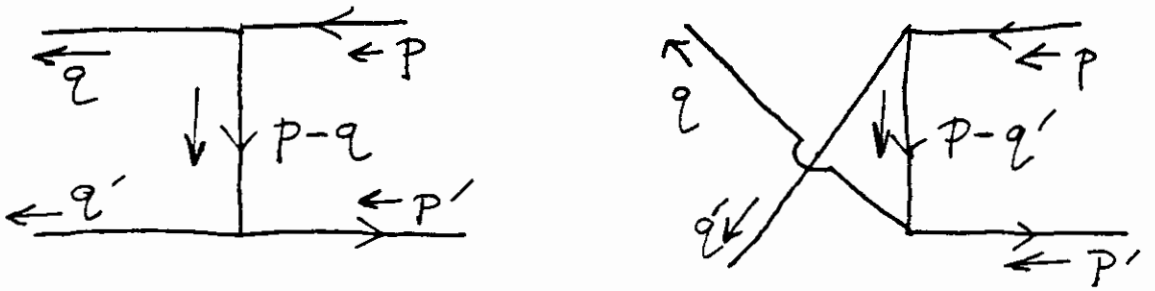
5

(10) $N + \bar{N} \to \phi + \phi$ "nucleon-anti-"nucleon" annihilation. In 1930 this was sensational.



The first graph is a Yukawa potential. The second is is an exchange Yukawa potential.

Our next topic is a discussion of the connection between Yukawa potentials, exchange Yukawa potentials and energy eigenstate poles in relativistic scattering theory.

In NRQM there is absolutely no connection between these things. You can have any one (or two) without having all three.

We'll develop some formalism which will be useful later to describe the connection.

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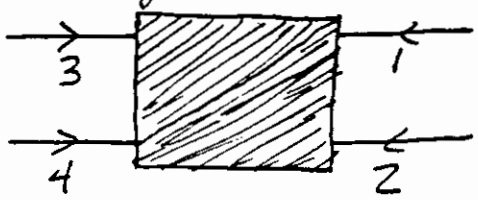
Brackets because this has nothing to do with symmetries and particles in the sense we have discussed them.

CROSSING [SYMMETRY] 6

Imagine a general $2 \to 2$ scattering process, $1+2 \to \bar{3}+\bar{4}$. We'll denote the amplitude (or some contribution to the amplitude) by



where the numbers on the lines tell you what type of particles propogate along those lines (with the arrows), and you aren't supposed to worry about what types of interactions are hidden from view when the lines go behind the shield.

The particle of type 1 is incoming, it's momentum is p_1, the particle of type 2 is also incoming, with momentum p_2, the particle of type $\bar{3}$ is outgoing, with momentum p_3, and the particle of type $\bar{4}$ is outgoing with momentum p_4. This you can tell because of our convention of putting incoming lines on the right and outgoing lines on the left, we read graphs as if time flowed from right to left, in analogy with the way we write down matrix elements.

Now we are going to abandon that convention.

They who is to say this is not the amplitude for $3+4 \to \bar{1}+\bar{2}$, reading left to right, or $3+1 \to \bar{2}+\bar{4}$ reading top to bottom, (or even $1 \to \bar{2}+\bar{3}+\bar{4}$?).

Well, we have another method for fixing a convention, which is useful, (but not necessary) for discussing crossing.

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7

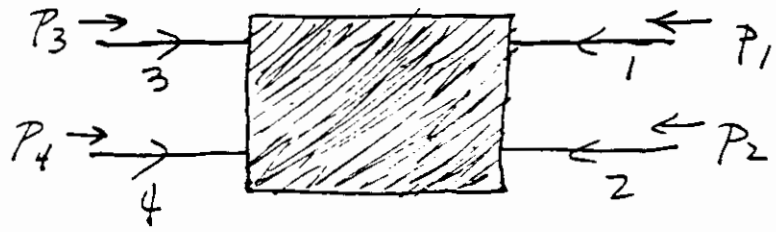
I ignore everything written about of stories talking about negative energy particles.

The honest to goodness physical momenta in the theory always are on the upper sheet of their mass hyperboloids, i.e.

$$p_3^2 = m_3^2 \quad \text{or} \quad p_3^{0\,2} = m_3^2 + |\vec{p}_3|^2 \quad \underline{\text{and}} \quad p_3^0 > 0$$

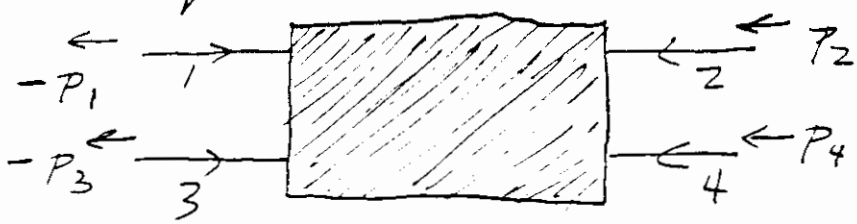
There are no negative energy states in our theories. Thus there will be no confusion if we flag a momentum by sending it to minus itself. If someone gives you a momentum, p, with $p^0 < 0$, you know what they are really giving you is a physical momentum, $-p$, and a wink, a flag, an extra bit of information. We will use that extra bit of information to specify whether a particle is incoming or outgoing.

We will orient all momenta inward on our general $2 \to 2$ graph:



the orientation on the page no longer matters

and if, say, p_1^0 and p_3^0 are less than zero, and p_2^0 and p_4^0 are greater than zero, what this actually stands for is (what we used to mean by)



which is the amplitude for $2+4 \to 1+3$. Note that the notation has been set up, so that in all cases, the energy momentum conserving delta function is $\delta^{(4)}(p_1+p_2+p_3+p_4)$

Mathematically, instead of graphically, what we have defined is a new function of three momenta. ($P_1+P_2+P_3+P_4$ is restricted to be zero. If you like you could think of the function as a function of four momenta, which is zero whenever $P_1+P_2+P_3+P_4$ is not equal to zero.) Just to keep an air of symmetry we'll display all four momenta in the function.

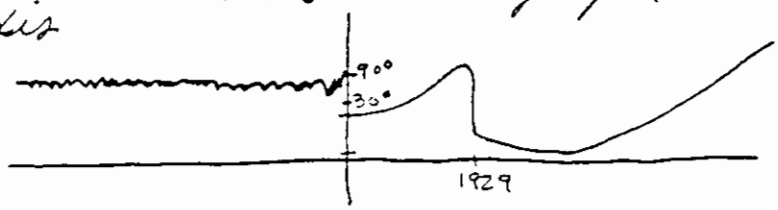
$$iA(P_1, P_2, P_3, P_4)$$

But don't think of A_{fi} should be thought of as a function of parameters that parametrize the surface $P_1^2=P_2^2=P_3^2=P_4^2=m^2$, $P_1+P_2+P_3+P_4=0$. 3 independent momenta ⇒ 6 components invariants 4 constraints ⇒ 2 parameters

This function is the amplitude for a particle of type 2 with momentum P_2, and a particle of type 4 with momentum P_4, and scatters into a particle of type 1 with momentum $-P_1$, and a particle of type $\bar{3}$ with momentum $-P_3$, when P_2° and P_4° are >0 and P_1° and P_3° are <0. It is also the amplitude for a bunch of other processes when the time components of the three independent momenta take on their various possible signs. Another way of writing the amplitude for $2+4 \to 1+\bar{3}$ using this function is to take all the momenta, P_1, P_2, P_3 and P_4 to be their honest to goodness physical values ($P_1^\circ, P_2^\circ, P_3^\circ$ and $P_4^\circ > 0$) and write

$$iA(-P_1, P_2, -P_3, P_4)$$

There is no reason we can't assemble the amplitudes for all these different processes into a single process like this, but there is also no obvious reason it is any more useful than graphing the Dow Jones on the positive axis and the temperature in Miami on the same graph on the negative real axis



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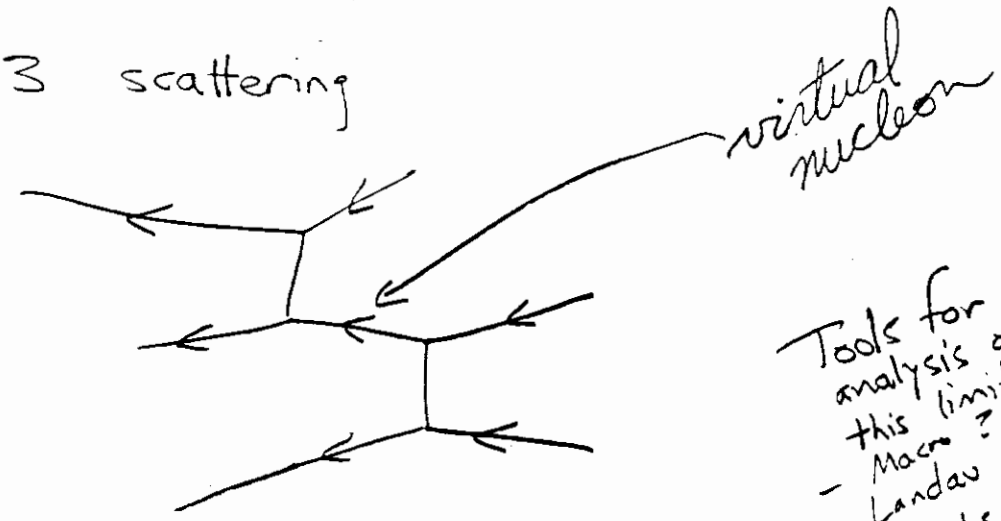
FOOD FOR THOUGHT:

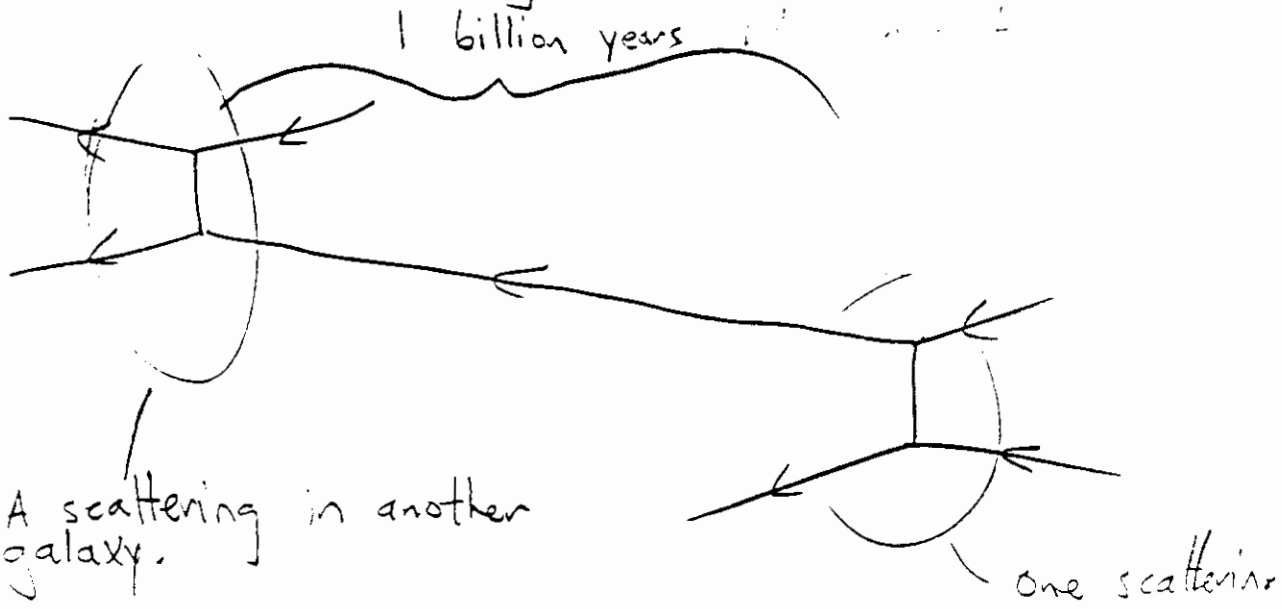
$3 \to 3$ scattering



Tools for analysis of this limit
- Macro ? causality?
- Landau rules
- Graphs with single poles

Also think of this as

$2 \to 2$ scattering followed 1 billion light years later by another $2 \to 2$ scattering.



There must be some appropriate limit where $2 \to 2$ followed by $2 \to 2$ is a limit of $3 \to 3$

The virtual internal line must somehow become almost real.

That's why you can get a vague description of virtual particles by thinking about them as real particles.

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Lets define three relativistic invariants to describe $2 \to 2$ scattering processes.

$$s \equiv (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t \equiv (p_1+p_3)^2 = (p_2+p_4)^2$$
$$u \equiv (p_1+p_4)^2 = (p_2+p_3)^2$$

If particle 3 is outgoing, $-p_3$ is its actual 4-momentum.

For the process $1+2 \to \bar{3}+\bar{4}$, $\sqrt{s}$ is the total COM energy, $-t$ is the momentum transfer squared, and $-u$ is the crossed momentum transfer squared. I have made these last two choices arbitrarily. If $1 \neq 2$ and $\bar{3} \neq 4$ and if $1 = \bar{3}$ or $2 = \bar{4}$, the choice is standard. If $1 \neq 2$ and $\bar{3} \neq 4$ and if $1 = \bar{4}$ or $2 = \bar{3}$, the choice is bassackwards; I ought to call $-u$ the momentum transfer^2 and $-t$ the crossed momentum transfer^2. In all other cases, anybody's designation is arbitrary.

Now there are only two relativistic invariants describing a $2 \to 2$ scattering process of spinless particles. They are often taken as the COM total energy and scattering angle. s, t and u are three relativistic invariants. They must be redundant. Here is a (nice symmetric) derivation of their interdependence.

$$2(s+t+u) = (p_1+p_2)^2+(p_3+p_4)^2+(p_1+p_3)^2+(p_2+p_4)^2+(p_1+p_4)^2+(p_2+p_3)^2$$
$$= 3\sum_{a=1}^{4} m_a^2 + 2\sum_{a>b} p_a \cdot p_b$$

Now use $0 = (\sum_a p_a)^2 = \sum_a m_a^2 + 2\sum_{a>b} p_a \cdot p_b$ to see

$$2(s+t+u) = 2\sum_{a=1}^{4} m_a^2 \quad \text{i.e.} \quad s+t+u = \sum_a m_a^2$$

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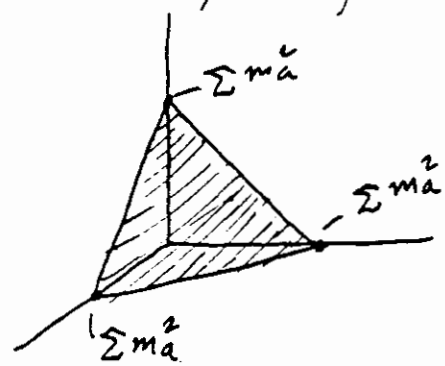
There is a symmetrical way of graphing three variables in the plane, when they are restricted like this. Look at the plane in s-t-u space, $s+t+u = \sum_a m_a^2$



Looking down perpendicularly at this plane, you get the idea of representing s, t and u in the plane by

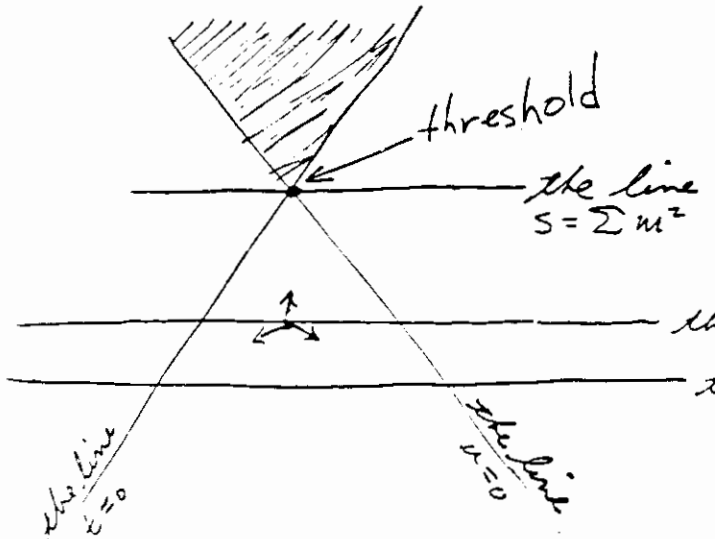
$$s = \vec{r}\cdot\hat{e}_s + \frac{1}{3}\sum_a m_a^2$$

$$t = \vec{r}\cdot\hat{e}_t + \frac{1}{3}\sum_a m_a^2$$

$$u = \vec{r}\cdot\hat{e}_u + \frac{1}{3}\sum_a m_a^2$$

unit vectors at 120° angles

Each vector $\vec{r}$ in the plane gives you a triple s, t, u, and since $\hat{e}_s+\hat{e}_t+\hat{e}_u$ is obviously 0 (rotational invariance) the set satisfies $s+t+u = \sum_a m_a^2$ we have a "Mandelstam-Kibble plot"



When all four masses are equal, $m_a^2 = m^2$, $a = 1, 2, 3, 4$, the shaded area, $s > 4m^2$, $u < 0$, $t < 0$ is the physically accessible region for the process $1+2 \to \bar{3}+\bar{4}$ and the process $3+4 \to \bar{1}+\bar{2}$

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Uniqueness Property of the Twofold Vacuum Expectation Value*

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(Received June 14, 1960)

It is shown under general assumptions that if the one-body Green's function equals its free-field value the theory is that of a free field.

We would like to point out a simple and remarkable generalization of previously known theorems[1,2] regarding the vacuum expectation values of relativistic field theory operators.[3] Namely, if the vacuum expectation value of the simple two-point function coincides with that for a free field at equal times, then all vacuum expectation values coincide with the corresponding free-field values.[4] The proof is essentially elementary. Consider the states,

$$\Psi=\int d^4x[(-\Box^2+m^2)f(x)]\phi(x)|0\rangle \tag{1}$$

where $f(x)$ is an arbitrary space-time function with reasonable properties such that surface terms vanish upon integration by parts. Then Ψ has zero norm since

$$(-\Box^2+m^2)\langle 0|\phi(x)\phi(y)|0\rangle=0. \tag{2}$$

(By assumption, $\langle 0|\phi\phi|0\rangle$ coincides with the free-field function at equal times, and therefore, by analyticity, at arbitrary times.) Consequently $\Psi=0$ if the metric is positive definite. Define

$$j(x)=(-\Box^2+m^2)\phi(x). \tag{3}$$

We see that from $\Psi=0$ if follows that

$$j(x)|0\rangle=0. \tag{4}$$

Accordingly, it is true that

$$\langle 0|j(x)\phi(y_1)\cdots\phi(y_n)|0\rangle=0. \tag{5}$$

It follows that

$$\langle 0|\phi(y_1)\cdots j(x)\cdots\phi(y_n)|0\rangle=0 \tag{6}$$

since both matrix elements are boundary values of the same analytic function of complex variables.[5]

We see immediately that:

(1) The operators satisfy the free-field equations.

(2) Canonical commutation relations imply all the matrix elements are equivalent to the free field ones.

(3) The weaker assumption of the asymptotic condition also yields the same result. (This follows directly from the Yang-Feldman equation.[3])

Although the result is therefore more general, we will assume the canonical commutation rules for simplicity.

It is clear that the same result follows with the assumption that any single vacuum expectation value with an even number of operators coincides with the corresponding free field expectation value. For by the same method we can then construct a null state and hence show that with a single $\phi(x)$ replaced by $j(x)$ we get a zero operator. It will then follow that all higher vacuum expectation values coincide with the free field functions. Finally, by reduction using commutators at equal times, all matrix elements must equal the free ones.

* This work is supported in part by funds provided by the U. S. Atomic Energy Commission, the Office of Naval Research and the Air Force Office of Scientific Research.

[1] R. Haag, Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. 29, No. 12 (1955).

[2] O. W. Greenberg, Phys. Rev. 115, 706 (1959).

[3] We have been informed by R. Jost that the same result has been obtained independently by both R. Jost and B. Schroer.

[4] D. W. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. 31, No. 5 (1957).

[5] D. Feldman and C. N. Yang, Phys. Rev. 79, 972 (1950).

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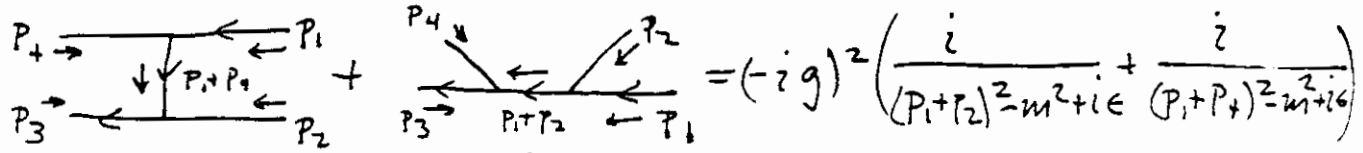
In an abuse of the scattering term "channel" the process $1+2 \to \bar{3}+\bar{4}$ is called the s-channel, and the crossed process $1+3 \to \bar{2}+\bar{4}$ and $1+4 \to \bar{2}+\bar{3}$ are called the t channel and u channel respectively, because $\sqrt{t}$ and $\sqrt{u}$ are the total COM energy in these processes. In model 3, the lowest order scattering amplitude for the process $N+\phi \to N+\phi$ was (using our new wacky conventions, $p_3^0 < 0$; $p_4^0 < 0$).



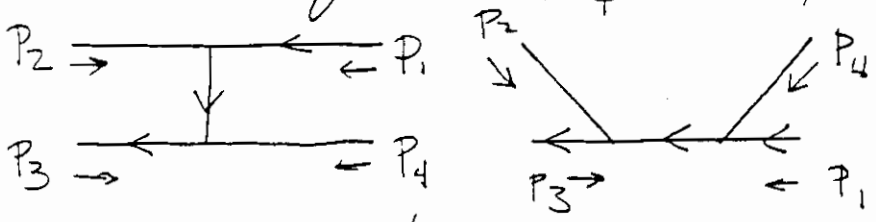
$$= (-ig)^2 \left(\frac{i}{(p_1+p_2)^2 - m^2 + i\epsilon} + \frac{i}{(p_1+p_4)^2 - m^2 + i\epsilon} \right)$$

$$= (-ig)^2 \left(\frac{i}{s - m^2 + i\epsilon} + \frac{i}{u - m^2 + i\epsilon} \right)$$

energy eigenstate pole — exchange Yukawa interaction

The lowest order scattering amplitude for the corresponding u channel process (I'm thinking of $1=N$ $2=\phi$ $3=N$ $4=\phi$) is

$N + \phi \to N + \phi$



$$= (-ig)^2 \left(\frac{i}{s - m^2 + i\epsilon} + \frac{i}{u - m^2 + i\epsilon} \right)$$

exchange Yukawa — energy eigenstate pole

How about that: the amplitudes are the same although the interpretations of the two terms are different.

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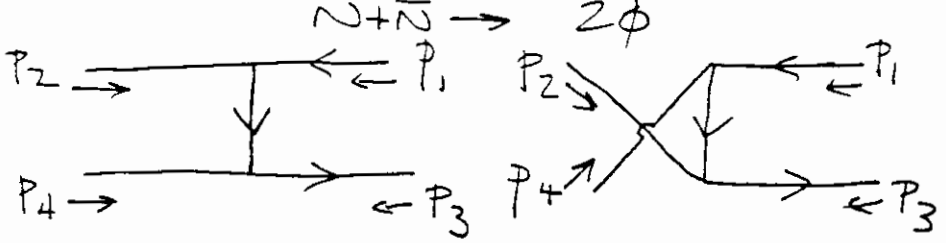
What about the t channel process

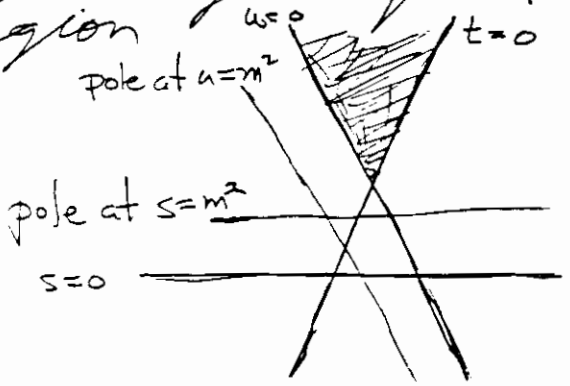
$N+\bar{N} \to 2\phi$



$$= (-ig)^2 \left(\frac{i}{s-m^2+i\epsilon} + \frac{i}{u-m^2+i\epsilon} \right)$$

Again the amplitude is the same although the interpretation is different. (I'd rather not assign a NR interpretation to the two graphs because nonrelativistically $N+\bar{N} \to 2\phi$ can't occur.)

Every one of these amplitudes is the exact same function of s, t and u. That is: the first amplitude is only defined in the shaded region (for simplicity take $\mu = m$)



where s, the COM energy squared, is greater than $4m^2$ and t and u are less than zero.

The second amplitude is only defined when $u > 4m^2$ and s and t are less than zero. What we have observed is that if we analytically continue an amplitude for some process of its physical region to the physical region of some other process we get the amplitude for that other process.

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From our picture of the s-t-u plane it may look to you like the analytic continuation can't be performed even for the simple $O(g^2)$ amplitudes we have discussed because the poles in s and u show up as lines which cut off one physical region from another. This is wrong because you can go around these poles by letting the variables become complex. Furthermore they are avoidable singularities, that is it doesn't matter how you go around them, you get the same analytic continuation. This brings up a tougher question: at this order in perturbation theory our amplitudes just have poles, but at higher orders they will have branch cuts, so can the analytic continuation from one physical region to another still be performed and if so, do you get the correct amplitude? The answer is yes and you do get the amplitude for the physical process by analytically continuing the amplitude for another, but you must follow specific prescriptions when going around the essential singularities. More on this March 19 [3].

Given this relation between amplitudes for different processes, we have related energy eigenstate poles, Yukawa interactions and exchange Yukawa interactions, three things which had no connection in nonrelativistic quantum mechanics. These effects are one in the same. A pole in s in an s channel process looks like an energy eigenstate pole. In the u channel process that same pole looks like an exchange Yukawa potential

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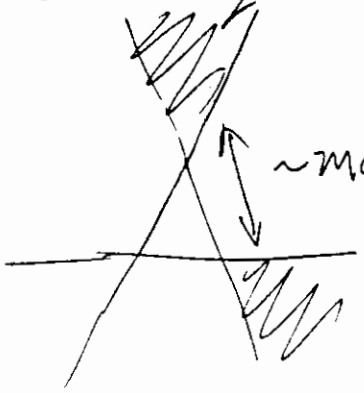
They are two aspects of the same analytic function restricted to two disconnected regions of the plane. The next thing to ask is how do we lose the relationship when we take the non relativistic limit, $c \to \infty$. As $c \to \infty$ the three physical regions on the Mandelstam plot which are separated by a distance of $O(mc^2)$ get very far apart. They are only finite c near each other for finite c. The other thing we do in the NR limit is chuck terms of order $(v/c)^2$ which of course is an arbitrarily good approximation in this limit. The problem is that even if you have an excellent approximation to an analytic function if you analytically continue the approximation a long ways you may get something that doesn't remotely resemble the analytic function. An example will suffice. Consider e^x on the real axis for $x < -1$ million. In that region 0 is a wonderful approximation to the function. But now analytically continue this wonderful approximation to $x = +1$ million and you discover you have completely missed the boat. It is in this way that the connection between different amplitudes is lost when you take the non-relativistic limit and chuck those teensy but important terms of $O(\frac{v}{c})^2$ and higher.



Although this has been illustrated only for $2 \to 2$ scattering it applies to any process.

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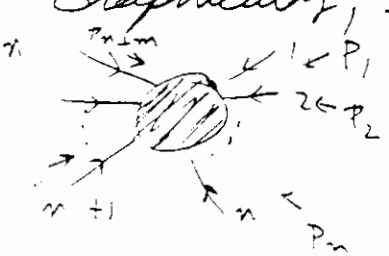
CPT symmetry

There are three scattering processes related to $1+2 \to \bar{3}+\bar{4}$ by various crossings. They are

$$3+4 \to \bar{1}+\bar{2}$$

$$2+\bar{4} \to \bar{1}+3 \quad \text{and} \quad 2+3 \to \bar{1}+\bar{4}$$

Note that the physical region for $3+4 \to \bar{1}+\bar{2}$ is in the same region as that of $1+2 \to \bar{3}+\bar{4}$. There is no need to do any analytic continuation to show that the amplitude for these two processes are the same. All you have to do is note that they are related by $p_a \to -p_a$ $a=1,\dots,4$ and that all the Feynman rules are quadratic in the momenta so there is no way this operation can change the amplitude. Although we haven't discussed theories without parity invariance, if there is any grace in the world, parity violating interactions will involve an ϵ tensor contracted with four momenta and that too is an even power of momenta. This is an argument to all orders in perturbation theory that even in a parity violating theory the amplitude for these two processes are equal. The argument applies to any process n particles $\to m$ particles by the same argument. The Feynman rules are invariant under $p_a \to p_a$ $a=1,\dots,n+m$. This equality has nothing to do with analytic continuation. Graphically, if



is calculated in a Lorentz invariant theory it is invariant under $p_a \to -p_a$ $a=1,\dots,n+m$. This is the CPT theorem.

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Why is it called the CPT theorem? I'll only explain why in the 2→2 case so I don't have to invent some notation. The most general 2→2 process is

$$1+2 \longrightarrow \bar{3}+4$$

p_1 p_2 p_3 p_4 ← physical 4 momenta amplitude = $a(p_1, p_2, -p_3, -p_4)$

If I charge conjugate the incoming and outgoing states I get a related process:

$$\bar{1}+\bar{2} \longrightarrow 3+4$$

In a charge conjugation invariant theory this process would be have the same amplitude but in general it doesn't. Now lets consider the time reversed process. That would be

$$3+4 \longrightarrow \bar{1}+\bar{2}$$

because if you run a movie backwards the products of a reaction become the reagents and the reagents become the products. Furthermore, what once went North now goes south and what once went up now goes down, that is, the velocities are reversed. If we also apply parity we undo the reversal of velocities and the final process is

$$3+4 \longrightarrow \bar{1}+\bar{2}$$

p_3 p_4 p_1 p_2 amplitude = $a(-p_1, -p_2, p_3, p_4)$

↑ physical momenta

whether or not these three operations individually affect the amplitude we have shown (previous page) that the combined effect of all three operations (CPT) can't change the amplitude on general grounds. If CPT violation is ever observed (Lagrangian) quantum field theory is cooked. (CONTRAST) If C violation is observed we must write down C non-invariant interaction.

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PHASE SPACE AND THE S MATRIX 12

Our job is to make contact with the numbers experimenters measure. To do this, we square our S matrix elements and integrate over the possible final states a detector might register, and we get a probability that a counter will advance. Our S matrix elements are proportional to $\delta^{(4)}(P_f - P_i)$. Squaring them is senseless. What went wrong?

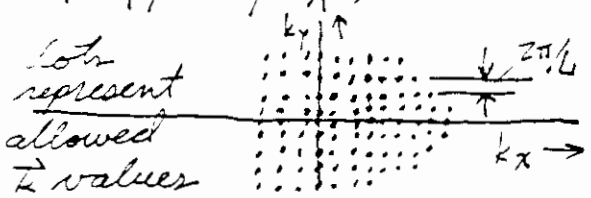
What went wrong is that the states we are using are not normalizable. They extend throughout all of space. The scattering process occurs at every point in space, and since two plane wave states never get far apart no matter how long you wait, the scattering process goes on for all time.

A half-assed way to salvage the situation is to put the system in a box, so that we can normalize the plane-wave states, and to turn the interaction on for a finite amount of time T. A more satisfying way to salvage the situation is to build wave packets, which are normalizable and do get far apart in the far past/future. We are in a hurry so we'll put the world in a box of volume V.

The states in a box of volume V with periodic boundary conditions are

$$|\vec{k}_1, \ldots, \vec{k}_n\rangle \quad \text{where} \quad k_{i\,x,y,z} = \frac{2\pi n_{i\,x,y,z}}{L}$$

$$L^3 = V$$



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The states $|\vec{k}_1, \ldots, \vec{k}_n\rangle$ are built up from the vacuum by creation operators

$$|\vec{k}_1, \ldots, \vec{k}_n\rangle = a^+_{\vec{k}_1} \cdots a^+_{\vec{k}_n} |0\rangle \qquad a_{\vec{k}}|0\rangle = 0$$

$$[a_{\vec{k}}, a^+_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} = \langle \vec{k} | \vec{k}' \rangle \qquad [a^+_{\vec{k}}, a^+_{\vec{k}'}] = 0$$

$$[a_{\vec{k}}, a_{\vec{k}'}] = 0$$

↑ Kronecker delta

The free field in the box has the expansion

$$\phi(x) = \sum_{\vec{k}} \left(\frac{a_{\vec{k}} e^{-ik\cdot x}}{\sqrt{2E_p}\sqrt{V}} + \frac{a^+_{\vec{k}} e^{ik\cdot x}}{\sqrt{2E_p}\sqrt{V}} \right)$$

We want to know, what is the probability of making a transition to some infinitesimal volume of phase space specified by $d^3k_1 \ldots d^3k_n$ (n final particles). It is the probability of going to one of the final states in that infinitesimal region times the number of states in that infinitesimal region,

$$\frac{d^3k_1}{(2\pi)^3} V \cdots \frac{d^3k_n}{(2\pi)^3} V.$$

Let's look at the transition probability to go to one of the final states in that region

$$|\langle f | (S-1) | i \rangle|^2$$

We will restrict our attention to the two simplest and most important initial states: one particle and two particle initial states. We will normalize the two particle initial state unconventionally.

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We will consider

$$|i\rangle = \begin{cases} |\vec{k}\rangle & \text{Decay} \\ \text{or} & \\ |\vec{k}_1, \vec{k}_2\rangle \sqrt{V} & \text{Scattering} \end{cases}$$

$|\vec{k}\rangle$ and $|\vec{k}_1, \vec{k}_2\rangle$ are box normalized. Why the factor $\sqrt{V}$? Without it, each particle has probability 1 of being somewhere in the box. The probability that they are both near a given point and can scatter is $\propto \frac{1}{V^2}$. Of course they could both be near any point in the box so the probability that they will scatter from anywhere is $\propto \frac{1}{V}$. With the factor $\sqrt{V}$ you can think of one particle as having probability 1 of being in any unit volume, and the other as having probability one of being somewhere in the box. With these conventions, we expect

$$\text{In}\langle f|\, S-1\,|i\rangle = i\mathcal{A}_{fi}^{VT}\,(2\pi)^4\,\delta^{(4)}_{VT}(p^i - p^f)$$

$$\times\left(\prod_{\text{final particles}} \frac{1}{\sqrt{2E_f}\sqrt{V}}\right)\left(\prod_{\text{initial particles}} \frac{1}{\sqrt{2E_i}}\right)\frac{1}{\sqrt{V}}$$

where $(2\pi)^4\,\delta^{(4)}_{VT}(p) \equiv \int_V d^3x \int_T dt\, f(t)\, e^{ip\cdot x}$

The extra factors you have never seen before come from the expansion of the fields. The coefficient of the creation operator $a^\dagger_{\vec{k}}$, which annihilates

a particle on the left and the coefficient of the annihilation operator, $a_{\vec{k}}$, which annihilates a particle on the right is

$$\frac{1}{\sqrt{V}\sqrt{2\omega_{\vec{k}}}}$$

You get one of these factors for every particle that is annihilated on the left or right by the fields.

We did not get these before because the coefficient of $a(k)$ and $a(k)^+$ in the field is $\frac{1}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}}$ which exactly cancels the factor we get when $a(k)$ hits the relativistically normalized state $|k'\rangle$

$$a(k)\,|k'\rangle = (2\pi)^3 2\omega_{\vec{k}}\,|0\rangle$$

The product over initial states in our formula has had the single factor of $\frac{1}{\sqrt{V}}$ pulled out in the decay case, and the two factors of $\frac{1}{\sqrt{V}}$ cancelled to just one by the $\sqrt{V}$ we put in by hand in the scattering case, leaving again a single factor of $\frac{1}{\sqrt{V}}$ which we have explicitly put in as the last factor of the formula.

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To get the transition probability, we square the transition amplitude. We multiply this by the the number of states in the infinitesimal region of final state phase space to get the differential transition probability. To get something that does not depend on the time we turn the interaction on for, T, we divide by T. You should get

DIFFERENTIAL TRANSITION PROBABILITY

UNIT TIME

$$= \frac{1}{VT} |a_{fi}^{VT}|^2 \left((2\pi)^4 \delta^{(4)}_{VT}(p^i - p^f) \right)^2$$

$$\times \prod_{\substack{\text{final} \\ \text{particles}}} \frac{d^3k_f}{(2\pi)^3 2E_f} \prod_{\substack{\text{initial} \\ \text{particles} \\ (1 \text{ or } 2)}} \frac{1}{2E_i}$$

Now we take the limits $V, T \to \infty$

$$a_{fi}^{VT} \xrightarrow[VT\to\infty]{} a_{fi} \qquad |a_{fi}^{VT}|^2 \to |a_{fi}|^2$$

$$\delta^{(4)}_{VT} \to \delta^{(4)} \qquad \left((2\pi)^4 \delta^{(4)}_{VT} \right)^2 \to$$ HERE'S WHERE WE HAVE TO BE CAREFUL.

Recall: $\delta^{(4)}_{VT}$ is a function concentrated near the origin. It becomes more and more so as $V, T \to \infty$. Also it is normalized to 1 for all V, T.

$$\int d^4p\, \delta^{(4)}_{VT}(p) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} dt\, f(t) \int_V d^3x \underbrace{\int d^4p\, e^{ip\cdot x}}_{(2\pi)^4 \delta^{(4)}(0)} = 1$$

For these two reasons we say $\lim_{VT\to\infty} \delta^{(4)}_{VT}(p) = \delta^{(4)}(p)$

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What about $\left(\delta^{(4)}_{VT}(P)\right)^2$?

In the limit $V,T\to\infty$, it is concentrated about $P=0$ just as "surely" as $\delta^{(4)}_{VT}(P)$ is. We can find its normalization

$$\int d^4p\left[\delta^{(4)}_{VT}(P)\right]^2 = \frac{1}{(2\pi)^8}\int dt\int dt'\, f(t)f(t')\int_V d^3x\int_V d^3x'\underbrace{\int d^4p\, e^{ip\cdot x}e^{ip\cdot x'}}_{(2\pi)^4\delta^{(4)}(x+x')}$$

The F.T. Power thm can be used to save a couple of steps.

$$= \frac{1}{(2\pi)^4}\int dt\,|f(t)|^2\int_V d^3x = \frac{1}{(2\pi)^4}VT$$

For these two reasons we say

$$\lim_{V,T\to\infty}\frac{1}{VT}(2\pi)^4\left(\delta^{(4)}_{VT}(P)\right)^2 = \delta^{(4)}(P)$$

Thank God a factor of $\frac{1}{VT}$ appears in our formula for the differential transition probability per unit time so we can take the $V,T\to\infty$ limit to get

DIFFERENTIAL TRANSITION PROBABILITY / UNIT TIME

$$= |a_{fi}|^2(2\pi)^4\delta^{(4)}(P_f-P_i)\underbrace{\prod_{\text{final particles}}\frac{d^3k_f}{(2\pi)^3 2E_f}}_{\text{L.I measure on the mass hyperboloid}}\prod_{\text{initial particles 1 or 2}}\frac{1}{2E_i}$$

This factor which is manifestly Lorentz invariant is called the "invariant density of states", D, or the "relativistic density of final states".

Note that you have no excuse for not getting the (2π)'s right. Every 2π goes with a δ function and every $\frac{1}{2\pi}$ goes with a k integration.

NOV. 4 APPLICATIONS OF 1/16

$$\frac{\text{DIFFERENTIAL TRANSITION PROBABILITY}}{\text{UNIT TIME}} = |a_{fi}|^2 D \prod_{\substack{\text{initial} \\ \text{particles} \\ (1 \text{ or } 2)}} \frac{1}{2E_i}$$

$$D = (2\pi)^4 \delta^4(P_f - P_i) \prod_{\text{final particles}} \frac{d^3k_f}{(2\pi)^3 2E_f}$$

(1) DECAY $\quad d\Gamma = \frac{\text{DIFF DECAY PROB}}{\text{UNIT TIME}} = \frac{1}{2E}|a_{fi}|^2 D$

The total decay probability per unit time is $d\Gamma$ summed and integrated over all possible final states.

$$\frac{\text{DECAY PROBABILITY}}{\text{UNIT TIME}} = \frac{1}{2E}\left(\sum_{\substack{\text{final} \\ \text{states}}} \int |a|^2 D\right)$$

~~It is manifestly obvious that~~ (~~manifestly obvious~~ "manifestly obvious" is manifestly redundant) It is obvious that the circled part is L.I.

We'll evaluate the decay probability rate in the rest frame of the decaying particle. This is the "decay width", Γ.

$$\Gamma = \frac{\text{Rest decay probability}}{\text{unit time}} = \frac{1}{2m}\sum \int |a|^2 D$$

Since the $\sum \int |a|^2 D$ is L.I.

$$\frac{\text{DECAY PROBABILITY}}{\text{UNIT TIME}} = \frac{m}{E}\Gamma = \frac{d\tau}{dt}\Gamma$$

where τ is the particle's proper time. The shelf life of a moving particle is longer, its decay rate is slower, exactly by a factor of elapsed proper time / elapsed observer time.

(2) CROSS SECTIONS

These $\vec{v}$'s are initial state particle velocities and energies.

$$d\sigma = \frac{\text{DIFF TRANS PROB}}{\text{UNIT TIME} \times \text{UNIT FLUX}} = \frac{1}{4E_1E_2}|a_{fi}|^2 D \frac{1}{|\vec{v}_1 - \vec{v}_2|}$$

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(2) CROSS SECTIONS (CONT'D)

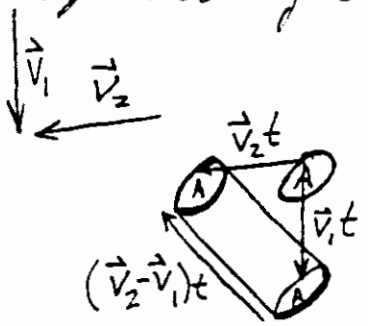
The factor $\frac{1}{|\vec{v}_1 - \vec{v}_2|}$ takes care of the per unit flux. Let's understand this factor with our conventions

OUR CONVENTION $\qquad |i\rangle = \sqrt{v}\,|\vec{k}_1, \vec{k}_2\rangle$

The transition probability per unit time is some mess

$$\frac{\text{t.p.}}{\text{u.t.}} = (\text{some mess})$$

Let's suppose the particle with momentum $\vec{k}_2$ presents some area, A to the particle beam with momentum $\vec{k}_1$. Think of the particles with momentum $\vec{k}_1$ as having probality 1 of being in any volume and the particle with momentum $\vec{k}_2$ as being located somewhere in the whole box with probability one.



$\vec{v}_2 t$ is the vector displacement of the particle with momentum $\vec{k}_2$ in a time t. $\vec{v}_1 t$ is the motion of the beam in time t. The orientation of A is so as to be $\perp$ to $\vec{v}_2 - \vec{v}_1$, that is, so as to catch the most flux. The cylinder is the volume swept out in the beam in a time t. Its volume is

$$|(\vec{v}_2 - \vec{v}_1)|\,t\,A$$

The flux is thus $|\vec{v}_2 - \vec{v}_1|$

and the $\qquad \frac{\text{t.p.}}{\text{u.t.} \times \text{unit flux}} = \frac{(\text{some mess})}{|\vec{v}_2 - \vec{v}_1|}$

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(2) CROSS SECTIONS (CONT'D)

ANOTHER CONVENTION is to take $|i\rangle = |\vec{k}_1, \vec{k}_2\rangle$

Then the transition probability per unit time would have come out as

$$\frac{\text{t.p.}}{\text{u.t.}} = \frac{1}{V}(\text{some mess})$$

↖ this is the same "(some mess)" as on the previous page, whatever it is.

The flux for this normalization is however

$$\frac{1}{V}\,|\vec{v}_1 - \vec{v}_2|$$

So $$\frac{\text{t.p.}}{\text{unit time} \times \text{unit flux}} = \frac{\frac{1}{V}(\text{some mess})}{\frac{1}{V}\,|\vec{v}_1 - \vec{v}_2|}$$ is the same ✓

I want to emphasize that this is the Nonrelativistic velocities and NR velocity addition formula that appears here. If two beams approach with speed c head on the flux is 2c.

The total cross section is

$$\sigma = \frac{1}{|\vec{v}_1 - \vec{v}_2|}\,\frac{1}{4E_1E_2}\sum_{\text{final states}} |a_{fi}|^2\, D$$

If $\vec{v}$ is parallel or antiparallel to $\vec{v}_2$ and the total crossection really has the interpretation of an area, then it should be unaffected by boosts along $\vec{v}_1$ and $\vec{v}_2$

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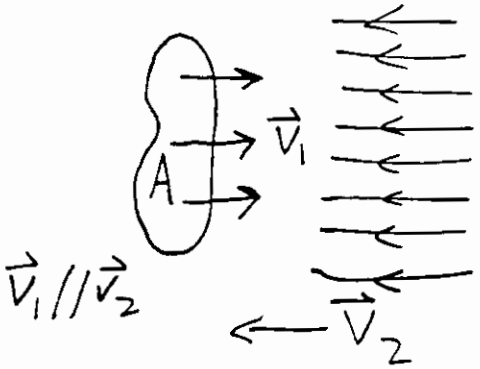
(2) CROSS SECTIONS (CONT'D)



The total number of particles that smash into the area A will depend on the velocity of the observer. It will be proportional to the flux in that frame. However the idea of a perpendicular area should be Lorentz invariant for boosts along the $\vec{V}_1 - \vec{V}_2$ direction. Since

$$\sigma = \frac{1}{|\vec{V}_1 - \vec{V}_2|} \frac{1}{4E_1E_2} \sum |a_{fi}|^2 D$$

is supposed to have the interpretation of an area it should be unaffected by these boosts. The indicated term is invariant under any Lorentz transformation. What about the factors in front. Take $\vec{V}_1$ and $\vec{V}_2$ to be along the x direction.

$$P_1 = (E_1, P_{1x}, 0, 0) \qquad P_2 = (E_2, P_{2x}, 0, 0)$$

Then

$$E_1E_2|\vec{V}_1 - \vec{V}_2| = E_1E_2\left|\frac{P_{1x}}{E_1} - \frac{P_{2x}}{E_2}\right| = |P_{1x}E_2 - P_{2x}E_1|$$

$$= |\epsilon^{23\mu\nu} P_{1\mu} P_{2\nu}|$$ which is obviously invariant under rotations in the 0-1 plane (boosts along $\vec{V}_1$)

This justifies the interpretation of the cross section as an area.

(3) D for a two body final state in the COM frame.

D contains both δ functions and integrals that can be trivially performed by using those δ functions! We can do them once and for all. Of course this turns independent variables (in $|a_{fi}|^2$ and in where they are kinematically) into dependent variables.

In the center of mass frame $\vec{P}^i = 0$ and $E^i = E_T$ while

$$D = \frac{d^3p_1 d^3p_2}{(2\pi)^6 4E_1E_2} (2\pi)^4 \delta^{(3)}(\vec{p}_1+\vec{p}_2)\,\delta(E_1+E_2-E_T)$$

(final particle energies and momenta)

$$= \frac{d^3p_1}{(2\pi)^3 4E_1E_2}\, 2\pi\, \delta(E_1+E_2-E_T)\Big|_{\vec{p}_2=-\vec{p}_1}$$

The $\delta^{(3)}(\vec{p}_1+\vec{p}_2)$ is used to do the $\vec{p}_2$ integration. You must now remember that $\vec{p}_2$ depends on $\vec{p}_1$ wherever it appears (in E_2 or in $|a_{fi}|^2$).

Let's rewrite d^3p_1 as $p_1^2 dp_1\, d\Omega_1$ and use the energy delta function to do the p_1 integration. Thought of as functions of $\vec{p}_1$ and $E_2^2 = \vec{p}_2^{\,2}+m_2^2 = \vec{p}_1^{\,2}+m_2^2$, we have

$$E_1^2 = \vec{p}_1^{\,2}+m_1^2 \qquad E_1\,dE_1 = p_1\,dp_1$$
$$E_2\,dE_2 = p_2\,dp_2$$

$$\frac{\partial(E_1+E_2)}{\partial p_1} = \frac{p_1}{E_1}+\frac{p_1}{E_2} = \frac{p_1 E_T}{E_1E_2},$$

so

$$D = \frac{1}{16\pi^2 E_1E_2}\, d\Omega_1\, \frac{p_1^2}{\left|\frac{\partial(E_1+E_2)}{\partial p_1}\right|} = \frac{1}{16\pi^2}\,\frac{p_1\,d\Omega_1}{E_T}$$

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(3) (a) $2 \to 2$ scattering in the COM frame.

In the COM frame

$$4E_{i1}E_{i2}|\vec{v}_1 - \vec{v}_2| = 4|E_{i2}\vec{p}_{i1} - E_{i1}\vec{p}_{i2}|$$

$$= 4|E_{i2}\vec{p}_{i1} + E_{i1}\vec{p}_{i1}| = 4E_T p_{i1} = 4E_T p_i$$

The i subscript reminds you that these are initial particle momenta and energies

$$d\sigma = \frac{1}{16\pi^2} \frac{p_f \, d\Omega_1}{E_T} \frac{1}{4E_T p_i}$$

$$= \frac{1}{64\pi^2 E_T^2} d\Omega_1 \frac{p_f}{p_i} |a_{fi}|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_T^2} \frac{p_f}{p_i} |a_{fi}|^2$$

Note that for an exothermic reaction we can have $p_i = 0$ while $p_f \neq 0$.

$\frac{d\sigma}{d\Omega}$ and hence σ can be infinite even when the amplitude a_{fi} is finite.

This is why they slow down the neutrons in atomic piles.

It is simple to understand this. As $p_i \to 0$ the amount of time the two particles spend in the danger zone near each other, which goes as $\frac{1}{p_i}$ becomes infinite.

We maximize the chance of neutron capture in the pile by making the neutron cruise out of the pile as slowly as possible.

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3)(b) Contact with elastic $2 \to 2$ scattering in NRQM[7]

Our formula for $\frac{d\sigma}{d\Omega}$ is

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_T^2} \frac{P_f}{P_i} |a_{fi}|^2 = \frac{1}{64\pi^2 E_T^2} |a_{fi}|^2$$

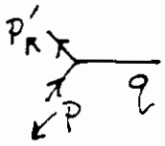
for elastic scattering. Compare this with

$$\frac{d\sigma}{d\Omega} = |f|^2 \quad \text{from NRQM and see}$$

$$|f| = \frac{1}{8\pi E_T} |a_{fi}|$$

Well get the phase when we do the optical thm.

3)(c) Example, Model 3, $\mathcal{L}' = -g\phi^* \psi \phi$ $\quad \not{\tau} = -ig$



$$= (-ig)(2\pi)^4 \delta^{(4)}(p + p' - q)$$

$$iA = -ig + O(g^3)$$ pretty simple, couldn't be simpler

from page 1 and the bottom of page 5,

$$\Gamma = \frac{1}{2\mu} \sum \int |a|^2 D = \frac{1}{2\mu} g^2 \int \frac{P_1 \, d\Omega_1}{16\pi^2 E_T}$$

$E_T = \mu$

$$= \frac{g^2 P_1}{8\pi \mu^2} = \frac{g^2}{8\pi \mu^2} \sqrt{(\tfrac{1}{2}\mu)^2 - m^2}$$

$$= \frac{g^2}{16\pi \mu^2} \sqrt{\mu^2 - 4m^2}$$

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(4) OPTICAL THEOREM

The optical theorem in NRQM is based on a simple idea. There is an incoming wave incident on a target, and an outgoing wave. The outgoing wave is the superposition of the incoming wave that passes right through the target and goes off in the forward direction, and the scattered wave which goes off in all directions. Since there is some probability that a particle in the beam is scattered off, and since probability is conserved, there must be a decrease in the intensity in the beam in the forward direction. The total probability for scattering in all directions but exactly forward, which mathematically is 0, the total cross section, must be equal to the decrease in probability of going exactly in the forward direction, which mathematically is an interference term between the wave that passes right through and the scattered wave in the forward direction.

There is nothing in this argument that is nonrelativistic, so we should be able to get an analog of the optical theorem in our relativistic scattering theory.

The mathematical statement of conservation of probability in the scattering process is

$$SS^+ = 1$$

We want to make a statement about our amplitudes, A_{fi}, which are proportional to matrix elements of $S-1$, so we'll rephrase the conservation of probability as

$$(S-1)(S-1)^+ = \overbrace{SS^+}^{=1} - S - S^+ + 1 = -(S-1) - (S-1)^+$$

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Now,
$$\langle f|(S-1)|i\rangle = i\,a_{fi}\,(2\pi)^4\,\delta^4(P_f-P_i)$$
and
$$\langle f|(S-1)^\dagger|i\rangle = -i\,a_{if}^*\,(2\pi)^4\,\delta^4(P_f-P_i)$$

Our rephrased statement of the conservation of probability has matrix elements of

$$\langle f|(S-1)(S-1)^\dagger|i\rangle = -\langle f|(S-1)|i\rangle - \langle f|(S-1)^\dagger|i\rangle$$

By inserting a complete set of intermediate states, the left hand side (LHS) becomes

$$LHS = \sum_{\substack{\text{intermediate}\\ \text{states } |m\rangle}} \langle f|(S-1)|m\rangle\langle m|(S-1)^\dagger|i\rangle$$

$$= \sum_{\substack{\text{intermediate}\\ \text{states with}\\ n_m \text{ particles}}} \frac{1}{n_m!}\int \frac{d^3k_1}{(2\pi)^3\,2E_1}\cdots\frac{d^3k_{n_m}}{(2\pi)^3\,2E_{n_m}} \times a_{fm}\,a_{im}^*\,(2\pi)^4\delta^4(P_f-P_m)(2\pi)^4\delta^4(P_m-P_i)$$

← an overcounting factor if the n_m particles are identical

Because of the $\delta^4(P_m-P_i)$, we can replace the P_m in $\delta^4(P_f-P_m)$ by P_i, so that we explicitly have that the LHS is proportional to $\delta^4(P_f-P_i)$.

The RHS of the rephrased statement of the conservation of probability is

$$RHS = -i\,a_{fi}\,(2\pi)^4\delta^4(P_f-P_i) + i\,a_{if}^*\,(2\pi)^4\delta^4(P_f-P_i)$$

Both the LHS and RHS are proportional to $(2\pi)^4\,\delta^4(P_f-P_i)$.

Comparing the LHS with the RHS, we have

$$\sum_{\substack{\text{intermediate states with } n_m \text{ particles}}} \frac{1}{n_m!} \int \frac{d^3k_1}{(2\pi)^3 2E_1} \cdots \frac{d^3k_{n_m}}{(2\pi)^3 2E_{n_m}} (2\pi)^4 \delta^4(P_m - P_i) a_{fm} a_{i}^{*}$$

$$= -i a_{fi} + i a_{if}^{*} = 2 \operatorname{Im} a_{fi}$$

The circled factor is what we would call the invariant density of states for the process $i \to m$, D_m (see page 1). If we choose $f = i$ we get

($\frac{1}{n_m!}$ ← overcounting factor if the n_m particles are identical)

$$\sum_{\substack{\text{intermediate states with } n_m \text{ particles}}} \frac{1}{n_m!} \int D_m |a_{im}|^2 = 2 \operatorname{Im} a_{ii}$$

This says the total transition probability per unit time is equal to twice the imaginary part of the forward scattering amplitude.

Statement might be off by a factor of E_T^2

If the process has a two particle initial state, we can rewrite this as a statement about crossections. In the COM frame this says (see the bottom of page 3 and the top of page 6)

$$2 \cancel{4} E_T p_i \sigma = \cancel{2} \operatorname{Im} a_{ii}$$

Since the LHS is zero till $O(q^4)$ for $2N \to 2N$ scattering, we see that the RHS must be zero till $O(q^4)$.

This proves that the forward scattering amplitude for $2N \to 2N$ is real at $O(q^4)$

In NRQM the optical theorem is for elastic scattering, i.e.

$$\frac{p}{4\pi} \sigma = \operatorname{Im} f|_{\theta = 0} \qquad (p_i = p_f = p)$$

Barring a different θ dependence in the phase convention we can finally state

$$f = \frac{1}{8\pi E_T} a$$

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(5) 3 BODY FINAL STATE PHASE SPACE IN THE COM FRAME

$$D = \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\,\delta^{(3)}(\vec{p}_1+\vec{p}_2+\vec{p}_3)\cdot\delta(E_1+E_2+E_3-E_T)$$

$$= \frac{1}{(2\pi)^5}d^3p_1 d^3p_2 \frac{1}{8E_1E_2E_3}\,\delta(E_1+E_2+E_3-E_T)$$

The momentum conserving δ function has been used to eliminate $\vec{p}_3$ i.e. From now on $\vec{p}_3$ and E_3 are not independent variables.

$$\vec{p}_3 = -(\vec{p}_1+\vec{p}_2) \qquad E_3 = \sqrt{(\vec{p}_1+\vec{p}_2)^2+m_3^2}$$

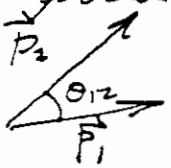
Now we'll rewrite d^3p_1 as $p_1^2\,dp_1\,d\Omega_1$

Instead of writing d^3p_2 as $p_2^2\,dp_2\,d\Omega_2$

Let's rewrite it as $d^3p_2 = p_2^2\,dp_2\,d\Omega_{12}$

$$= p_2^2\,dp_2\,d\varphi_{12}\,d\cos\theta_{12}$$

Where φ_{12} is an azimuthal angle about $\vec{p}_1$ and θ_{12} is a polar angle measured from $\vec{p}_1$!



We are going to use the energy conserving δ function to do the θ_{12} integration. E_3 depends on θ_{12}!

$$E_3^2 = p_1^2 + 2\vec{p}_1\cdot\vec{p}_2 + p_2^2 + m_3^2 = p_1^2+p_2^2+m_3^2+2p_1p_2\cos\theta_{12}$$

Therefore $\frac{\partial E_3}{\partial\cos\theta_{12}} = \frac{p_1p_2}{E_3}$ and thus $d\cos\theta_{12}\,\delta(E_1+E_2+E_3-E_T) = \frac{E_3}{p_1p_2}$

θ_{12} is now a dependent variable.

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$$D = \frac{1}{(2\pi)^5} p_1^2 dp_1 d\Omega_1 \, p_2^2 dp_2 d\varphi_{12} \frac{E_3}{p_1 p_2} \frac{1}{8 E_1 E_2 E_3}$$

$$= \frac{1}{256\pi^5} \underbrace{\frac{p_1 dp_1}{E_1}}_{dE_1} \underbrace{\frac{p_2 dp_2}{E_2}}_{dE_2} d\Omega_1 d\varphi_{12} \quad \text{(VALID IN COM FRAME)}$$

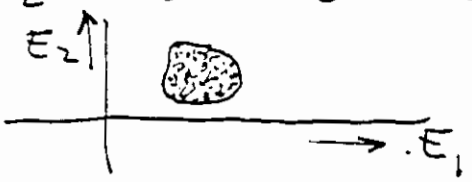
amazing simple result if you use the right variables

Suppose the amplitude a is independent of Ω_1 and φ_{12}, as in the decay of a spinless meson (at rest), or for a particle with spin decaying, if you average over initial spin states, then we can do the angular integrations (which give $8\pi^2$) to get

$$\frac{1}{32\pi^3} |a|^2 \, dE_1 dE_2$$

as the differential transition probability per unit time into an energy range E_1 for particle 1 and E_2 for particle 2.

If I make a plot of experimental data points as a function of E_1 and E_2, they will be distributed according to $|a|^2$ because $dE_1 dE_2$ is the Euclidean measure on the plane



There will be a kinematically imposed boundary to the sprinkling of dots, but within those boundaries, the sprinkling of data is directly proportional to $|a|^2$.

Our next topic is the beginning of a discussion of Green's functions, scattering with wave packets, and the LSZ reduction formula.

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FEYNMAN DIAGRAMS WITH EXTERNAL LINES OFF THE MASS SHELL

We'll restrict ourselves (for notational simplicity only) to considering diagrams in which only one type of scalar meson appears on the external lines. (By scalar, I just mean uncharged, with no Lorentz indices on its field, that is, no spin, not a specification of its parity transformation properties. A parity need not even exist for the formalism we are about to develop to be applicable. ("Charged scalar" means charged, but no spin. If I really wanted to specify parity's effect I would say "scalar under parity transformations," or whatever.)) We'll still let particles of all types run around on the internal lines.

Let a blob like this,

$$\equiv \tilde{G}^{(4)}(k_1, \ldots, k_4)$$

represent a sum of Feynmann diagrams. It could be all Feynmann diagrams to some order in perturbation theory, or in our imagination the sum of all diagrams to all orders in perturbation theory.

CAN WE ASSIGN ANY MEANING TO THIS BLOB IF THE MOMENTA ON THE EXTERNAL LINES ARE UNRESTRICTED, OFF THE MASS SHELL, MAYBE NOT EVEN SATISFYING $k_1 + k_2 + k_3 + k_4 = 0$?

We are going to come up with three affirmative answers to this question. Something neat is that they all agree.

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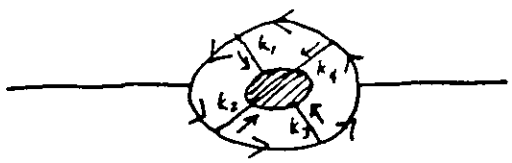
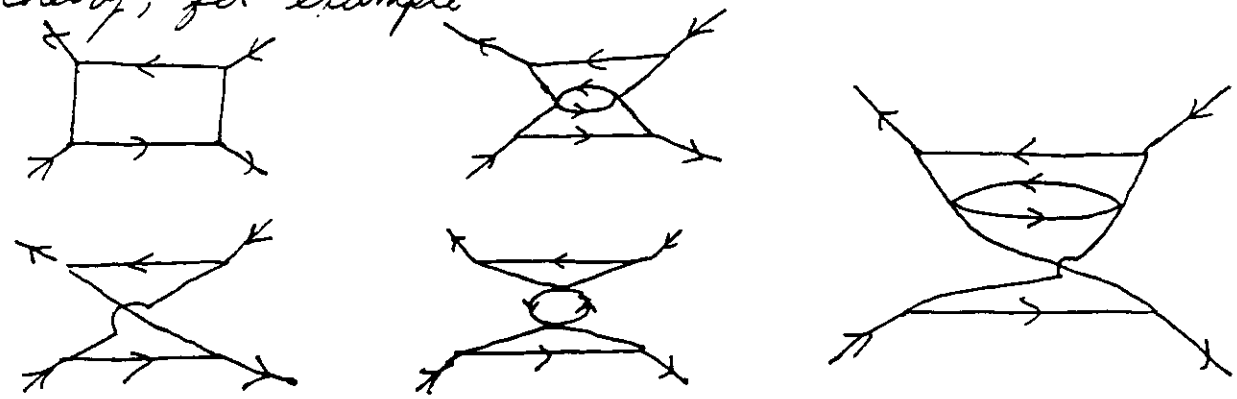
ANSWER 1 The blob could be an internal part of a more complicated graph

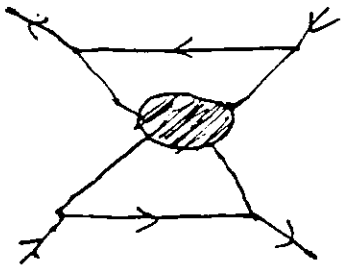


The Feynman rules instruct you to label all internal momenta arbitrarily and integrate over them.

Suppose in our study of other graphs in the theory, for example



all of which have the form



,

that we have already summed the blob, in our work calculating those graphs to some order. Then it would be nice not to repeat that work when calculating  , it would be nice to just plug the result in from a table of blobs.

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or a sum of internal parts in bigger graphs 15

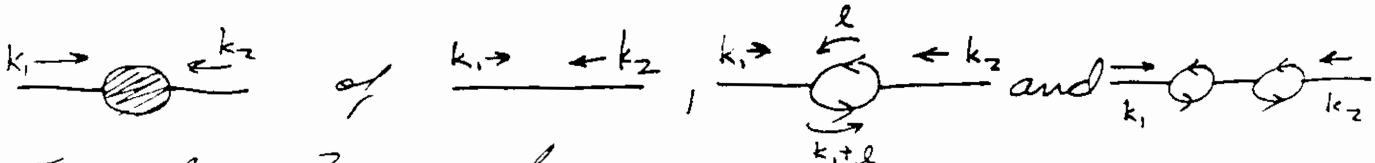
So we have one sensible, even useful definition of the blob. We will define it to be what it would be if it were an internal part of a bigger graph. Our Feynman rules for the bigger graph, which has its external lines on the mass shell then tell us exactly how to define the blob.

We still have a couple of conventional choices to make in defining a blob. We could include or not include the n propogators that hang off $\tilde{G}^{(n)}(k_1, \ldots, k_n)$ and we could include or not include the overall energy momentum conserving δ function. We'll include it all.

Here's a simple example. A big graph that contains $\tilde{G}^{(2)}(k_1, k_2)$ is

More explicitly various contributions of this type are

These correspond to contributions to



To order g^2 we have

$$\xrightarrow{k_1} \text{blob} \xleftarrow{k_2} \equiv \tilde{G}^{(2)}(k_1, k_2) = (2\pi)^4 \delta^{(4)}(k_1+k_2)\left[\frac{i}{k_1^2-\mu^2+i\epsilon} + \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(k_1+l)^2-m^2+i\epsilon} \frac{i}{l^2-m^2+i\epsilon} \cdot \frac{i}{k_1^2-\mu^2+i\epsilon} \cdot \frac{i}{k_2^2-\mu^2+i\epsilon}\right]$$

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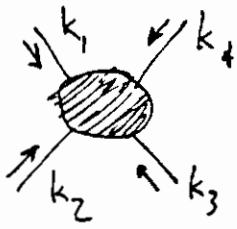
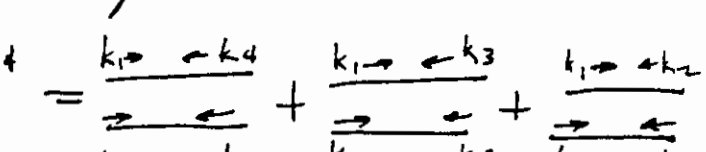
Because of the overall energy momentum conserving delta function, which enforces $k_1 = -k_2$, there is some ambiguity in the way to write down the contributions to $\tilde{G}^{(2)}(k_1, k_2)$ to $O(g^2)$. It could just as well have been written

$$(2\pi)^4 \delta^{(4)}(k_1+k_2)\left[\frac{i}{k_2^2-\mu^2+i\epsilon} + \left(\frac{i}{k_2^2-\mu^2+i\epsilon}\right)^2 \int \frac{d^4l}{(2\pi)^4} \frac{i}{(-k_2+l)^2-m^2+i\epsilon} \frac{i}{l^2-m^2+i\epsilon}\right]$$

We can also write down a few contributions to

$\tilde{G}^{(4)}(k_1, k_2, k_3, k_4) =$ 

$=$ 

$+ O(g^2)$

$$= (2\pi)^4 \delta^{(4)}(k_1+k_4) \frac{i}{k_1^2-\mu^2+i\epsilon} (2\pi)^4 \delta^{(4)}(k_2+k_3) \frac{i}{k_2^2-\mu^2+i\epsilon} + 2 \text{ perms}$$
$$+ O(g^2)$$

Since the second δ function enforces $k_2 = -k_3$ we can rewrite the first δ function as

$$\delta^{(4)}(k_1+k_2+k_3+k_4)$$

if you like to display over all energy momentum conservation explicitly.

One thing we can do with these blobs is to recover S matrix elements. We cancel off the external propagators and put the momenta back on their mass shells.

$$\langle k_1', k_2' | (S-1) | k_1, k_2 \rangle = \prod_{r=1,2,1',2'} \frac{k_r^2-\mu^2}{i} \tilde{G}^{(4)}(-k_1', -k_2', k_1, k_2)$$

← to cancel the external propagators we had included in $\tilde{G}$

Because of the four factors of zero out front when the momenta are on mass shell, the graphs that we wrote out above do not contribute. Indeed, they should not contribute to $S-1$. ✓

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Fourier transform (convention of Nov. 6):

$$f(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{ik\cdot x}$$

This is a little unfortunate because $e^{-iEt + i\vec{k}\cdot\vec{x}}$ $(E>0)$ is generally called a positive frequency plane wave (because $i\frac{\partial}{\partial t}$ acting on it gives E and $\frac{1}{i}\frac{\partial}{\partial \vec{x}}$ acting on it gives $\vec{k}$) and thus if $\tilde{f}(k)$ has support for positive k^0, $f(x)$ has negative frequency.

A source with positive frequencies creates particles while a source with negative frequencies absorbs particles. This is summed up in the Feynman rule

$\Longleftrightarrow$ $i\tilde{p}(-k)$

See page 5 of next lecture if you don't know or remember how to get this Feynman rule.

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ANSWER 1 (CONT'D) We have found one meaning for our blob. We can use it to obtain another function, its Fourier transform.

Using the Fourier transform convention

$$f(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}(k)\, e^{ik\cdot x}$$

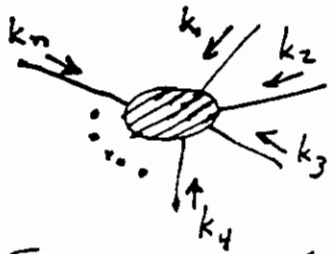
$$\tilde{f}(k) = \int d^4x\; f(x)\, e^{-ik\cdot x}$$

The power theorem is $\int d^4x\, F(x) g(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{f}(k)\, \tilde{g}(-k)$

(which you'll notice has a different sign in the exponent from what we used on OCT. 21, pp. 10ff) we have

$$G^{(n)}(x_1, \ldots, x_n) = \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}\, e^{ik_1\cdot x_1 + \cdots + ik_n\cdot x_n}\, \tilde{G}^{(n)}(k_1, \ldots, k_n)$$

where $\tilde{G}^{(n)}(k_1, \ldots, k_n) =$



IF THE SUM DOES NOT EXIST THEN THE BLOB REPRESENTS A FORMAL POWER SERIES IN g.

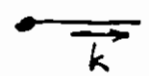
LET'S MAKE THIS DEFINITE

This is the sum of (all) Feynman diagrams to all orders and the blob includes factors for the external propagators and the factor for overall energy momentum conservation, $(2\pi)^4\delta^{(4)}(k_1 + \cdots + k_n)$.

ANSWER 2 Consider modifying $\mathcal{H}$, say in model 3,

$$\mathcal{H} \to \mathcal{H} - \rho(x)\phi(x) \qquad \left(\mathcal{L} \to \mathcal{L} + \rho(x)\phi(x)\right)$$

where $\rho(x)$ is a specified c number source, not an operator. This adds a new vertex, a model 1 type vertex



$= i\tilde{\rho}(-k)$

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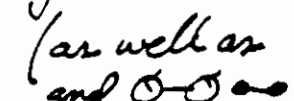
ANSWER 2 (CONT'D)

This new Feynman rule was just quoted in class, let's see it arise in a simple example.

Let's suppose we have got the original Hamiltonian's vacuum counterterm all calculated out to some high order in perturbation theory so that there are no corrections to $\langle 0|\mathcal{S}|0\rangle$ to this high order. The modification of the Hamiltonian (density) $\mathcal{H} \to \mathcal{H} - \rho(x)\phi(x)$ spoils this. There are now contributions to $\langle 0|\mathcal{S}|0\rangle$ proportional to ρ^n at low orders in g. at order ρ and order g we have ↓. at order ρ and $O(g^3)$ we have ↙

Unfortunately, these are not interesting simple examples, because unless $\tilde{\rho}(0)$ is nonzero they vanish because of energy-momentum conservation.

at order ρ^2 and order g^0 we have —●—.
at order ρ^2 and order g^2 we have

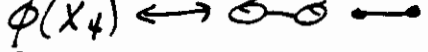
(as well as 
and)

This is a nice simple example, let's look at it. It comes from the term second order in ρ and second order in g in

$$S = U_I(\infty,-\infty) = T\, e^{-i\int d^4x\,(g\,\psi^*\psi\phi - \rho\phi)}$$

i.e. $$\frac{(-ig)^2}{2!}\,\frac{(i)^2}{2!}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4\; \rho(x_3)\rho(x_4)$$
$$T\left(\psi^*\psi\phi(x_1)\,\psi^*\psi\phi(x_2)\;\phi(x_3)\;\phi(x_4)\right)$$

The process is vacuum → vacuum, so we are looking for the completely contracted terms in the Wick expansion of this time ordered product. They are

$\psi^*\psi\ \phi(x_1)\ \psi^*\psi\ \phi(x_2)\ \phi(x_3)\ \phi(x_4)$ ↔ 

$\psi^*\psi\ \phi(x_1)\ \psi^*\psi\ \phi(x_2)\ \phi(x_3)\ \phi(x_4)$
$\psi^*\psi\ \phi(x_1)\ \psi^*\psi\ \phi(x_2)\ \phi(x_3)\ \phi(x_4)$ } ↔ 

$\psi^*\psi\ \phi(x_1)\ \psi^*\psi\ \phi(x_2)\ \phi(x_3)\ \phi(x_4)$ ↔ 

$\psi^*\psi\ \phi(x_1)\ \psi^*\psi\ \phi(x_2)\ \phi(x_3)\ \phi(x_4)$
and $\psi^*\psi\ \phi(x_1)\ \psi^*\psi\ \phi(x_2)\ \phi(x_3)\ \phi(x_4)$ } ↔ 

The last two are the ones I want to look at in detail. They differ by an exchange of $x_1 \leftrightarrow x_2$ only, and since these are dummy variables of integration they together make a contribution to $\langle 0|S|0\rangle$ of

$$(-ig)^2 \frac{(i)^2}{2!} \int d^4x_1 d^4x_2 d^4x_3 d^4x_4\ \rho(x_3)\rho(x_4)$$
$$\psi^*(x_1)\psi(x_2)\ \psi(x_1)\psi^*(x_2)\ \phi(x_1)\phi(x_3)\ \phi(x_2)\phi(x_4)$$

$$\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x_1-x_2)} \frac{i}{p^2-m^2+i\epsilon} \qquad \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot(x_1-x_2)} \frac{i}{q^2-m^2+i\epsilon}$$

$$\int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x_1-x_3)} \frac{i}{k^2-\mu^2+i\epsilon} \qquad \int \frac{d^4l}{(2\pi)^4} e^{il\cdot(x_2-x_4)} \frac{i}{l^2-\mu^2+i\epsilon}$$

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We have

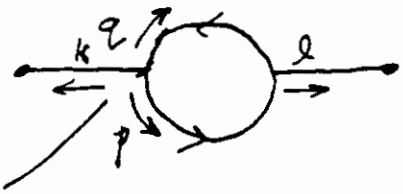
$$\frac{1}{2!}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{k^2-\mu^2+i\epsilon}\frac{i}{l^2-\mu^2+i\epsilon}$$

$$\times(-ig)^2(i)^2\int d^4x_1 d^4x_2 d^4x_3 d^4x_4\,\rho(x_3)\rho(x_4)$$
$$e^{ix_1\cdot(p+q+k)}e^{ix_2\cdot(-p-q+l)}e^{-ix_3\cdot k}e^{-ix_4\cdot l}$$

Now if you go back to pages 2, 3 and 4 of Oct. 28 you'll see that when we have a factor

$e^{ix_1\cdot(p+q+k)}$ (especially the very bottom of page 2)

that corresponds to a picture with p, q and k flowing out of x_1. Our picture for the integral at hand is



this vertex, corresponding to space-time point x_1 (but I hate to label it as such because it is just a dummy integration variable which is going to be integrated over, and in our combinatoric arguments for Feynman diagrams, we have kept the vertices unlabelled) is creating a nucleon with momentum p, creating an antinucleon with momentum q, and creating a meson with momentum k.

The x integrals are easy to perform, we have

$$\frac{1}{2!}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}\frac{i}{q^2-m^2+i\epsilon}\frac{i}{k^2-\mu^2+i\epsilon}\frac{i}{l^2-\mu^2+i\epsilon}$$
$$\times(-ig)^2(i)^2\,\tilde{\rho}(k)\tilde{\rho}(l)\,(2\pi)^4\delta^{(4)}(p+q+k)(2\pi)^4\delta^{(4)}(-p-q+l)$$

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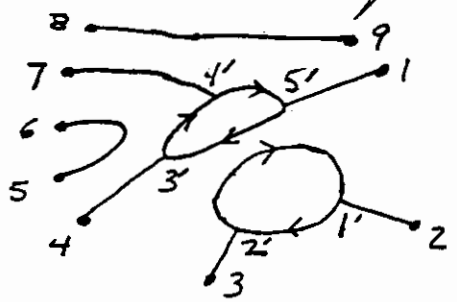
6

Suppose we have a graph in the Wick expansion with n powers of $\rho\phi$ and m powers of $g\psi^*\psi\phi$, which comes with a $\frac{1}{n!}\frac{1}{m!}$ from the exponential.



In this example $n = 9$ and $m = 5$

Each of the $m!$ permutations of the model 3 vertices, keeping the source vertices fixed, is a new term in the Wick expansion. They are all uniquely identified by the way they are attached to the source vertices. There are no model 3 vertices that are not somehow attached to a source vertex. That would be a disconnected bubble, which is a contribution to $\langle 0|S|0\rangle$ with the source off, and by assumption, the vacuum energy counterterm has been adjusted so that there are no corrections to $\langle 0|S|0\rangle$ with the source off.

Now what about the $n!$ permutations of source vertices. Some of them make no new contributions to the Wick expansion. For example $5 \leftrightarrow 6$ and $8 \leftrightarrow 9$, but also $2 \leftrightarrow 3$, because that has already been counted as $1' \leftrightarrow 2'$.

any of the $n!$ permutations that do make a new contribution to the Wick expansion, that is, that have not already been counted in the permutations of the m model 3 vertices, are accounted for in another way.

For example, $7 \leftrightarrow 8$ or $6 \leftrightarrow 6$ gives a new term in the Wick expansion.

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This is just what you would have directly written down using our old Feynman rules supplanted by

$$\overleftarrow{\underset{k}{\bullet}} = i\tilde{\rho}(k)$$

except for that $1/2!$ out front, which I'll explain in a moment.

This was a moderately interesting graph to show how the Feynman rule comes out. If you know how the Feynman rule comes out, but you want to check whether it's $\tilde{\rho}(k)$ or $\tilde{\rho}(-k)$ just look at the lowest order (in ρ, zeroth order in g) contribution to

$$\langle 0|S|k\rangle = \langle 0|U_I(\infty,-\infty)|k\rangle = \langle 0|\left[\mathbb{I} + i\int d^4x\,\rho(x)\phi(x)\right]|k\rangle + \cdots$$

($|k\rangle$ is relativistically normalized, and I'll use the relativistically normalized creation and annihilation ops in the expansion for $\phi(x)$.)

$$= 0 + \overleftarrow{\underset{k}{\bullet}} + \cdots$$

But, $\langle 0|\phi(x)|k\rangle = \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} e^{-ik\cdot x} \underbrace{\langle 0|a(k')|k\rangle}_{(2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}-\vec{k}')}$

$$= e^{-ik\cdot x}$$

So,

$$\overleftarrow{\underset{k}{\bullet}} = i\int d^4x\,\rho(x)e^{-ik\cdot x} = i\tilde{\rho}(k)$$

← Short argument on page 8

Now for that $\frac{1}{2!}$ out front. Earlier (OCT. 28, p.6) we sung and danced about how there were no symmetry factors in model 3. That argument still goes through, and it still only applies to diagrams where <u>each connected part has at least one external line</u>. What we have here is a vacuum to vacuum diagram, no external lines, and we now have to worry about symmetry factors.

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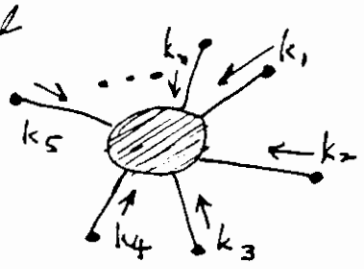
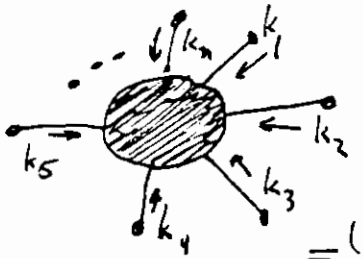
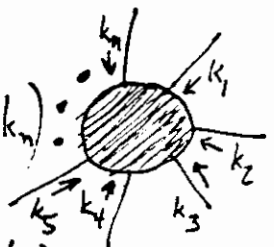
Let 

denote the sum of all diagrams to all orders in g (i.e. all m) and at nth order in ρ that contribute to $\langle 0|S|0\rangle$

What these combinatoric arguments say is



$= \frac{(i)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \tilde{\rho}(-k_1) \cdots \tilde{\rho}(-k_n)$ 

$$= \frac{(i)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \tilde{\rho}(-k_1) \cdots \tilde{\rho}(-k_n) \tilde{G}(k_1, \ldots, k_n)$$

Having gone all the way back to Wick expansion arguments to show the combinatorics are right for this, I'll try to make a shorter argument.

The source creates n mesons, which are distinguishable by virtue of the fact that they all carry different momenta $k_1, \ldots, k_n$. They interact in all possible ways. That gives us $(i)^n \tilde{\rho}(-k_1) \cdots \tilde{\rho}(-k_n)\, \tilde{G}(k_1, \ldots, k_n)$. Now we integrate over all momenta $k_1, \ldots, k_n$, and in doing so we make an overcounting $1/n!$.

BEST ARGUMENT

One last way of arguing this. Instead of considering this as an nth order calculation in ρ, temporarily think of it as a first order calculation in each of n different sources $\rho_1(x), \ldots, \rho_n(x)$. Then the diagram where source 1 creates a particle with momentum k_1 and source 2 creates a particle with momentum k_2 really is distinguishable from a diagram where source 1 creates k_2 and source 2 creates k_1. There is no overcounting when you integrate over all momenta. That contribution to $\langle 0|S|0\rangle$ would be

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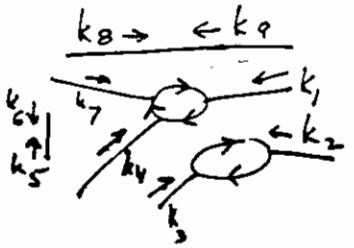
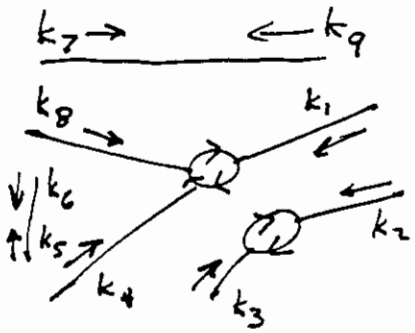
To see the accounting work, look at the messy example on the previous page in momentum space. It is

$$\frac{(-i)^9}{9!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_9}{(2\pi)^4}\,\tilde{\rho}(-k_1)\cdots\tilde{\rho}(-k_9)\times$$



Feynman diagram with external lines off the mass shell

Instead of using the permutation $7 \leftrightarrow 8$ to partially cancel off the $\frac{1}{9!}$ out front consider it as the same mess w $9!$ with a new diagram in the integrand

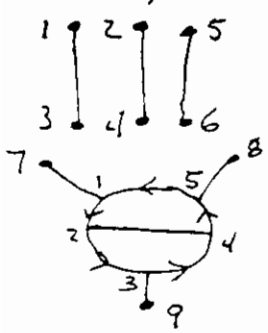


Of course when the momenta are integrated over, this is identical to the integral above

Now both of these would be counted in $\tilde{G}(k_1,\ldots,k_9)$ and in fact every permutation of the 9 source vertices that leads to a new term in the Wick expansion corresponds to a diagram in $\tilde{G}(k_1,k_2,\ldots,k_9)$.

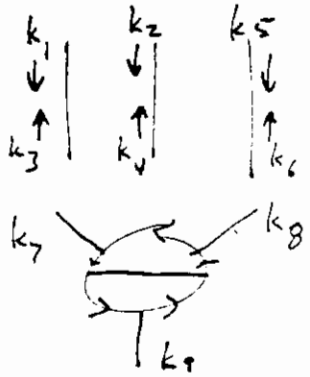
Of course there are diagrams with $n=9$ and $m=5$ that are not of the same pattern V (differ by more than a permutation of vertices).

For example



as the one drawn

There is a Feynman diagram in $\tilde{G}(k_1,\ldots,k_9)$ for this too



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$$(i)^n \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}\, \tilde{\rho}_1(-k) \cdots \tilde{\rho}_n(-k_n)\, \tilde{G}(k_1, \ldots, k_n)$$

How does this imagined calculation differ from ours? Well, in the exponential $\tilde{\rho}(k_1) \cdots \tilde{\rho}_n(x_n)$ comes with coefficient 1, while $\rho(x_1) \cdots \rho(x_n)$ comes with coefficient $\frac{1}{n!}$.

To all orders

$$\langle 0|S|0\rangle = 1 + \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \tilde{\rho}(-k_1) \cdots \tilde{\rho}(-k_n) \tilde{G}^{(n)}(k_1, \ldots, k_n)$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int d^4x_1 \cdots d^4x_n\, \rho(x_1) \cdots \rho(x_n)\, G^{(n)}(x_1, \ldots, x_n)$$

This is the second answer to our question. The Fourier transform of the sum of Feynman diagrams with n external lines off the mass shell is a Green's function (that's what G stands for). In a theory with linear response only $G^{(1)} \neq 0$. From conservation of probability alone, you can see that the response of a quantum-mechanical system is linear. Green introduced the first Green's function in the early 19th century. From a prescribed charge distribution, $\rho(\vec{x})$, his Green's function gave you the electrostatic potential, $\phi(\vec{x})$

$$\phi(\vec{x}) = \int d^3x' G(\vec{x}, \vec{x}')\, \rho(\vec{x}') \qquad G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|}$$

satisfies $\nabla^2 \phi = -\vec{\nabla} \cdot \vec{E} = -4\pi\rho \quad (\vec{E} = -\vec{\nabla}\phi)$

Let's explicitly note that the vacuum to vacuum transition amplitude depends on ρ by writing

$\langle 0|S|0\rangle_\rho$ ← don't confuse the ρ with a p

$\langle 0|S|0\rangle_\rho$ is a functional of ρ. You give me a function on spacetime, $\rho(x)$, and I give you back a number, $\langle 0|S|0\rangle_\rho$. Actually, it is just a function of an infinite number of variables, the value of the source at each spacetime point, and the nomenclature "functional" is redundant, we could just say "function". Mathematicians don't call a vector in an infinite dimensional space "vectorial". $\langle 0|S|0\rangle_\rho$ comes up often enough it gets a name, $Z[\rho]$

$$Z[\rho] \equiv \langle 0|S|0\rangle_\rho$$

↖ The square brackets remind you that this is a function of a function, ρ.

$Z[\rho]$ is called a generating functional for the Green's functions because, via the infinite dimensional generalization of a Taylor series, we have

$$\left.\frac{\delta^n Z[\rho]}{\delta\rho(x_1)\cdots\delta\rho(x_n)}\right|_{\rho=0} = (i)^n G^{(n)}(x_1,\ldots,x_n)$$

↑ The δ instead of a ∂ reminds you that you are taking a partial derivative of Z with respect to $\rho(x)$, holding a (4-d)continuum of other variables fixed. These are called functional derivatives

Ex nihil omnes — all Green's fns., and hence all S matrix elements

All physical information about the system is coded in the vacuum persistence amplitude in the presence of an external source ρ.

$Z[\rho]$ is called a generating functional in terms of two variables, x and y, which when you Taylor expand in one variable, the coefficients are a set of functions, like the Legendre polynomials in the other. Sometimes it is very useful to put a whole set of functions in one neat package like that.

an example will be the ward identities, which are a statement about Green's functions resulting from a symmetry. It can be put very compactly in

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Because of our great theorem

$$\Sigma \text{ all Wick diagrams} = :e^{\Sigma \text{ connected diagrams}}:$$

The sums are sums of normal ordered terms.

Apply this to a model which has had a source added. Take the vacuum expectation value of both sides. The LHS is just $\langle 0|S|0\rangle_\rho$, i.e. $Z[\rho]$. We have

$$Z[\rho] = \langle 0| :e^{\Sigma \text{ connected diagrams}}: |0\rangle$$

$$= e^{\langle 0|\Sigma \text{ connected diagrams}|0\rangle}$$

This is true because the terms in the sum in the exponential are themselves normal ordered, convince yourself. Taking the natural logarithm,

$$\ln Z[\rho] = \langle 0|\Sigma \text{ connected diagrams}|0\rangle$$

$$= \sum_{n=1}^{\infty} \frac{(i)^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} \tilde{\rho}(-k_1)\cdots\tilde{\rho}(-k_n) \times \tilde{G}_c(k_1,\ldots,k_n)$$

$\tilde{G}_c$ is the sum of all connected Feynman diagrams, with $k_1,\ldots,k_n$ possibly off shell, including the overall energy momentum conserving δ function, and the external propagators, which blow up on mass shell.

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"Vacuum Expectation Value" 12

ANSWER 3 One more way of interpreting $G^{(n)}(x_1,\dots,x_n)$. By a cunning trick we will show that $G^{(n)}(x_1,\dots,x_n)$ is a VEV of a string of Heisenberg fields. As we did in obtaining answer 2 / time ordered

Let

$$\mathcal{H} \longrightarrow \mathcal{H} - \rho\phi(x)$$

$$\mathcal{H}_0 + \mathcal{H}' \longrightarrow \mathcal{H}_0 + \mathcal{H}' - \rho\phi(x)$$

as far as Dyson's formula is concerned, you can break the Hamiltonian up into a "free" and interacting part in any way you please. Let's take the "free" part to be

$$\mathcal{H}_0 + \mathcal{H}' \text{ and the interaction to be } -\rho\phi(x)$$

I put quotes around "free", because in this new interaction picture, the fields evolve according to

$$\phi(\vec{x},t) = e^{iHt}\,\phi(\vec{x},0)\,e^{-iHt}$$

$$H = \int d^3x\, \mathcal{H} \qquad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}''$$

These fields are not free. They do not obey the free field equations of motion. You can't define a contraction for these fields, and thus you can't do Wick's theorem! These fields are what we would have called Heisenberg fields if there was no source. For this reason we'll subscript them with an H.

Let's see what this tells us about $Z[\rho]$.

$$Z[\rho] = \langle 0|S|0\rangle_\rho = \langle 0|\,T\, e^{+i\int d^4x\,\rho(x)\,\phi_H(x)}\,|0\rangle$$

first expand $= 1 + \sum_{n=1}^{\infty} \frac{(+i)^n}{n!} \int d^4x_1 \cdots d^4x_n\, \rho(x_1)\cdots\rho(x_n)\langle 0|\,T\phi_H(x_1)\cdots\phi_H(x_n)\,|0\rangle$

and we read off $G^{(n)}(x_1,\dots,x_n) = \langle 0|\,T(\phi_H(x_1)\cdots\phi_H(x_n))\,|0\rangle$

To summarize, We have found three meanings for the (sum of all) Feynman diagrams with (n) external lines off the mass shell.

It is a handy blob we can plaster into the interior of a larger diagram.

Its Fourier transform (times $(+i)^n$) is the coefficient of the nth order $\frac{}{n!}$ term in ρ in the expansion of the vacuum to vacuum to persistence amplitude in the presence of a source, ρ.

Its Fourier transform is the VEV of a time ordered string of Heisenberg fields.

This can all be taken as motivation, because we are going to start from scratch and do a

REFORMULATION OF SCATTERING THEORY

No more turning on and off function

$= \int d^3x \, \mathcal{H}$

Imagine you have a well-defined theory, with a time independent Hamiltonian, H (the turning on and off function is gone for good), whose spectrum is bounded below, whose lowest lying state, is not part of a continuum, and the Hamiltonian has actually been adjusted so that this state, $|0\rangle_P$, the physical vacuum, satisfies

$$H|0\rangle_P = 0$$

↑ don't confuse P with ρ

The vacuum is translationally invariant and normalized to one

$$\vec{P}|0\rangle_P = 0 \qquad \text{and} \qquad {}_P\langle 0|0\rangle_P = 1$$

Now let $\mathcal{H} \longrightarrow \mathcal{H} - \rho(x)\phi(x)$ and define

$$Z[\rho] \equiv {}_{\rho}\langle 0|\, S\, |0\rangle_{\rho} \Big|_{\text{in the presence of the source } \rho}$$

$$= {}_{\rho}\langle 0|\, U(\infty, -\infty)\, |0\rangle_{\rho} \qquad ?!$$

↑ Schrödinger picture evolution operator for the Hamiltonian $\int d^3x (\mathcal{H} - \rho\phi)$

and define,

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^n} \frac{\delta^n Z[\rho]}{\delta\rho(x_1) \cdots \delta\rho(x_n)}$$

Two Questions

1. Is $G^{(n)}$ defined this way (the F.T.) of the sum of all Feynman graphs? Let's call the $G^{(n)}$ defined as the sum of all Feynman graphs $G_F^{(n)}$ and the Z which generated those Z_F. The question is: Is $G^{(n)} = G_F^{(n)}$? or equivalently, Is $Z = Z_F$?

   Answer will be "yes."

2. Are S-1 matrix elements obtained from Green's functions in the same way as before? For example, is

$$\langle k_1', k_2' | (S-1) | k_1, k_2 \rangle = \prod_{\substack{a=1,2 \\ 1',2'}} \frac{k_a^2 - \mu^2}{i}\, \tilde{G}(-k_1', k_2', k_1, k_2)$$

?

Answer will be "almost."

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Answer to question 1, Is $G^{(n)} = G_F^{(n)}$?

Using Dyson's formula, in the exact same way as on we did on the bottom of p. 12 gives

$$G^{(n)}(x_1, \ldots, x_n) = {}_p\langle 0|T(\phi_H(x_1)\cdots\phi_H(x_n))|0\rangle_p$$

Does $Z_F[\rho]$, the generating functional you get by blindly summing graphs generate the same Green's fns.?

$\mathcal{H}$ splits up into $\mathcal{H}_0 + \mathcal{H}'$. Let $\mathcal{H}_I = \mathcal{H}'(\phi_I)$. The thing which after Wick's theorem and a combinatoric engagement or two had a graphical expansion is

$$Z_F[\rho] = \lim_{t_\pm \to \pm\infty} \langle 0| T e^{-i\int_{t_-}^{t_+} d^4x [\mathcal{H}_I - \rho\phi_I]} |0\rangle$$

(Bare, eigenstate of H_0: $H_0|0\rangle = 0$)

We used to adjust the constant part of $\mathcal{H}_I$ to eliminate vacuum bubbles in our old scattering theory. when $\rho = 0$. That is, we adjusted the vacuum energy counterterm, so that the vacuum to vacuum graphs (with no source vertices) summed to zero. There is an equivalent way of throwing away the vacuum bubbles. You divide out of $Z_F[\rho]$ the sum of all vacuum to vacuum graphs with no source vertices explicitly, and then you don't have to worry about a vacuum energy c.t. i.e. you divide by the same thing with $\rho = 0$.

$$Z_F[\rho] = \lim_{t_\pm \to \pm\infty} \frac{\langle 0| T e^{-i\int_{t_-}^{t_+} d^4x [\mathcal{H}_I - \rho\phi_I]} |0\rangle}{\langle 0| T e^{-i\int_{t_-}^{t_+} d^4x \mathcal{H}_I} |0\rangle}$$

To get $G_F^{(n)}(x_1, \ldots, x_n)$, we do n functional derivatives w.r.t. ρ and then set $\rho = 0$ (and divide by i^n).

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$$G_F^{(n)}(x_1,\dots,x_n) = \lim_{t_\pm \to \pm\infty} \frac{\langle 0|T\left[\phi_I(x_1)\cdots\phi_I(x_n) e^{-i\int_{t_-}^{t_+} d^4x\, \mathcal{H}_I}\right]|0\rangle}{\langle 0|T e^{-i\int_{t_-}^{t_+} d^4x\, \mathcal{H}_I}|0\rangle}$$

We have got a little work to do to show this is the same as $G^{(n)}$ on the top of the previous page. Fortunately, both these expressions are manifestly symmetric under the $n!$ permutations of the $x_1,\dots,x_n$, so it suffices to prove they are equal for one ordering which for convenience we choose so that

$$x_1^0 > x_2^0 > \cdots > x_n^0 \qquad \text{or for short} \quad t_1 > t_2 > \cdots > t_n$$

The time ordering in the expression for $G^{(n)}$ is just lexicographic ordering

$$G^{(n)}(x_1,\dots,x_n) = {}_P\langle 0|\ \phi_H(x_1)\cdots\phi_H(x_n)|0\rangle_P$$
$$= {}_P\langle 0|\phi_H(x_1)\cdots\phi_H(x_n)|0\rangle_P$$

Using the standard shorthand for $e^{-i\int_{t_a}^{t_b} d^4x\, \mathcal{H}_I} \equiv U_I(t_b,t_a)$, the time ordering in the expression for $G_F^{(n)}$ is

$$G_F^{(n)}(x_1,\dots,x_n) = \lim_{t_\pm\to\pm\infty} \frac{\langle 0|U_I(t_+,t_1)\phi_I(x_1)U_I(t_1,t_2)\phi_I(x_2)\cdots\phi_I(x_n)U_I(t_n,t_-)|0\rangle}{\langle 0|U(t_+,t_-)|0\rangle}$$

at least in the $\lim_{t_\pm\to\pm\infty}$ when $t_+ > t_1 > \cdots > t_n > t_-$. Convince yourself.

Everywhere $U_I(t_a,t_b)$ appears, rewrite it as

$$U_I(t_a,0)\,U_I(0,t_b)$$

and then use

$$\phi_H(x_i) = U_I(t_i,0)^+ \phi_I(x_i)\, U_I(t_i,0)$$
$$= U_I(0,t_i)\,\phi_I(x_i)\,U_I(t_i,0)$$

to get

I usually put "convince yourself" when I haven't written enough to make something clear, but if I wrote more it would take just as long to figure out what I was saying as it would take to convince yourself.

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* see below You see it doesn't really matter from this point on that it is a bare vacuum that $U(0,t_-)$ is acting on. What we are showing is that for two arbitrary fixed states $\lim_{t_-\to-\infty} \langle\varphi|U(0,t_-)|\psi\rangle = \langle\varphi|0\rangle_p{}_p\langle 0|\psi\rangle$

$$G_F^{(n)}(x_1,\ldots,x_n) = \lim_{t_\pm\to\pm\infty} \frac{\langle 0|U_I(t_+,0)\phi_H(x_1)\phi_H(x_2)\cdots\phi_H(x_n)U_I(0,t_-)|0\rangle}{\langle 0|U_I(t_+,0)U_I(0,t_-)|0\rangle}$$

Considering the whole mess sandwiched with $U_I(0,t_-)|0\rangle$ as some fixed state $\langle\varphi|$ let's work on in the numerator or denominator

$$\lim_{t_-\to-\infty}\langle\varphi|U_I(0,t_-)|0\rangle = \lim_{t_-\to-\infty}\langle\varphi|U_I(0,t_-)e^{iH_0t_-}|0\rangle$$

fancy way of inserting I

$$\overset{*}{=}\lim_{t_-\to-\infty}\langle\varphi|U(0,t_-)|0\rangle$$

* see above

↑ Schrödinger picture

Using an easily derivable relationship between the evolution operator in the various pictures. Oct. 16, p. 9

insert a complete set

$$=\lim_{t_-\to-\infty}\langle\varphi|U(0,t_-)\Big[|0\rangle_p{}_p\langle 0| + \sum_{\text{all other eigenstates of the full Hamiltonian, } H} |n\rangle\langle n|\Big]|0\rangle$$

$$H|0\rangle_p = 0 \qquad H|n\rangle = E_n|n\rangle$$

$$= \langle\varphi|0\rangle_p{}_p\langle 0|0\rangle + \lim_{t_-\to-\infty}\sum_{\text{all other eigenstates}} e^{iE_nt_-}\langle\varphi|n\rangle\langle n|0\rangle$$

Now every state but the vacuum is part of a continuum. As long as $\langle\varphi|n\rangle\langle n|0\rangle$ is a continuous function, the limit vanishes. The integrand is a continuous function that oscillates more and more wildly as $t_-\to-\infty$. In the limit it integrates to zero. A similar argument shows

$$\lim_{t_+\to\infty}\langle 0|U_I(t_+,0)|\psi\rangle = \langle 0|0\rangle_p{}_p\langle 0|\psi\rangle$$

Physically what this theorem about oscillating integrands (the Riemann-Lebesque lemma) says is that if you look at a state in some fixed region (take its inner product with some fixed state $\langle\varphi|$) and you wait long enough, the only trace of it that will remain is its (true) vacuum component. All the one and multi-particle states will have run away.

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$$G_F^{(n)}(x_1,\dots,x_n) = \frac{\cancel{\langle 0|0\rangle_P}\,{}_P\langle 0|\phi_H(x_1)\cdots\phi_H(x_n)|0\rangle_P\,\cancel{{}_P\langle 0|0\rangle}}{\cancel{\langle 0|0\rangle_P}\,{}_P\langle 0|0\rangle_P\,\cancel{{}_P\langle 0|0\rangle}}$$

$$= G^{(n)}(x_1,\dots,x_n)$$

and there is no longer any reason to distinguish between them.

Question 2 Are S-1 matrix elements obtained from Green's functions in the same way as before?

By introducing a turning on and off function, we were able to show that

$$\langle l_1,\dots,l_s|(S-1)|k_1,\dots,k_r\rangle = \prod_{a=1,\dots,s}\frac{l_a^2-\mu^2}{i}\prod_{b=1,\dots,r}\frac{k_b^2-\mu^2}{i}\,\tilde{G}^{(r+s)}(-l_1,\dots,-l_s,k_1,\dots,k_r)$$

The real world does not have a turning on and off function. Is this formula right?

The answer is "almost."

We will show how to obtain S-1 matrix elements from Green's functions without resorting to perturbation theory. We will make no reference to free Hamiltonian, bare vacua, interaction picture fields, etc. Accordingly, take

$$\phi(x) \equiv \phi_H(x) \qquad |0\rangle \equiv |0\rangle_P$$

$|0\rangle$ is the ground state of the full Hamiltonian, which as usual we assume to be translationally invariant and not part of a continuum, i.e. normalizable.

$$P^\mu|0\rangle = 0 \qquad \langle 0|0\rangle = 1$$

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We will assume there are physical one meson states, $|k\rangle$, relativistically normalized,

$$H|k\rangle = \sqrt{\vec{k}^2+\mu^2}\,|k\rangle\,, \quad \vec{P}|k\rangle = \vec{k}|k\rangle$$

$$\langle k'|k\rangle = (2\pi)^3 2\omega_{\vec{k}}\,\delta^{(3)}(\vec{k}-\vec{k}')$$

The reason that the answer to question 2 is "almost" is because the field, ϕ, which enters the formula for S-1 matrix elements through $G^{(n)}$ ($G^{(n)}(x_1,\ldots,x_n) = \langle 0|T(\phi(x_1)\ldots\phi(x_n))|0\rangle$), does not have quite the right properties. First, it may have a VEV, and second, in general it is not normalized so as to create a one particle state from the vacuum with a standard amplitude. It is normalized to obey the canonical commutation relations. We correct for these things by defining a renormalized field, ϕ', in terms of ϕ.

More precisely, $\langle 0|\phi(x)|0\rangle$ may not be zero. However this VEV is independent of x by translational invariance.

$$\langle 0|\phi(x)|0\rangle = \langle 0|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|0\rangle = \langle 0|\phi(0)|0\rangle$$

We also have by translational invariance

$$\langle k|\phi(x)|0\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|0\rangle = e^{ik\cdot x}\langle k|\phi(0)|0\rangle.$$

traditionally called the "wave function renormalization"

By Lorentz invariance you can easily see that $\langle k|\phi(0)|0\rangle$ is independent of k. It is some number, $Z_3^{1/2}$, in general $\neq 1$,

$$Z_3^{1/2} \equiv \langle k|\phi(0)|0\rangle \quad \text{which we hope is} \quad \text{nonzero.}$$

We define a new field ϕ' which has zero VEV and is normalized to have a standard amplitude to create one meson

$$\phi'(x) = Z_3^{-1/2}\left(\phi(x) - \langle 0|\phi(0)|0\rangle\right)$$

$$\langle 0|\phi'(x)|0\rangle = 0 \qquad \langle k|\phi'(x)|0\rangle = e^{ik\cdot x}$$

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LSZ formula stated

Define the renormalized Green's functions, $G'^{(n)}$,

$$G'^{(n)}(x_1,\dots,x_n) \equiv \langle 0|T(\phi'(x_1)\cdots\phi'(x_n))|0\rangle$$

and $\tilde{G}'^{(n)}$, their Fourier transforms, then S-1 matrix elements are given by

$$\langle l_1,\dots,l_s|(S-1)|k_1,\dots,k_r\rangle$$

$$= \prod_{a=1,\dots s} \frac{l_a^2-\mu^2}{i} \prod_{b=1,\dots,r} \frac{k_b^2-\mu^2}{i} \tilde{G}'^{(r+s)}(-l_1,\dots,-l_s,k_1,\dots,k_r)$$

This is the Lehmann-Symanzyik-Zimmermann reduction formula.

- The only assumptions needed about the local scalar field ϕ' is that it satisfy

$$\langle 0|\phi'(x)|0\rangle = 0 \quad \text{and} \quad \langle k|\phi'(x)|0\rangle = e^{ik\cdot x}.$$

In particular, its relationship to $\phi(x)$, the field that appears in the Lagrangian with a standard kinetic term, is not used. Any field that satisfies these properties, whose Green's functions you have, gives you the S matrix.

The proof of the LSZ reduction formula is as follows. First we'll find a way of making one meson wave packets. The method will be inspired by the way a limiting process gave us the physical vacuum when we started with the bare vacuum. Once we know how to make one meson states we'll wave our arms some and describe how to get many meson in and out states. Then we'll be set to get S matrix elements in terms of the Green's functions of the renormalized fields that were used to create the in and out states.

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LSZ reduction formula proof

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A notation for normalizable wave packet states

$$|f\rangle \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} F(\vec{k})|k\rangle \qquad \left(\text{you can recover } F(\vec{k}) \text{ from } |f\rangle: \; F(\vec{k}) = \langle k|f\rangle\right)$$

Associated with each of these wave packets, we have a negative frequency solution of the K.G. eqn.

$$f(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} F(\vec{k}) e^{-ik\cdot x} \qquad (\Box + \mu^2) f(x) = 0$$

Note that as $|f\rangle \to |k\rangle$ (i.e. $F(\vec{k}') \to (2\pi)^3 2\omega_{\vec{k}} \delta^{(3)}(\vec{k}-\vec{k}')$)
$f(x) \to e^{-ik\cdot x}$

Define an operator which is a function of time only out of any operator which is a function of $\vec{x}$ and t (here taken to be ϕ'):

$$\phi' f(t) = i \int d^3x \left(\phi' \partial_0 f - f \partial_0 \phi'\right)$$

Those of you who are familiar with the initial value theorem of the K.G. eqn. will not find this a strange combination. It will turn out to create a particle in state $|f\rangle$, in the limit $t \to \pm\infty$.

From the properties of $\phi'(x)$ and $f(x)$

$$\langle 0|\phi' f(t)|0\rangle = 0 \qquad \text{and}$$

$$\langle k|\phi' f(t)|0\rangle = i \int d^3x \int \frac{d^3k'}{(2\pi)^3 2\omega_{\vec{k}'}} F(\vec{k}') \times \langle k|\left[\phi' \partial_0 e^{-ik'\cdot x} - e^{-ik'\cdot x} \partial_0 \phi'(x,t)\right]|0\rangle$$

$$= i \int d^3x \int \frac{d^3k'}{(2\pi)^3 2\omega_{\vec{k}'}} F(\vec{k}') \left[-i\omega_{\vec{k}'} e^{-ik'\cdot x} - e^{-ik'\cdot x} \partial_0\right] \underbrace{\langle k|\phi'(x,t)|0\rangle}_{e^{ik\cdot x}}$$

$$= i \int d^3x \int \frac{d^3k'}{(2\pi)^3 2\omega_{\vec{k}'}} F(\vec{k}') \left(-i\omega_{\vec{k}'} - i\omega_{\vec{k}}\right) e^{-ik'\cdot x + ik\cdot x}$$

$$= F(\vec{k}) \quad \text{independent of time}$$

A similar derivation except for one crucial minus sign shows

$$\langle 0|\phi'^f(t)|k\rangle = 0$$

In these few matrix elements, $\phi'^f(t)$ is acting like a creation operator for a physical meson wave packet. We will now take the limit $t \to \pm\infty$, and in this limit we'll see that many more matrix elements of $\phi'^f(t)$ look like the matrix elements of a creation operator.

Consider any state with two or more particles satisfying $P^\mu|n\rangle = P_n^\mu|n\rangle$.

$$\langle n|\phi'^f(t)|0\rangle = \langle n|\, i\int d^3x\,[\phi'(\vec{x},t)\,\partial_0 f - f\,\partial_0\phi'(\vec{x},t)]\,|0\rangle$$

$$= i\int d^3x\,[\partial_0 f - f\,\partial_0]\underbrace{\langle n|\phi'(\vec{x},t)|0\rangle}_{e^{iP_n\cdot x}\langle n|\phi'(0)|0\rangle}$$

$$= i\int d^3x\,[\partial_0 f - f\,iP_n^0]\,e^{iP_n\cdot x}\langle n|\phi'(0)|0\rangle$$

$$= i\int d^3x\left[(\partial_0 - iP_n^0)\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}e^{-ik\cdot x}F(\vec{k})\right]e^{iP_n\cdot x}\langle n|\phi'(0)|0\rangle$$

$$= i\int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}F(\vec{k})(-i\omega_{\vec{k}} - iP_n^0)\int\underbrace{d^3x\;e^{-ik\cdot x}\,e^{iP_n\cdot x}}_{(2\pi)^3\delta^{(3)}(\vec{k}-\vec{P}_n)\cdot e^{-i\omega_{\vec{k}}t + iP_n^0 t}}\langle n|\phi'(0)|0\rangle$$

$$= \frac{\omega_{\vec{P}_n} + P_n^0}{2\omega_{\vec{P}_n}}F(\vec{P}_n)\langle n|\phi'(0)|0\rangle\, e^{-i(\omega_{\vec{P}_n} - P_n^0)t}$$

The important thing to notice is that this matrix element contains $e^{-i(\omega_{\vec{P}_n} - P_n^0)t}$ and that $\omega_{\vec{P}_n} < P_n^0$ for any multiparticle state. A multiparticle state with momentum $\vec{P}_n$ always has more energy than a single particle state with momentum $\vec{P}_n$. A two particle state with $\vec{P} = 0$ can have any energy from 2μ to ∞. The one particle state with $\vec{P} = 0$ has energy $\omega_{\vec{P}} = \mu$.

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Now consider $\langle\psi|\phi'^f(t)|0\rangle$ in the limit $t\to\pm\infty$ where $|\psi\rangle$ is a definite (not varying with t) normalizable state. Insert a complete set.

$$\lim_{t\to\pm\infty}\langle\psi|\phi'^f(t)|0\rangle = \lim_{t\to\pm\infty}\langle\psi|\left(|0\rangle\langle 0| + \int\frac{d^3k}{(2\pi)^3 2\omega_k}|k\rangle\langle k| + \sum_{\substack{\text{multiparticle}\\ \text{states } |n\rangle}}|n\rangle\langle n|\right)\phi'_f(t)|0\rangle$$

$$= 0 + \int\frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}}\langle\psi|k\rangle F(\vec{k}) + \lim_{t\to\pm\infty}\sum_{|n\rangle}\langle\psi|n\rangle\frac{F(\vec{P}_n)(\omega_{\vec{P}_n}+P_n^0)}{2\omega_{\vec{P}_n}}\langle n|\phi'(0)|0\rangle e^{-i(\omega_{\vec{P}_n}-P_n^0)t}$$

$$= \langle\psi|f\rangle + 0$$

The integrals over the various continua of multiparticle states have integrands which oscillate more and more wildly as $t\to\pm\infty$. They integrate to zero in the limit by the Riemann-Lebesgue lemma.

The phases were arranged to cancel in the one particle state matrix elements only.

The analagous derivation showing $\lim_{t\to\pm\infty}\langle 0|\phi'^f(t)|\psi\rangle = 0$ goes through because the phases add (and thus obviously never cancel) for every momentum eigenstate, one or multiparticle in the inserted complete set.

Physically what we have shown is this: We have created a state which in part looks like a one meson wave packet, plus a little multiparticle garbage. If we look at this state in some definite region in space time (take its inner product with some definite state $|\psi\rangle$, and send the time of creation to $-\infty$, all the multiparticle states will run away. Of course the one particle state may run away too. We prevent this by modifying the state we create in such a way that the one meson packet always has the same relationship to the observer that the funny combination $\phi'^f(t)$. No multiparticle state has the right dispersion relation to keep the same relationship to the observer under this modification.