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What we have done so far has been rigorous, at least it can be made rigorous with only a little effort. For the creation of multiparticle in and out states we are going to have to do some vigorous handwaving.

Consider two normalizable wave packets described by

$$F_1(\vec{k}) \text{ and } F_2(\vec{k}) \quad \text{in momentum space}$$

which have no common support. This excludes scattering at threshold.

What is $\lim_{t \to +(-)\infty} \langle \psi | \phi'^{f_2}(t) | f_1 \rangle$?

The handwaving requires we picture this in position space. Because F_1 and F_2 describe packets headed in different directions if you wait long enough (go back far enough) the two wave packets will be widely separated in position. Then the application of $\phi'^{f_2}(t)$ to $|f_1\rangle$ is, for all purpose to an observer near the f_2 packet, like an application of $\phi'^{f_2}(t)$ to the vacuum.

You may be bothered that a position space picture doesn't really exist, there is no $\vec{X}$ operator. However there is still some concept of localization up to a few Compton wavelengths. If a teensy exponential tail with $\frac{1}{e}$ distance $\frac{1}{m}$ is too big for you for some purpose, wait another thousand years.

This physical argument says

$$\lim_{t \to +(-)\infty} \langle \psi | \phi'^{f_2}(t) | f_1 \rangle = \langle \psi | f_1, f_2 \rangle^{out\,(in)}$$

By definition, the S matrix, is what tells you the probability amplitude that a state that looks a given state in the far past will look like another given state in the far future.

$$\langle f_3, f_4 | S | f_1, f_2 \rangle \equiv {}^{out}\langle f_3, f_4 | f_1, f_2 \rangle^{in}$$

$$= \lim_{t_4 \to \infty} \lim_{t_3 \to \infty} \lim_{t_2 \to -\infty} \lim_{t_1 \to -\infty} \langle 0 | \phi'^{f_3 \dagger}(t_3) \phi'^{f_4 \dagger}(t_4) \phi'^{f_2}(t_2) \phi'^{f_1}(t_1) | 0 \rangle$$

Note that in the limits, this is time ordered and thus we have succeeded in writing S matrix elements in terms of the renormalized Green's functions. So we could quit now, but we are going to massage this expression. In doing so, we'll extend the idea of an S matrix element. Physically there is no way to create plane wave states. Thus there is no way to measure or define S matrix elements of plane wave states. However, after we get done massaging the RHS of the above equation, we will get an expression that you can put plane wave states into without getting nonsense. We'll make this the definition of S matrix elements of plane wave states, $\langle k_3, k_4 | (S-1) | k_1, k_2 \rangle$. The utility of this object is that you can integrate it, smear it a little, to recover physically measurable S matrix elements. Now I'll tell you the answer, that is, what we will soon show is a sensible definition for

$$\langle k_3, k_4 | (S-1) | k_1, k_2 \rangle$$

$$= \int d^4x_1 \cdots d^4x_4 \, e^{ik_3\cdot x_3 + ik_4 \cdot x_4 - ik_1\cdot x_1 - ik_2 \cdot x_2} (i)^4 \prod_r (\Box_r + \mu^2) \langle 0 | T(\phi'(x_1) \cdots \phi'(x_4)) | 0 \rangle$$

That looks unfamiliar and messy, but it actually isn't. Recall that

$$\langle 0 | T(\phi'(x_1) \cdots \phi'(x_4)) | 0 \rangle \equiv G'(x_1, \ldots, x_4) = \int \frac{d^4 l_1}{(2\pi)^4} \cdots \frac{d^4 l_4}{(2\pi)^4} e^{i l_1 \cdot x_1 + \cdots + i l_4 \cdot x_4} \tilde{G}'(l_1, \ldots, l_4).$$

If you substitute this in the expression for $\langle k_3\, k_4 | (S-1) | k_1, k_2 \rangle$, it collapses to

$$\langle k_3, k_4 | (S-1) | k_1, k_2 \rangle = \prod_r \frac{k_r^2 - \mu^2}{i} \tilde{G}'(k_1, k_2, -k_3, -k_4)$$

This says that an S-1 matrix element is equal to a Green's function with the external propogators removed. This is almost exactly the result that came out of our low budget scattering theory, with the only difference being that the Green's function is of renormalized fields. The result we will first obtain won't be an expression for $\langle k_3, k_4 | (S-1) | k_1, k_2 \rangle$. We get that by abstracting the expression for $\langle f_3 f_4 | (S-1) | f_1 f_2 \rangle$ which looks just like the expression for $\langle k_3 k_4 | (S-1) | k_1, k_2 \rangle$ stated ~~on the previous page~~ above except $e^{-ik_1 \cdot x_1}$ is replaced by $f_1(x_1)$, $e^{-ik_2 \cdot x_2}$ by $f_2(x_2)$, $e^{ik_3 \cdot x_3}$ by $f_3^*(x_3)$ and $e^{ik_4 \cdot x_4}$ by $f_4^*(x_4)$. That is what we will show is

$$\langle f_3, f_4 | (S-1) | f_1 f_2 \rangle = \int d^4x_1 \cdots d^4x_4 \, f_3^*(x_3) f_4^*(x_4) f_1(x_1) f_2(x_2)$$

$$(i)^4 \prod_r (\Box_r + \mu^2) \langle 0 | T(\phi'(x_1) \cdots \phi'(x_4)) | 0 \rangle$$

Let's get on with the proof, beginning with a lemma.

Given any function, $f(x)$, satisfying $(\Box+\mu^2)f(x)=0$, and $f\to 0$ as $|x|\to\infty$ and a general field A, then

$$
\begin{aligned}
i\int d^4x\, f(\Box+\mu^2)A &= i\int d^4x[f\partial_0^2 A + A(-\nabla^2+\mu^2)f] \\
&= i\int d^4x(f\partial_0^2\, A - A\partial_0^2 f) \\
&= \int dt\,\partial_0 \underbrace{\int d^3x\; i(f\partial_0 A - A\partial_0 f)}
\end{aligned}
$$

this is something that appears often enough that it it is worth giving it a name. It is a function of time only, call it $-A^f(t)$.

$$
= -\int dt\,\partial_0 A^f(t) = (\lim_{t\to-\infty} - \lim_{t\to\infty})\, A^f(t)
$$

also, if A is hermitian, note difference in sign from conjugating the i in the defn of $A^f(t)$.

$$
i\int d^4x\, f^*(x)(\Box+\mu^2)A = (\lim_{t\to\infty} - \lim_{t\to-\infty})\, A^{f+}(t)
$$

Now we'll apply this equation to the RHS of the equation we want to prove. First to the x_1 integration, you get

$$
\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right)\int d^4x_2\, d^4x_3\, d^4x_4\, f_2(x_2) f_3^*(x_3) f_4^*(x_4)
$$

$$
(i)^3 \prod_{r=2,3,4} (\Box_r+\mu^2)\langle 0|T(\phi'^{f_1}(t_1)\phi'(x_2)\phi'(x_3)\phi'(x_4))|0\rangle
$$

We push a time derivative through a time-ordered product in these steps. It is OK in the limit we have. Then the x_2 integration

$$
\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right)\left(\lim_{t_2\to-\infty} - \lim_{t_2\to\infty}\right)\int d^4x_3\, d^4x_4\, f_3^*(x_3) f_4^*(x_4)
$$

$$
(i)^2(\Box_3+\mu^2)(\Box_4+\mu^2)\langle 0|T(\phi'^{f_1}(t_1)\phi'^{f_2}(t_2)\phi'(x_3)\phi'(x_4))|0\rangle
$$

Etcetera.

note the difference in sign

$$
\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right)\left(\lim_{t_2\to-\infty} - \lim_{t_2\to\infty}\right)\left(\lim_{t_3\to\infty} - \lim_{t_3\to-\infty}\right)\left(\lim_{t_4\to\infty} - \lim_{t_4\to-\infty}\right)
$$

$$
\langle 0|T(\phi'^{f_1}(t_1)\phi'^{f_2}(t_2)\phi'^{f_3+}(t_3)\phi'^{f_4+}(t_4))|0\rangle
$$

If we had reduced the integrals in some other order, we would have a different order of limits here. All 4! orderings lead to the same result however, and we'll just do the order we have arrived at.

When $t_4 \to -\infty$, it is the earliest time and thus the time ordering puts it on the right. However $\phi'^{f_4 +}(t_4)$ with the vacuum on the right and any other state on the left vanishes in the limit. When $t_4 \to +\infty$, it is the latest time and the time ordering puts it on the left, and we get the matrix element of $\langle f_4|$ with the rest of the mess. We have

$$\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right)\left(\lim_{t_2\to-\infty} - \lim_{t_1\to\infty}\right)\left(\lim_{t_3\to\infty} - \lim_{t_3\to-\infty}\right)\langle f_4|T(\phi'^{f_1}(t_1)\phi'^{f_2}(t_2)\phi'^{f_3 +}(t_3))|0\rangle$$

The exact same considerations apply to the t_3 limits except we get the matrix element of ${}^{out}\langle f_3, f_4|$ with the remaining mess. 'out' because both fields are applied to the vacuum in the far future. Doing the t_2 limits does not result in such a simplification. We get

$$\left(\lim_{t_1\to-\infty} - \lim_{t_1\to\infty}\right)\left({}^{out}\langle f_3, f_4|\phi'^{f_1}(t_1)|f_2\rangle - \lim_{t_2\to\infty}{}^{out}\langle f_3, f_4|\phi'^{f_2}(t_2)\phi'^{f_1}(t_1)|0\rangle\right)$$

The first term was expected. The second term looks real bad. Let's compartmentalize our ignorance by just giving a name to this state we have created

$$\langle\psi| \equiv \lim_{t_2\to\infty}{}^{out}\langle f_3 f_4|\phi'^{f_2}(t_2)$$

On to the evaluation of the t_1 limit. We get

$${}^{out}\langle f_3, f_4|f_1 f_2\rangle^{in} - {}^{out}\langle f_3 f_4|f_1 f_2\rangle^{out} - \cancel{\langle\psi|f_1\rangle} + \cancel{\langle\psi|f_1\rangle}$$

The last two terms cancel, because there is no difference between a matrix element of $\lim_{t_1\to+\infty}\phi'^{f_2}(t_2)|0\rangle$ and $\lim_{t_1\to-\infty}\phi'^{f_2}(t_2)|0\rangle$. ☺. The two terms remaining are exactly what we wanted to get. We have obtained

$$\langle f_3, f_4|(S-1)|f_1, f_2\rangle$$

REMARKS

(1) The mathematical expression we started with makes sense even for f_1, f_2, f_3, and f_4 plane waves. We'll make that expression the definition of an S-1 matrix element of plane waves. Of course you only get something physically measurable when you integrate, smear, the expression. The situation is very analogous to $V(\vec{x}-\vec{y})$ in the expression $U=\int d^3x\, d^3y\, V(\vec{x}-\vec{y})\,\rho(\vec{x})\,\rho(\vec{y})$. No one can build a point charge, and thus no one can make a charge distribution that directly measures $V(\vec{x}-\vec{y})$, that is, one for which the interaction energy is $V(\vec{x}-\vec{y})$. All you can do is measure U for various charge distributions. Then you can abstract to the notion of $V(\vec{x}-\vec{y})$, "the potential energy of between two point charges." You only recover something physically measurable when you integrate, smear, the expression for S-1 matrix elements of plane waves. The formula analogous to $U=\int d^3x\, d^3y\, V(\vec{x}-\vec{y})\,\rho(x)\,\rho(y)$ is

$$\langle f_3, f_4 | (S-1) | f_1, f_2 \rangle = \int \frac{d^3k_1}{(2\pi)^3\, 2\omega_{\vec{k}_1}} \cdots \frac{d^3k_4}{(2\pi)^3\, 2\omega_{\vec{k}_4}} F_3^*(\vec{k}_3) F_4^*(\vec{k}_4) F_1(\vec{k}_1) F_2(\vec{k}_2) \times \langle k_3, k_4 | (S-1) | k_1, k_2 \rangle$$

(2) The proof only required that the field you begin with have a nonzero vacuum to one particle matrix element. Then you shift that field by some constant, and multiply it by another constant to get the renormalized field, whose Green's functions are what actually entered the proof. There is thus a many to one correspondence between fields and particles. From the point of view of the reduction formula, $\tilde{\phi} = \phi + \frac{1}{2} g \phi^2$ is just as good a field (at least except for one exceptional value of g that makes the vacuum to one particle matrix element of $\tilde{\phi}$ vanish). You do not have to begin with one of the fields that seemed to be fundamental in the Lagrangian.

(3) There is no problem in principle of obtaining scattering matrix elements of composite particles and bound states. In the QCD theory of the strong interactions, the mesons are bound states of a quark and an antiquark. If $q(x)$ is a quark field, you would expect $\bar{q}q(x)$ to have a nonvanishing vacuum to one meson matrix element. "All" you need to calculate 2⟶2 meson scattering then would be

$$G'^{(4)}(x_1, x_2, x_3, x_4) \equiv \langle 0 | T(\bar{q}q(x_1)\bar{q}q(x_2)\bar{q}q(x_3)\bar{q}q(x_4)) | 0 \rangle$$

where $\bar{q}q'(x)$ is the renormalized field. Of course noone has gotten $G'^{(4)}$.

(Remark (5) contains the proof of the fact used in Problem 18(c) DUE DEC. 4. At the same time as rewriting this page, I have rewritten the 4 previous pages) NOV. 13

(4) If we have some exact knowledge of the position space properties of a field, it may be possible to use these properties in the LSZ formula to get some exact knowledge about S-1 matrix elements

(5) Using methods of the same type as those used in the derivation of the LSZ formula, other formulas can be derived. For example, one can "stop half way" in the reduction formula and obtain

$$\langle k_3, k_4 | (S-1) | k_1, k_2 \rangle = \int d^4x_3 d^4x_4 \, e^{ik_3\cdot x_3} e^{ik_4 \cdot x_4} (i)^2 (\Box_3+\mu^2)(\Box_4+\mu^2) \langle 0 | T(\phi'(x_3)\phi'(x_4)) | k_1, k_2 \rangle^{in}$$

This is used to derive theorems about the production of "soft" (low energy) photons

We can also derive use LSZ methods to derive expressions for the matrix elements of fields between in and out states. For example, I can show

$$^{out}\langle k_1, \dots, k_n | A(x) | 0 \rangle = \int d^4x_1 \cdots d^4x_n \, e^{ik_1\cdot x_1 + \cdots + ik_n \cdot x_n} (i)^n \prod_r (\Box_r+\mu^2) \langle 0 | T(\phi'(x_1)\cdots\phi'(x_n) A(x)) | 0 \rangle$$

where $\phi'(x)$ is a correctly normalized field that can create the outgoing mesons and $A(x)$ is an arbitrary field. Of course this is really an abstraction of

$$^{out}\langle f_1, \dots, f_n | A(x) | 0 \rangle = \int d^4x_1 \cdots d^4x_n \, f^*(x_1) \cdots f^*(x_n) (i)^n \prod_r (\Box_r+\mu^2) \langle 0 | T(\phi'(x_1)\cdots\phi'(x_n) A(x)) | 0 \rangle$$

applying the methods of page 3 to the RHS we get

$$\left(\lim_{t_1\to\infty} - \lim_{t_1\to-\infty}\right) \cdots \left(\lim_{t_n\to\infty} - \lim_{t_n\to-\infty}\right) \langle 0 | T(\phi'^{f_1 \dagger}(t_1) \cdots \phi'^{f_n \dagger}(t_n) A(x)) | 0 \rangle$$

Just as easily as we evaluated the t_3 and t_4 limits on the top of page 4, these limits can be evaluated to get

$$^{out}\langle f_1, \dots, f_n | A(x) | 0 \rangle$$

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a second look at model 3 and its renormalization

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu_0^2}{2}\phi^2 + \partial_\mu \psi^* \partial^\mu \psi - m_0^2 \psi^* \psi - g_0 \psi^* \psi \phi$$

The upshot of what we have done so far is that some 0 subscripts have been added to the Lagrangian. The coefficient of $-\frac{1}{2}\phi^2$ in $\mathcal{L}$ may not be the meson mass squared, m_0^2 may not be the charged meson (nucleon) mass squared. Furthermore g_0 may not be what we want to call the coupling constant. In real electrodynamics there is a parameter e, defined by some experiment. It would be lucky, extremely lucky, if that were the coefficient of some term in the electrodynamics Lagrangian. In general it isn't. We'll subscript the coupling constant, compute the conventionally defined coupling constant from it, and then invert the equation to eliminate g_0, which is not directly measured, from our expressions for all other quantities of interest. Also, when calculating our scattering matrix elements, we need Green's functions. What we have a perturbative expansion for if we treat $-g_0 \psi^* \psi \phi$ as our interaction Lagrangian, is the Green's functions of ϕ. Those Green's functions aren't exactly what we are interested in. We want the Green's functions of ϕ', the field satisfying

$$\langle 0 | \phi' | 0 \rangle = 0 \qquad \langle p | \phi'(0) | 0 \rangle = 1 \qquad \phi' = Z_3^{-1/2}(\phi - \langle \phi \rangle_0)$$

So along the way in calculating quantities of interest, we'll have to calculate the Green's functions of ϕ' from the Green's functions of ϕ. This determination of ϕ', m, μ and g from m_0, μ_0, g_0 and the above conditions, and then the plugging in of the inverse of these equations into other quantities of interest sounds like a mess. It can be avoided.

We rewrite $\mathcal{L}$ with six new parameters, $A, \ldots, F$.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi')^2 - \frac{\mu^2}{2}\phi'^2 + \partial_\mu \psi^{*\prime} \partial^\mu \psi' - m^2 \psi^{*\prime}\psi'$$
$$- g\psi^{*\prime}\psi'\phi' + \mathcal{L}_{CT}$$

$$\mathcal{L}_{CT} = A\phi' + \frac{B}{2}(\partial_\mu \phi')^2 - \frac{C}{2}\phi'^2 + D\partial_\mu \psi^{*\prime}\partial^\mu \psi' - E\psi^{*\prime}\psi' - F\psi^{*\prime}\psi'\phi'$$
$$+ \text{const}$$

The six new parameters are going to be determined order by order in perturbation theory by six renormalization conditions.

① $\langle 0|\phi'|0\rangle = 0$

② $\langle q|\phi'(0)|0\rangle = 1$
↖ one meson

③ $\langle p|\psi'(0)|0\rangle = 1$
↖ one anti-nucleon

④ The meson mass is μ

⑤ the nucleon mass is m

⑥ g agrees with the conventionally defined g.

Six unknowns, six conditions.

Of course, if you actually wanted to know the relationship of ϕ' to ϕ, the field whose kinetic term has coeffecient 1 in the Lagrangian (and thus obeys the canonical commutation relations), and has no linear term, you can read it off. You can also read off the bare meson mass, the bare nucleon mass, g_0, and the relationship of ψ' to ψ.

$$\frac{1}{2}(1+B)(\partial_\mu \phi')^2 + \frac{1}{2}(\mu^2 + C)\phi'^2 + A\phi' + \text{const}$$
$$= \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu_0^2}{2}\phi^2$$

$$Z_3 = 1 + B, \quad \mu_0^2 = \frac{1}{Z_3}(\mu^2 + C), \text{ etc.}$$

This parenthetical remark should be emphasized more. It is ϕ that satisfies $[\phi, \dot{\phi}] = i\delta^{(3)}$ with coefficient 1, not ϕ'!

See Oct. 23 pp. 13–15 for these same ideas expressed before we knew/worried about Z_3 and $\langle 0|\phi|0\rangle$.

What you have now is a perturbation theory for the quantities you are really interested in in terms of the conditions on ϕ' and ψ', and experimentally input parameters.

The difference in the two kinds of perturbation theory is what you call the interaction Lagrangian. We'll be taking $-g\psi^{*\prime}\psi'\phi' + \mathcal{L}_{CT}$ as the interaction. This is called renormalized perturbation theory.

Instead of computing scattering matrix elements μ^2, m^2 and g from μ_0, m_0, and g_0, the wrong parameters to hold fixed, and then inverting to get S matrix elements in terms of μ^2 m^2 and g, we compute everything in terms of the right quantities, the experimentally input parameters, μ^2, m^2 and g.

This procedure has a bonus. As long as you stick to observable quantities expressed in terms of physical parameters, you avoid the infinities which plague quantum field theory.

There are three technical obstacles we will have to overcome to implement this program.

(1) There are derivative interactions in $\mathcal{L}_{CT}$.

(2) Renormalization conditions 4, 5 and 6 are not expressed in terms of Green's functions, the things we usually compute.

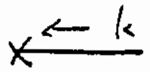
(3) We have to make contact with the committee definition of g. Then we may still have to worry about defining it in terms of Green's functions (2).

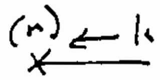
Nov. 18 Lets go into more detail on how $\langle 0|\phi'|0\rangle = 0$ determines A.

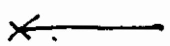
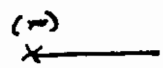
A is going to be some power series in g

$A = \sum_r A_r \qquad A_r \propto g^r$

Diagrammatically,

 corresponds to $iA(2\pi)^4\delta^{(4)}(k)$

 $iA_r(2\pi)^4\delta^{(4)}(k)$

 $= \sum_r$ 

I'll now explain how to determine A order by order in perturbation theory.

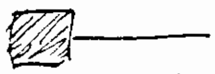
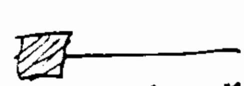
Suppose that we know all Feynman graphs and have determined all counterterms to order g^n.

To determine A to order g^{n+1}, that is to get A_{n+1} we apply the renormalization condition $\langle 0|\phi'|0\rangle = 0$. Graphically,

 $= 0$ We demand this for all values of g, so the coefficent of g^{n+1} in its power series must vanish

↑ at order g^{n+1}

We can break  at $O(g^{n+1})$ into two parts.

 $= \sum$  $+ \sum$ 

↑ at order g^{n+1} — graphs of order g^{n+1} with more than one vertex behind the shield — graphs of order g^{n+1} with only one vertex

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2

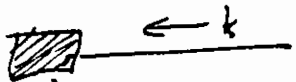
The graphs with more than one vertex behind the shield have a special property. If they are going to be of order g^{n+1} every vertex has to be of order g^n or less. Thus these graphs only contain known stuff, by hypothesis.

The graphs with only one vertex at order g^{n+1} with one external line also have a special property. There is only one of them.

$$\overset{(n+1)}{\times}\!\xleftarrow{\;\;k\;\;}$$

Setting

$$\overset{(n+1)}{\times}\!\xleftarrow{\;\;k\;\;} = -\sum_{\substack{\text{graphs with}\\ \text{more than one vertex behind}\\ \text{the shield of order } g^{n+1}}}$$

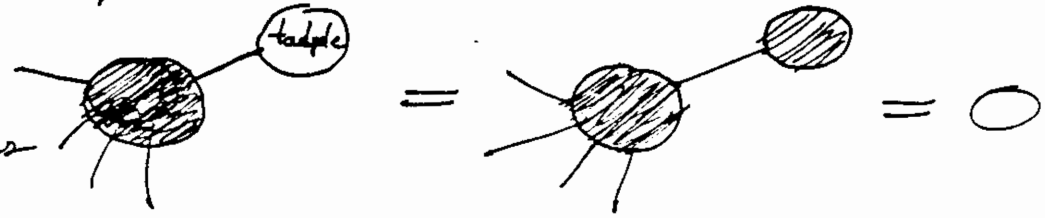


determines A_{n+1}. We can cancel completely a potentially momentum dependent sum of graphs by adjusting a single number because it is always $\propto \delta^{(4)}(k)$. $\xleftarrow{k} = 0$ for all k. Now there is a nice simplification in all graphs because of this cancellation. Suppose we have

a part of a graph that is connected to the rest of the graph by one and only one line is called a tadpole. Physical Reviews editors rejected spermion.

Consider the same anything but summed over all possible tadpoles that can be attached to that same line.

$$\sum_{\substack{\text{all tadpoles}\\ \text{including}\\ \text{counterterms}}}$$



The total result is that you can just ignore all tadpoles. (Unless you cared about A.)

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3

The program for determining $B, \ldots, F$ successively will be similar, but first we have to surmount three obstacles before we can do anything.
↑ Nov. 13, p. 9

(1) Problems with derivative couplings and why they don't arise here.

In the presence of a derivative interaction,

$$\pi^\mu \neq \partial^\mu \phi, \quad \text{in general.}$$

This means that the interaction Hamiltonian is not just $-\mathcal{L}_I$, the interaction Lagrangian.

A second problem is that to get from Dyson's formula to Feynman diagrams, we had to employ the Wick expansion which turns time ordered products of free fields into normal ordered products. The Wick expansion does not apply to derivatives of fields, and we can't pull the derivatives out of the time ordered product

$$T(\partial_\mu \phi(x) \cdots) \neq \partial_\mu T(\phi(x) \cdots)$$

and then apply the Wick expansion

Later we will develop a new method to deal with these problems.

For now, we'll just note that these two problems frequently cancel out, and that in a few simple examples, we can explicitly show this.

Act as if $\mathcal{H}_I = -\mathcal{L}_I$ and as if $\partial_\mu \to ik_\mu$

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4

What do I mean by "cancel out"?

If you are naive, and you act as if $\mathcal{H}_I = -\mathcal{L}_I$ and as if $T(\partial_\mu \phi \cdots) = \partial_\mu T(\phi \cdots)$ you get the right answer.

A simple example

Take the simplest field theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2$$

and introduce a new field $\phi' = Z_3^{-1/2}\phi$, take Z_3 to be arbitrary. In terms of ϕ'

$$\mathcal{L} = Z_3\left[\frac{1}{2}(\partial_\mu \phi')^2 - \frac{\mu^2}{2}\phi'^2\right]$$

$$= \frac{1}{2}(\partial_\mu \phi')^2 - \frac{\mu^2}{2}\phi'^2 + (Z_3 - 1)\left[\frac{1}{2}(\partial_\mu \phi')^2 - \frac{1}{2}\mu^2 \phi'^2\right]$$

The Green's functions of ϕ' are simply related to the Green's functions of ϕ, because $\phi' = Z_3^{-1/2}\phi$.

$$\langle 0|T(\phi(x_1)\cdots\phi(x_n))|0\rangle = Z_3^{n/2}\langle 0|T(\phi'(x_1)\cdots\phi'(x_n))|0\rangle$$

Will show that this holds perturbatively using the naive method above. Actually, first we will only show it for one Green's function, but will be more general in a moment.

Define a connected Green's function, $\tilde{G}_c^{(n)}(k_1, \ldots, k_n)$ to be the sum of all connected graphs with n external lines that contribute to $\tilde{G}^{(n)}(k_1, \ldots, k_n)$. The only nonzero connected Green's function for one scalar field with no interactions is

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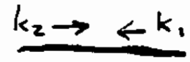
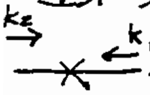
← only contribution is 

$$G_c^{(2)}(k_1,k_2)=(2\pi)^4\delta^4(k_1+k_2)\,\frac{i}{k_1^2-\mu^2+i\epsilon}$$

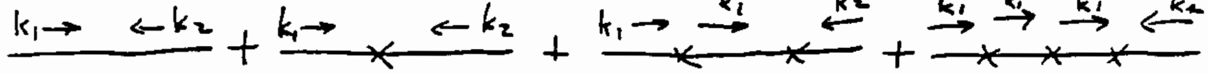
That's the right answer in this exactly soluble theory. What does naive perturbation theory give for $\tilde{G}_c^{(2)\prime}(k_1,k_2)$, the sum of all connected graphs that contribute to $\tilde{G}^{(2)\prime}(k_1,k_2)$.

The interaction is $(Z_3-1)\left[\frac{1}{2}(\partial_\mu\phi')^2-\frac{1}{2}\mu^2\phi'^2\right]$. It has as Feynman rule

 $\leftrightarrow i(Z_3-1)\,\frac{1}{2}\,2!\left((-ik_1^\mu)(-ik_{2\mu})-\mu^2\right)(2\pi)^4\delta^{(4)}(k_1+k_2)$

$$=-i(Z_3-1)(-k_1^2+\mu^2)(2\pi)^4\delta^{(4)}(k_1+k_2)$$

The connected con graphs contributing to $G_c^{(2)\prime}(k_1,k_2)$ are



+ . . . birds on a rail

$$=(2\pi)^4\delta^{(4)}(k_1+k_2)\,\frac{i}{k_1^2-\mu^2+i\epsilon}\left[1+\frac{-i(Z_3-1)(-k_1^2+\mu^2)\,i}{k_1^2-\mu^2+i\epsilon}+\left(\frac{-i(Z_3-1)(-k_1^2+\mu^2)\,i}{k_1^2-\mu^2+i\epsilon}\right)^2+\dots\right]$$

$$=(2\pi)^4\delta^{(4)}(k_1+k_2)\,\frac{i}{k_1^2-\mu^2+i\epsilon}\left[1-(Z_3-1)+(Z_3-1)^2-\dots\right]$$

$$=(2\pi)^4\delta^{(4)}(k_1+k_2)\,\frac{i}{k_1^2-\mu^2+i\epsilon}\,\frac{1}{1+(Z_3-1)}$$

$$=Z_3^{-1}(2\pi)^4\delta^{(4)}(k_1+k_2)\,\frac{i}{k_1^2-\mu^2+i\epsilon}$$

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We have shown,

$$G_c^{(2)\prime}(k_1,k_2) = Z_3^{-1}\, G_c^{(2)}(k_1,k_2)$$

using a naive method, but this agrees with the right result.

A slightly less simple example.

Consider a theory of one scalar meson with arbitrary nonderivative self interactions

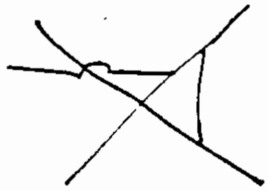
$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}\mu^2\phi^2 + \sum_{r=3}^{N} g_r\phi^r$$

Again let $\quad \phi' = Z_3^{-1/2}\phi$

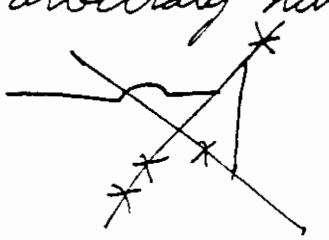
$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi')^2 - \tfrac{1}{2}\mu^2\phi'^2 + (Z_3-1)\left[\tfrac{1}{2}(\partial_\mu\phi')^2 - \tfrac{\mu^2}{2}\phi'^2\right]$$
$$+ \sum_{r=3}^{N} g_r\,\phi'^r\, Z_3^{r/2}$$

We are going to compute a general Green's function in this theory in terms of the Green's functions of the theory without Z_3 by making a graphical relation.

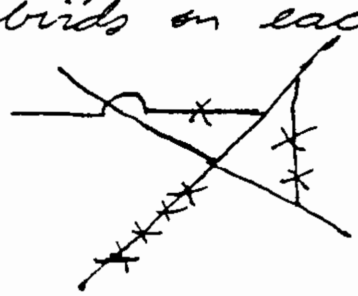
Consider any graph in the Green's function of ϕ.



Corresponding to this graph, the progenitor, there are a whole bunch of graphs in the Green's function of ϕ' which look just like this graph except there are an arbitrary number of birds on each line.



and



etc.

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We can sum up this bunch of graphs with our naive Feynman rule. The only affect of all these birds is to replace each internal and external propagator by Z_3^{-1} times the free propagator. There is also an effect on the value of the graph coming from all those $Z_3^{r/2}$ factors at the vertices. Suppose there are n external lines, I internal lines and V_r vertices with r legs. The graphs we have summed give

$$\underbrace{Z_3^{-n}\, Z_3^{-I}}_{\substack{\text{product of } n+I \\ \text{independent geometric series}}} \prod_r Z_3^{rV_r/2} = Z_3^{-n-I+\sum_r rV_r/2}$$

times ~~the graph it corresponds to~~ progenitor's contribution to $G^{(n)}$. It looks like the contributions to $G'^{(n)}$ depend on Z_3 in a graph dependent way, but we aren't done yet.

There is a conservation law, conservation of ends.

Every external line ends on a vertex. Every internal line has both ends on a vertex. Every r-legged vertex connects to r of these ends. Therefore

$$n + 2I = \sum_r rV_r$$

or

$$-n - I + \sum_r \frac{rV_r}{2} = -\frac{n}{2}$$

The graphs we have summed give $Z_3^{-n/2}$. Since all the contributions to $G'^{(n)}$ have this factor

$$G'^{(n)} = Z_3^{-n/2}\, G^{(n)}$$ as expected.

This is the right result. It justifies the naive treatment we will apply to $\mathcal{L}_{CT}$ in Model 3.

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Overcoming the second obstacle, that renormalization conditions ②, ③, ④ and ⑤ aren't expressed in terms of Green's functions (we will worry about ⑥ later). This will require a study of $G^{(2)}$:

$$G^{(2)\prime}(k_1, k_2) =$$

$$= \int d^4x\, d^4y\, e^{-ik_1\cdot x - ik_2\cdot y} \langle 0 | T(\phi'(x)\phi'(y)) | 0 \rangle$$

Now, $T(\phi'(x)\phi'(y)) = \theta(x^0 - y^0)\,\phi'(x)\phi'(y) + \theta(y^0 - x^0)\,\phi'(y)\,\phi'(x)$

so it is sufficient to study $\langle 0|\phi'(x)\phi'(y)|0\rangle$ and then take this combination at the end. $\langle 0|\phi'(x)\phi'(y)|0\rangle$ is called a Wightman function.

$$\langle 0|\phi'(x)\phi'(y)|0\rangle = \sum_{\substack{\text{complete}\\ \text{set of intermediate}\\ \text{momentum eigenstates, } |n\rangle \\ P_\mu |n\rangle = P_{n\mu}|n\rangle}} \langle 0|\phi'(x)|n\rangle\langle n|\phi'(y)|0\rangle$$

Now, $\langle 0|\phi'(x)|n\rangle = \langle 0| e^{iP\cdot x}\phi'(0)e^{-iP\cdot x}|n\rangle$
$= e^{-iP_n\cdot x}\langle 0|\phi'(0)|n\rangle$

so,

$$\langle 0|\phi'(x)\phi'(y)|0\rangle = \sum_{|n\rangle} e^{-iP_n\cdot(x-y)} |\langle 0|\phi'(0)|n\rangle|^2$$

$$= \underbrace{\cancel{|\langle 0|\phi'(0)|0\rangle|^2}}_{0} + \int \frac{d^3p}{(2\pi)^3 2\omega_{\vec p}} e^{-ip\cdot(x-y)} \underbrace{|\langle 0|\phi'(0)|p\rangle|^2}_{1}$$

one meson

$$+ \sum_{\substack{\text{all other momentum}\\ \text{eigenstates } |n\rangle \text{ besides vacuum and one meson}}} e^{-iP_n\cdot(x-y)} |\langle 0|\phi'(0)|n\rangle|^2$$

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We have broken up the sum into vacuum, one meson and all other intermediate states and applied renormalization conditions ① and ②. We have an name for

$$\int \frac{d^3p}{(2\pi)^3 2\omega_{\vec p}} e^{-ip\cdot(x-y)} \quad \text{it is} \quad \Delta_+(x-y,\mu^2)$$

μ^2, the physical meson mass^2 is what comes out here. It is in $\omega_{\vec p}$ ($\omega_{\vec p} = \sqrt{\vec p^2+\mu^2}$), and it comes from inserting physical one meson momentum eigenstates.

Let's massage the sum over all other momentum eigenstates.

$$\sum_{\text{all other } |n\rangle} e^{-iP_n\cdot(x-y)} |\langle 0|\phi'(0)|n\rangle|^2$$

$$= \sum_{\text{all other } |n\rangle} e^{-iP_n\cdot(x-y)} \int d^4p \, \delta^4(p-P_n) \, |\langle 0|\phi'(0)|n\rangle|^2$$

The integral over p is just a fancy way of writing 1, but now we can do something tricky with it. Take $e^{-iP_n\cdot(x-y)}$ inside the p integration and rewrite it as $e^{-ip\cdot(x-y)}$. We have

$$\int d^4p \, e^{-ip\cdot(x-y)} \underbrace{\sum_{\text{all other } |n\rangle} \delta^{(4)}(p-P_n) |\langle 0|\phi'(0)|n\rangle|^2}$$

To agree with unfortunate but longstanding conventions, we are abandoning our 'every p integration gets a $\frac{1}{2\pi}$, every δ function gets a 2π' rule.

This is a manifestly Lorentz invariant function of p, that vanishes when $p_0 < 0$. It is conventionally called

$$\frac{1}{(2\pi)^3} \sigma(p^2)\theta(p^0)$$

$$= \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot(x-y)} \sigma(p^2)\theta(p^0)$$

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The density $\sigma(p^2)$ has some definite properties.[10] It is always ≥ 0. In perturbation theory, it equals zero if $p^2 \leq \min(4m^2, 4\mu^2)$, because there are no bound states in P.T. Outside of P.T., it still is zero for $p^2 <$ mass^2 of the lightest neutral bound state, call it $\mu^2+\epsilon$, $\epsilon > 0$. (If the lightest neutral bound state has a mass less than the meson mass, then that is what we would be calling the meson.)

So what we have found so far is

$$\langle 0|\phi'(x)\phi'(y)|0\rangle = \Delta_+(x-y,\mu^2) + \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot(x-y)}\sigma(p^2)\theta(p^0)$$

$$= \Delta_+(x-y,\mu^2) + \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot(x-y)} \int_0^\infty da^2\, \delta(a^2-p^2)\, \sigma(a^2)\theta(p^0)$$

$$= \Delta_+(x-y,\mu^2) + \int_0^\infty da^2\, \sigma(a^2)\, \Delta_+(x-y, a^2)$$

Sometimes $\rho(a^2) \equiv \delta(a^2-\mu^2) + \sigma(a^2)$ is used

$$\sigma(a^2) \geq 0, \qquad \sigma(a^2) = 0 \quad \text{for} \quad a^2 < \mu^2+\epsilon$$

This is the Lehmann – Källén spectral decomposition
"Chalain"

We can use this to make a statement about Z_3 using $\phi' = Z_3^{-1/2}\phi$ and the fact that ϕ obeys the canonical commutation relations.

$$\langle 0|[\phi'(\vec{x},t), \dot{\phi}'(\vec{y},t)]|0\rangle = Z_3^{-1} i\delta^{(3)}(\vec{x}-\vec{y}) \quad \text{by c.c.r.}$$

$$\langle 0|[\phi'(\vec{x},t), \dot{\phi}'(\vec{y},t)]|0\rangle = i\delta^{(3)}(\vec{x}-\vec{y}) + \int_0^\infty da^2\, \sigma(a^2)\, i\delta^{(3)}(\vec{x}-\vec{y})$$

by using $\frac{\partial}{\partial y^0}\Delta_+(\vec{x}-\vec{y}) = i\delta^{(3)}(\vec{x}-\vec{y})$

$$\therefore \quad Z_3^{-1} = 1 + \int_0^\infty da^2\, \sigma(a^2) \geq 1$$

$\therefore$ In general $Z_3 < 1$. We will show that if $Z_3 = 1$, you have free field theory later in the course.

$Z_3 = 1 + Ag^2 + O(g^4)$

* $[\phi_{in}, \dot{\phi}_{in}] = i\delta^{(3)}(\vec{x}-\vec{y})$ because ϕ, π can be changed to $\phi_{in}\ \pi_{in}$ by a canonical trans.

$D'(k^2)$ $(k^2-\mu^2)D'(k^2) = i$ at $k^2=\mu^2$

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We set out to study $G^{(2)\prime}(k,k')$. What we have shown implies that

$$G^{(2)\prime}(k,k') = (2\pi)^4\,\delta^{(4)}(k+k') \times\left(\frac{i}{k^2-\mu^2+i\epsilon} + \int_0^\infty da^2\,\sigma(a^2)\,\frac{i}{k^2-a^2+i\epsilon}\right)$$

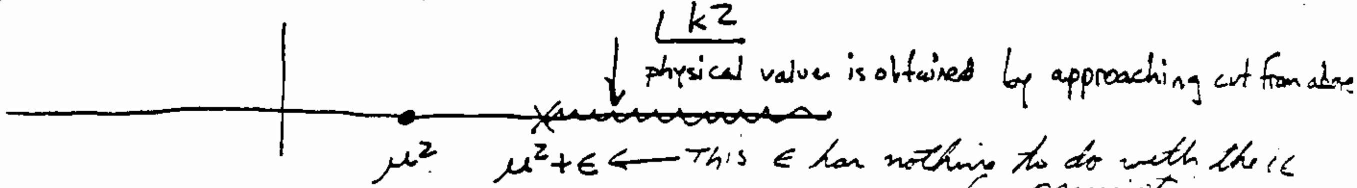
$$= (2\pi)^4\,\delta^{(4)}(k+k')\; D'(k^2)$$

$D'(k^2)$ = "renormalized propogator"

$$= \frac{i}{k^2-\mu^2+i\epsilon} + \int_0^\infty da^2\,\sigma(a^2)\,\frac{i}{k^2-a^2+i\epsilon}$$

Note that ~~Useful fact for problem 4?~~
$[iD'(p^2)]^* = -iD'(p^{2*})$
Schwarz reflection property

This is a highly nontrivial expression. It defines a function everywhere in the complex k^2 plane (even though the propogator was not originally defined there. The function is analytic except at $k^2=\mu^2$ where it has a pole with residue i and along the positive real axis beginning at $k^2=\mu^2+\epsilon$, where it has a branch cut. The value on the positive real axis is given by the $i\epsilon$ prescription, which says you take the value just above the cut.



Our renormalization conditions, ② and ④ are encoded in this function.

④ The meson mass is $\mu \iff D'$ has a pole at μ^2
② $\langle 0|\phi'(0)|q\rangle = 1 \iff$ The residue at this pole is i

look back and see where ② was used in the derivation of the expression for D'

(2) $\text{Im}\,k^2 \to 0^+$
(3) $-iD'(k^2) = [-iD'(k^{2*})]^*$
(4) $\lim_{\epsilon\to 0^+}\left(\frac{1}{x-i\epsilon} - \frac{1}{x+i\epsilon}\right) = 2\pi i\,\delta(x)$
On
$\text{Im}[-iD'(k^2)] = -\pi\sigma(k^2)$

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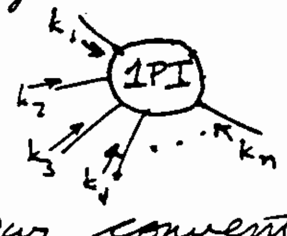
We are going to keep massaging $G^{(2)\prime}$ to find a slicker statement of our renormalization conditions

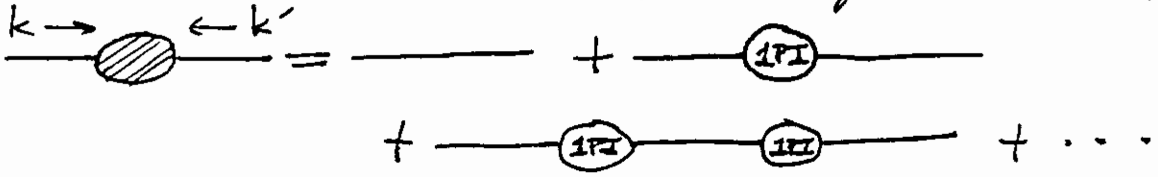
Define another new kind of Green's function, the one particle irreducible Green's function. Again it will be defined graphically.



≡ the sum of all connected graphs that cannot be disconnected by cutting a single internal line

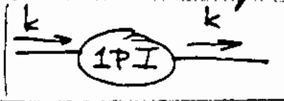
Our convention will be that this does not include the overall energy momentum conserving δ function or the external propogators.

The cute thing about this Green's function when $n=2$ is the following expression for $G^{(2)\prime}$



That is, nothing can happen, or we can have an interaction, but before we get to the other external line there is never a point where we get just one line, or there is only one line like that or ...

By def'n, the LHS is $(2\pi)^4 \delta^{(4)}(k+k') D'(k^2)$.
If we define



$= -i\Pi'(k^2)$ $\qquad \Pi'(k^2) =$ "self-energy"

we can sum the series on the RHS. It is

$$\frac{i}{k^2-\mu^2+i\epsilon}\left[1+\frac{\Pi'(k^2)}{k^2-\mu^2+i\epsilon}+\left(\frac{\Pi'(k^2)}{k^2-\mu^2+i\epsilon}\right)^2+\cdots\right](2\pi)^4\delta^{(4)}(k+k')$$

$$\frac{i}{k^2-\mu^2+i\epsilon}\;\frac{1}{1-\frac{\Pi'(k^2)}{k^2-\mu^2+i\epsilon}}\,(2\pi)^4\delta^{(4)}(k+k') = \frac{i}{k^2-\mu^2-\Pi'(k^2)+i\epsilon}(2\pi)^4\delta^{(4)}(k+k')$$

Now you can see why $\Pi'(k^2)$ is called the "self-energy." It is like a momentum dependent mass. Identifying the coefficient of $(2\pi)^4\delta^{(4)}(k+k')$ on the LHS and RHS,

$$\boxed{D'(k^2) = \frac{i}{k^2-\mu^2-\Pi'(k^2)+i\epsilon}}$$

Now for the slick rephrasing of the renormalization conditions:

D' has a pole at $\mu^2 \iff \Pi'(\mu^2)=0$

The residue of this pole is $i \iff \left.\frac{d\Pi'}{dk^2}\right|_{k^2=\mu^2}=0$

Perhaps this is easier to see if you think of expanding $\Pi'(k^2)$ around $k^2=\mu^2$ in a power series:

$$\Pi'(k^2) = \Pi'(\mu^2) + \left.\frac{d\Pi'}{dk^2}\right|_{\mu^2}(k^2-\mu^2) + \cdots$$

These two terms must vanish or it screws up the location and residue of the pole.

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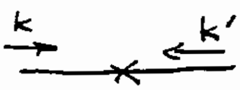
Having succeeded in expressing renormalization conditions (2) and (4) as statements about the 1PI two-point function, I'll now explain how to determine B and C order by order in perturbation theory (those of you that have taken quantum field theory once or twice before probably recognize that this is going to be a rerun of the argument for determining A).

$$\mathcal{L}_{CT} = \cdots + \frac{1}{2}B(\partial_\mu\phi')^2 - \frac{1}{2}C\phi'^2 + \cdots$$

$$= -i\Pi'(k^2) \qquad \Pi'(\mu^2) = 0$$

$$\left.\frac{d\Pi'}{dk^2}\right|_{\mu^2} = 0$$

We can express the Feynman rule for the B and C counterterms together as



corresponds to $i(2\pi)^4\delta^{(4)}(k+k')(-Bk\cdot k' - C)$

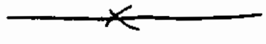
$$= i(2\pi)^4\delta^{(4)}(k+k')(Bk^2 - C)$$

Writing B and C as power series expansions

$$B = \sum_r B_r \qquad B_r \propto g^r$$

$$C = \sum_r C_r \qquad C_r \propto g^r$$

we can also write



$$= \sum_r$$

corresponds to $i(2\pi)^4\delta^{(4)}(k+k')(B_r k^2 - C_r)$

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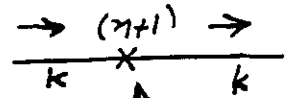
Assume everything is known to $O(g^n)$, including all the counterterms, and we'll show that B_{n+1} and C_{n+1} can be determined.

at order g^{n+1} = known stuff + 

known stuff: sum of all 1PI graphs with more than one vertex at order g^{n+1}

the only $O(g^{n+1})$ 1PI graph with only one vertex

$$iB_{n+1}\mu^2 - iC_{n+1} = -(\text{known stuff})\Big|_{\mu^2}$$

$$iB_{n+1} = -\frac{d(\text{known stuff})}{dk^2}\Bigg|_{k^2=\mu^2}$$

Similar arguments apply to the nucleon self energy.

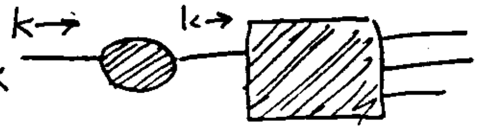
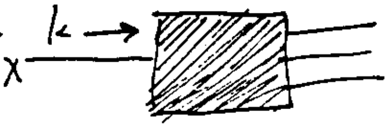
$= -i\Sigma'(p^2)$

which can be used to express renormalization conditions (3) and (5) as

$$\Sigma'(m^2) = 0$$

$$\frac{d\Sigma'}{dp^2}\Bigg|_{p^2=m^2} = 0$$

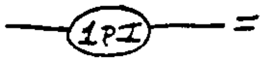
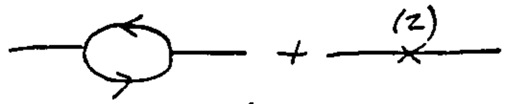
Of course these subtractions are not going to allow you to ignore corrections to the 1PI two point function. $\xrightarrow{k}$ (1PI) $\xrightarrow{k}$ has a complicated momentum dependance in general, which is not eliminated by just subtracting a constant and a term linear in k^2. However, this does allow you to ignore corrections <u>to lines on the mass shell</u>, that is external lines in the computation of S matrix elements. That is because in the computation of an S matrix element, the only thing that matters about an external line is the location and residue of its pole.

$\lim\limits_{k^2\to\mu^2} \frac{k^2-\mu^2}{i} \times$  $= \lim\limits_{k^2\to\mu^2} \frac{k^2-\mu^2}{i} \times$ 

The location and residue of the pole in the full propogator is, in renormalized perturbation theory, the exact same as that of the free propogator.

We can do some examples before worrying about obstacle (3), that is renormalization condition ⑥.

Calculation of $\pi'(k^2)$ to order g^2

$-i\pi'(k^2) =$  $=$ 

$$= -i\pi_f(k^2) + iB_2 k^2 - iC_2$$

where $-i\pi_f(k^2) \equiv$ 

The renormalization conditions are

$$\pi_f(\mu^2) - B_2\mu^2 + C_2 = 0 \qquad \left.\frac{d\pi_f}{dk^2}\right|_{\mu^2} - B_2 = 0$$

If you don't care what B_2 and C_2 are, these can be rephrased as

$$\pi'(k^2) = \pi_f(k^2) - \pi_f(\mu^2) - (k^2-\mu^2)\left.\frac{d\pi_f}{dk^2}\right|_{\mu^2}$$

We should check that B_2 and C_2 are real however.

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$$-i\Pi_f(k^2) =$$  $$= (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(q+k)^2 - m^2 + i\epsilon}$$

There are three problems in doing this integral.

(1) Not spherically symmetric. I suppose we could parametrize the integral with a polar angle measured from k but,

(2) We are in Minkowski space, and it isn't even spherical symmetry we have.

(3) The integral is divergent; at high q it looks like $\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4}$ which if the integral was spherically symetric would be $\sim \int \frac{q^3\,dq}{q^4}$, and if it was cut off at some large radius in momentum space Λ, would be $\sim \ln \Lambda$. (This is called log divergent.)

This last problem is the easiest to take care of.

$\Pi_f(k^2) - \Pi_f(\mu^2)$ is not divergent.

Renormalized perturbation theory, which was implemented to make expansions in the right parameters has saved us from this unexpected infinity.

To make this thing manifestly spherically symmetric (actually L.I.) we use Feynman's trick for combining two denominators.

$$\int_0^1 dx \frac{1}{[ax+b(1-x)]^2} = \frac{1}{b-a}\,\frac{1}{ax+b(1-x)}\Big|_0^1$$

$$= \frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}$$

apply this to the two denominators in Π_f, with

$$a = (q+k)^2 - m^2 + i\epsilon \qquad b = q^2 - m^2 + i\epsilon$$

$$-i\Pi_f(k^2) = g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{1}{[((q+k)^2-m^2+i\epsilon)x + (q^2-m^2+i\epsilon)(1-x)]^2}$$

$$= g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \frac{1}{[q^2 + xk^2 + 2k\cdot q x - m^2 + i\epsilon]^2}$$

$q' = q + kx$

$$= g^2 \int_0^1 dx \int \frac{d^4q'}{(2\pi)^4} \frac{1}{[q'^2 + k^2x - k^2x^2 - m^2 + i\epsilon]^2}$$

We could do this integral in a moment if we were living in Euclidean space. It is not spherically symmetric though.

So now will study integrals of the form

$$I_n(a) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + \underset{+i\epsilon}{a})^n} = \int \frac{d^3q\, dq^0}{(2\pi)^4} \frac{1}{(q^{0\,2} - \vec{q}^{\,2} + a)^n}$$

where a has a positive imaginary part.

(the case of interest has $n=2$, $a = k^2x(1-x) - m^2 + i\epsilon$)

$$\int d^4q \left[\frac{1}{(q^2+a)^2} - \frac{1}{(q^2+b)^2}\right] = \int d^4q \int_b^a dy \frac{d}{dy}\frac{1}{(q^2+y)^2}$$

$$= \int_b^a dy \int d^4q\, (-2)\frac{1}{(q^2+y)^3} = \int_b^a dy\, (-2)\frac{\pi^2}{\underbrace{\Gamma(3)}_{2}}\Gamma(1)\frac{1}{y} = \pi^2 \ln\frac{b}{a}$$

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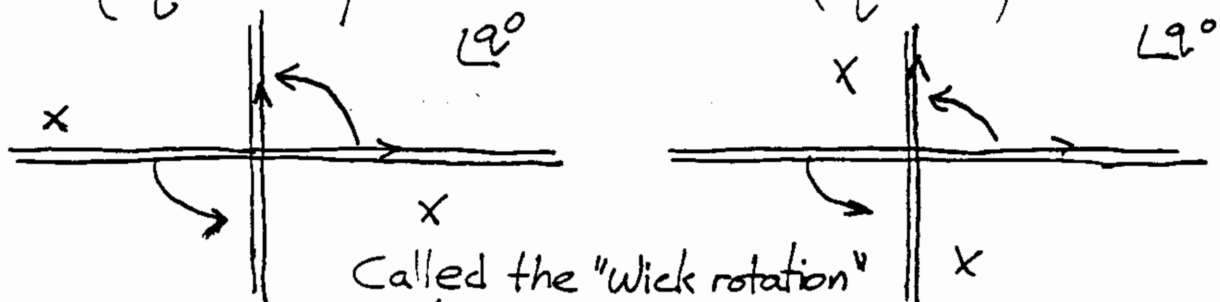
The location of the poles in the q^0 integration splits into two cases

$\mathrm{Re}(\vec{q}^{\,2} - a) > 0$ $\qquad\qquad$ $\mathrm{Re}(\vec{q}^{\,2} - a) < 0$



Called the "Wick rotation"

In either case, the contour can be rotated as shown, so that it runs up the imaginary q^0 axis. Because this rotation does not cross any poles the value of the integral is unchanged.
Now that q^0 runs from $-i\infty$ to $+i\infty$ define a new variable q_4 that runs from $-\infty$ to ∞

$$q_4 = -iq^0 \qquad dq^0 = i\,dq_4 \qquad d^4q = i\,d^4q_E$$
$$= i\,dq_4\,d^3q$$

$$I_n(a) = i\int \frac{d^4q_E}{(2\pi)^4}\frac{1}{[-q_4^2 - \vec{q}^{\,2} + a]^n} = i\int\frac{d^4q_E}{(2\pi)^4}\frac{1}{(-q_E^2 + a)^n}$$

This is now a spherically symmetric integral in 4-d Euclidean space. Using $V(S^3) = 2\pi^2$ and setting $z = q_E^2$, $q_E^3\,dq_E = \frac{1}{2}z\,dz$, we have

$$I_n(a) = i\,\frac{\pi^2}{(2\pi)^4}\int_0^\infty z\,dz\,\frac{1}{(-z+a)^n}$$

$$= \frac{i}{16\pi^2}\frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{da^{n-1}}\int_0^\infty z\,dz\,\frac{1}{-z+a} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}I_1(a)}{da^{n-1}}$$

limit $d \to 4$ $\pi^{d/2} \to \pi^2$ $\Gamma(\delta) = \Gamma(2) = 1$ $\Gamma(2-\frac{d}{2}) \Rightarrow \frac{1}{2-d/2}$

$a^{d/2-2} = e^{(\frac{d}{2}-2)\ln a} = 1 + (\frac{d}{2}-2)\ln a + \cdots$

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This is only a formal expression because $I_1(a) = \int_0^\infty z\,dz \frac{1}{-z+a}$ has a divergent part. If we cut the integral off at some large value Λ^2 (in a bit we'll send $\Lambda \to \infty$) we have

$$I_1(a) = \int_0^{\Lambda^2} dz\, \frac{z-a+a}{-z+a} = \int_0^{\Lambda^2} dz \left(-1 + \frac{a}{-z+a}\right)$$

For large z, the integrand is $-1 - \frac{a}{z} + O\left(\frac{a^2}{z^2}\right)$

I'll evaluate $I_1(a)$ in a way which is only valid when the integral is part of a convergent combination

$$\int_0^\infty z\,dz \sum_n \frac{c_n}{-z+a_n} \quad \text{where } \sum_n c_n = 0 \text{ and } \sum_n a_n c_n = 0$$

This will guarantee that those first two terms in the integrand of order 1 and $\frac{1}{z}$ have coefficient zero.

$$I_1(a) = \frac{-i}{16\pi^2} \lim_{\Lambda\to\infty} \int_0^{\Lambda^2} dz \left(1 + \frac{a}{z-a}\right)$$

$$= \frac{-i}{16\pi^2} \lim_{\Lambda\to\infty} \Big[z + a\ln(z-a) \Big]\Big|_0^{\Lambda^2}$$

$$= \frac{-i}{16\pi^2} \lim_{\Lambda\to\infty} \left[\Lambda^2 + a\ln\Lambda^2\left(1 + O\left(\frac{a}{\Lambda^2}\right)\right) - a\ln(-a) \right]$$

for our purposes

$$= \frac{i}{16\pi^2} a\ln(-a)$$

vanishes in convergent combinations

0 in $\Lambda \to \infty$ limit

$$\int d^d q \left[\frac{1}{(q^2+a)^2} - \frac{1}{(q^2+b)^2} \right] = \frac{\pi^2}{2-\frac{d}{2}} \left[1 + (\tfrac{d}{2}-2)\ln a - 1 - (\tfrac{d}{2}-2)\ln b + O\big((\tfrac{d}{2}-2)^2\big) \right]$$

$$\underset{d\to 4}{=} -\pi^2 \ln\frac{a}{b}$$

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What about $I_2(a)$?

$$-16\pi^2 i\, I_2(a) = \int_0^\infty z\,dz\ \frac{1}{(z-a)^2}$$

$$= \int_0^\infty dz\ \frac{z-a+a}{(z-a)^2} = \int_0^\infty dz\left(\frac{1}{z-a} + \frac{a}{(z-a)^2}\right)$$

For large z the integrand is $\frac{1}{z} + O\left(\frac{1}{z^2}\right)$
what follows is only valid in either of two cases

I. The integral is part of a convergent combination

$$\int_0^\infty z\,dz\ \Sigma_n \frac{c_n}{(z-a_n)^2} \quad \text{where } \Sigma_n c_n = 0$$

II. you plan to differentiate I_2 with respect to a to get I_3, I_4, etc.

As before we make sense of $I_2(a)$ by itself by cutting the integral off at some large value Λ^2. The limit $\Lambda \to \infty$ will be taken at end.

$$I_2(a) = \frac{i}{16\pi^2}\int_0^{\Lambda^2} dz\left(\frac{1}{z-a} + \frac{a}{(z-a)^2}\right)$$

$$= \frac{i}{16\pi^2}\left[\ln(z-a) - a\frac{1}{z-a}\right]_0^{\Lambda^2}$$

$$= \frac{i}{16\pi^2}\left[\ln(\Lambda^2)\left(1+\cancel{O\left(\frac{a}{\Lambda^2}\right)}\right) - \cancel{\frac{a}{\Lambda^2}\left(1+O\left(\frac{a}{\Lambda^2}\right)\right)} - \ln(-a) - \cancel{1}\right]$$

vanishes   0 in $\Lambda \to \infty$ limit   vanishes

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Let's see why the terms I claim vanish, vanish in either case.

I. The condition $\sum_n c_n = 0$ which was put in to make the coefficient of $\frac{1}{\epsilon}$ vanish makes the infinite terms as $\Lambda \to \infty$ vanish. It also gets rid of the -1, since that is independent of a.

II. $\ln \Lambda^2$ and 1 are both constants independent of a. Taking a derivative w.r.t. a eliminates these terms.

So $I_2(a) = \frac{-i}{16\pi^2} \ln(-a)$ for our purposes.

Note that if you take $-\frac{dI_1(a)}{da}$ to get $I_2(a)$ you get $\frac{-i}{16\pi^2}(\ln(-a)+1)$, and the 1 that vanishes in convergent combinations or when differentiated to get I_3, I_4, etc., can be checked.

Let's get I_3, I_4, etc. For $n \geq 3$,

$$I_n(a) = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1} I_1(a)}{da^{n-1}} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} \left(\frac{i}{16\pi^2} a \ln(-a) \right)$$

$$= \frac{i}{16\pi^2} \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-2}}{da^{n-2}} \left(\ln(-a) + 1 \right) = \frac{i}{16\pi^2} \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-3}}{da^{n-3}} \left(\frac{1}{a} \right)$$

$$= \frac{i}{16\pi^2} (-1)^{n-1} (-1)^{n-3} \frac{(n-3)!}{(n-1)!} \frac{1}{a^{n-2}} = \frac{i}{16\pi^2} \frac{1}{(n-1)(n-2)a^{n-2}}$$

These facts are summarized on the following table of integrals.

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Physics 253a Integral Table

The Minkowski-space integral,

$$I_n(a) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2+a)^n},$$

with n integer and Im a $>$ 0, is given by

$$I_n(a) = i\,\left[16\pi^2(n-1)(n-2)a^{n-2}\right]^{-1},$$

for n $\geq$ 3. For n = 1, 2,

$$I_1 = \frac{i}{16\pi^2}\, a \ln(-a) + \cdots,$$

and

$$I_2 = \frac{-i}{16\pi^2} \ln(-a) + \cdots,$$

where the triple dots indicate terms that cancel in a sum of such terms such that the total integrand vanishes for high q more rapidly than q^{-4}.

$\int_0^\infty$ [illegible] Re $z > 0$

$\Gamma(n+1) = n!$ $n = 0, 1, \ldots$

$z \to -n$ $\Gamma(z) = \frac{(-1)^n}{n!\,(z+n)}$ + analytic

$\int d^4q \int_0^\infty dt\, [illegible]\, e^{-t(q^2+a)}$

$\int_0^\infty dt\, t^{z-1-d/2}\, e^{-ta}$

[illegible] $= \int \frac{d^dq}{(q^2+a)^z}$

The logarithm is defined by $\int_1^x \frac{1}{s} ds$.
Q. What is it for complex x?
A. By analytic continuation by branch cut on negative [illegible] axis. Q. So branch cut does this [illegible] not [illegible] to [illegible] field [illegible].

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"

On page 5, we had an expression for Π_f to which we can apply our expression for I_2, with

$$a = k^2 x - k^2 x^2 - m^2 + i\epsilon$$

$$\Pi_f(k^2) = \frac{g^2}{16\pi^2}\int_0^1 dx \; \ln\left(-k^2 x(1-x) + m^2 - i\epsilon\right)$$

+ terms that vanish in convergent combs.

On page 3, we had an expression for Π' in terms of Π_f.

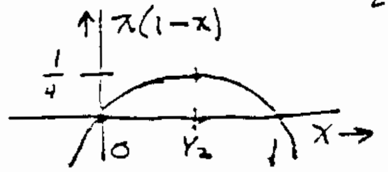
$$\Pi'(k^2) = \Pi_f(k^2) - \Pi_f(\mu^2) - (k^2-\mu^2)\left.\frac{d\Pi_f}{dk^2}\right|_{\mu^2}$$ This is a convergent combination

$$= \frac{g^2}{16\pi^2}\int_0^1 dx\left\{\ln\frac{-k^2x(1-x)+m^2-i\epsilon}{-\mu^2 x(1-x)+m^2} + \frac{(k^2-\mu^2)(+x(1-x))}{-\mu^2 x(1-x)+m^2}\right\}$$

This thing, $\Pi_f(\mu^2)$, which was subtracted off of $\Pi_f(k^2)$, corresponds to the mass counterterm in $\mathcal{L}$, $-\frac{1}{2}C\phi'^2$. It is infinite, C is infinite, the bare mass of the meson is infinite.

However, that is unimportant. The bare mass of the meson does not enter into any expression relating physical quantities. We should worry if whether this infinite term we have stuck into $\mathcal{L}$ is real.

The only way the expression for $\Pi_f(\mu^2)$ (and $\left.\frac{d\Pi_f}{dk^2}\right|_{\mu^2}$) gets an imaginary part is when the argument of the logarithm in the integral becomes negative, which can happen for ranges of x within $[0,1]$ if $\mu^2 > 4m^2$. This can be seen by graphing $x(1-x)$



Of course in this case we have no business treating the meson as a stable particle anyway.

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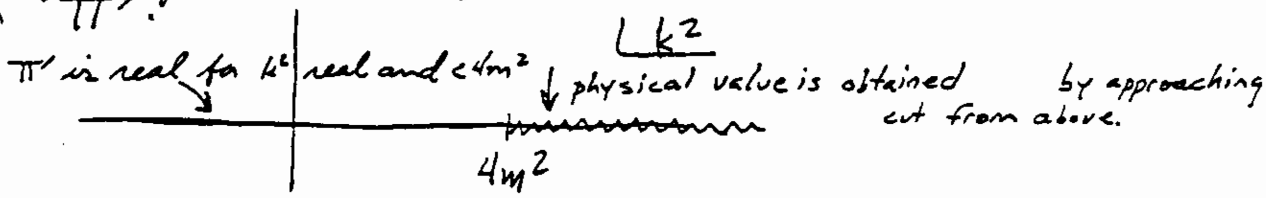
Recall that (NOV. 18, p. 11) we have already found the analytic structure of $D'(k^2)$, and

$$D'(k^2) = \frac{i}{k^2-\mu^2-\Pi'(k^2)+i\epsilon} \qquad [iD'(k^2)]^* = iD'(k^{2*}) \Rightarrow \Pi'(k^2)^* = \Pi'(k^{2*})$$

Let's look at the analytic structure of $\Pi'(k^2)$ to second order in perturbation theory, the function we have just obtained an expression for for real k^2, but which can be defined by this expression for complex k^2.

$$\Pi'(k^2) = \frac{g^2}{16\pi^2}\int_0^1 dx \left\{ \ln \frac{-k^2x(1-x)+m^2-i\epsilon}{-\mu^2x(1-x)+m^2} + \frac{(k^2-\mu^2)\,x(1-x)}{-\mu^2x(1-x)+m^2} \right\}$$

This expression is not only well defined, it is analytic for $\mathrm{Im}\,k^2 \neq 0$. It is also analytic for $\mathrm{Im}\,k^2 = 0$ $-\infty < k^2 < 4m^2$, but starting at $4m^2$, because the branch cut in the logarithm needs to be defined for ranges of $x \in [0,1]$, there is a branch cut in Π'.



The $i\epsilon$ prescription when k^2 is real and greater than $4m^2$ says to define the logarithm and hence Π' by approaching the cut from above. Compare this with the analytic structure of D' and you'll see that perturbation theory is satisfying formulas obtained outside of perturbation theory. (D' had a pole at μ^2. This is in agreement. When you invert D' to get Π' you get a zero.) This and the fact that our counterterm was real when $\mu^2 < 4m^2$ (a requirement necessary for the existence of a physical meson), are satisfying consistency checks. All right theories are internally consistent. (However, all internally consistent theories are not right.)

LOOP LORE

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How do you generalize the wonderful tricks done here to graphs with more propagators and more loops.

We'll introduce Feynman parameters, $n-1$ of them if there are n propagators, that combine all the propagators into one denominator. Then we'll be able to do a shift and a Wick rotation, and then a spherically symmetric integral. What remains and is very difficult is the integration over the Feynman parameters. In difficult but important applications those integrations are done accurately by computer.

Combining denominators

$$\prod_{r=1}^{n} \frac{1}{a_r + i\epsilon} = \prod_r \left[-i \int_0^\infty d\beta_r \, e^{i\beta_r (a_r + i\epsilon)} \right]$$

$$= (-i)^n \int_0^\infty d\beta_1 \cdots d\beta_n \, e^{i\Sigma_r \beta_r (a_r + i\epsilon)} \underbrace{\int_0^\infty d\lambda \, \delta(\lambda - \Sigma_s \beta_s)}_{\text{Fancy way of inserting 1 into the integrand}}$$

$$= (-i)^n \int_0^\infty d\lambda \int_0^\infty d\beta_1 \cdots d\beta_n \; e^{i\Sigma_r \beta_r (a_r + i\epsilon)} \; \delta(\lambda - \Sigma_s \beta_s)$$

Now for some rescalings. First rewrite

$$\delta(\lambda - \Sigma_s \beta_s) \quad \text{as} \quad \frac{1}{\lambda} \, \delta\!\left(1 - \frac{\Sigma \beta_s}{\lambda}\right)$$

Then introduce new integration variables $\alpha_i = \dfrac{\beta_i}{\lambda}$

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$$\prod_{r=1}^{n}\frac{1}{a_r+i\epsilon}=(-i)^n\int_0^\infty d\lambda\int_0^\infty d\alpha_1\cdots d\alpha_n\,\lambda^{n-1}e^{i\lambda\Sigma\alpha_r(a_r+i\epsilon)}\delta(1-\Sigma_i\alpha_i)$$

$$=\int_0^\infty d\alpha_1\cdots d\alpha_n\,\delta(1-\Sigma_i\alpha_i)\underbrace{(-i)^n\int_0^\infty\lambda^{n-1}d\lambda\, e^{i\lambda\Sigma\alpha_r(a_r+i\epsilon)}}_{\frac{(-i)^n}{[-i\Sigma_r\alpha_r(a_r+i\epsilon)]^n}(n-1)!}$$

So,

$$\prod_{r=1}^{n}\frac{1}{a_r+i\epsilon}=(n-1)!\int_0^1 d\alpha_1\cdots d\alpha_n\,\delta(1-\Sigma_i\alpha_i)\frac{1}{[\Sigma_r\alpha_r(a_r+i\epsilon)]^n}$$

where I have noticed that the δ function vanishes whenever any of the α_i are greater than one, and used that to stop the α_i integrations at 1. This is a nice symmetric form; easy to remember. However, we can use the δ function to do the integral over one of the Feynman parameters, leaving $n-1$ of them, as advertised.

$$\prod_{r=1}^{n}\frac{1}{a_r+i\epsilon}=(n-1)!\int_0^1 d\alpha_1\int_0^{1-\alpha_1}d\alpha_2\int_0^{1-\alpha_1-\alpha_2}d\alpha_3\cdots\int_0^{1-\alpha_1-\alpha_2-\cdots-\alpha_{n-2}}d\alpha_{n-1}$$

$$\frac{1}{\left[\Sigma_{r=1}^{n-1}\alpha_r a_r+(1-\Sigma_{r=1}^{n-1}\alpha_r)a_n+i\epsilon\right]^n}$$

This generalizes the result of p.5 of this lecture. Take $n=2$, $\alpha_1=x$ and you have

$$\frac{1}{A_1+i\epsilon}\,\frac{1}{A_2+i\epsilon}=\int_0^1 dx\,\frac{1}{[A_1x+A_2(1-x)+i\epsilon]^2}$$

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A shorter derivation using the Γ function

Uses: $\Gamma(x) \equiv \int_0^\infty dt\, t^{x-1} e^{-t}$ $\qquad \Gamma(n+1) = n!$

Feynmann parameters $\qquad A_j$ real $\quad \alpha_j > 0$

$$I = \int_0^\infty dt\, t^{\alpha-1} e^{-At} = \frac{1}{A^\alpha}\Gamma(\alpha)$$

$$\frac{1}{\prod_j (A_j^{\alpha_j})} = \prod_j \frac{\int_0^\infty dt_j\, t_j^{\alpha_j - 1} e^{-A_j t_j}}{\Gamma(\alpha_j)} \cdot \underbrace{\int_0^\infty ds\, \delta(s - \Sigma t_j)}_{\text{Fancy way of writing } I}$$

$t_j = s x_j$

$$= \int_0^\infty ds \prod_j \frac{\int_0^\infty s^{\alpha_j} dx_j\, x_j^{\alpha_j - 1} e^{-sA_j x_j}}{\Gamma(\alpha_j)} \cdot \frac{\delta(1 - \Sigma x_j)}{s}$$

$$= \prod_j \frac{\int_0^1 dx_j\, x_j^{\alpha_j - 1}}{\Gamma(\alpha_j)} \cdot \int_0^\infty ds\, s^{(\Sigma_j \alpha_j) - 1} e^{-s\Sigma_j x_j A_j}$$

$$= \prod_j \frac{\int_0^1 dx_j\, x_j^{\alpha_j - 1}}{\Gamma(\alpha_j)} \cdot \frac{\Gamma(\Sigma_j \alpha_j)}{(\Sigma_j x_j A_j)^{\Sigma_j \alpha_j}}$$

$$\boxed{\frac{1}{\prod_j A_j^{\alpha_j}} = \frac{\Gamma(\Sigma_j \alpha_j)}{\prod_j \Gamma(\alpha_j)} \int dx_j \cdots dx_j\, \delta(1 - \Sigma_j x_j) \frac{\prod_j x^{\alpha_j - 1}}{(\Sigma_j x_j A_j)^{\Sigma \alpha_j}}}$$

EXAMPLES ① $\displaystyle \frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}$

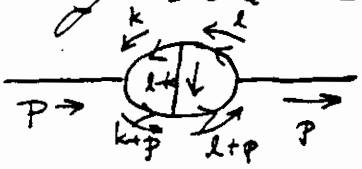
② Take $A_j = 1$ $\qquad \displaystyle \frac{\prod_j \Gamma(\alpha_j)}{\Gamma(\Sigma_j \alpha_j)} = \int \delta(1 - \Sigma x) \prod_j dx_j\, x_j^{\alpha_j - 1}$

generalized binomial expansion

③ Beta function $\qquad \displaystyle \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} = \int dx\, x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}$

Now that we have introduced the Feynman parameters into the integral, how do we make the loop integrations trivial

Suppose we have a graph with I internal lines, and L loops, that is L momentum integrals still left to be done after using the energy-momentum conserving δ fns.



has $L=2$ $I=5$

The integral to be done looks like

$$\int \frac{d^4k\, d^4l}{(k^2-m^2)(l^2-m^2)((k-l)^2-\mu^2)((k+p)^2-m^2)((l+p)^2-m^2)}$$

To this we would apply our denominator combining identity. Let's write down the general case. Call the independent loop momenta k_i, $i=1,\dots,L$, and the external momenta, q_j. All momenta on the I internal lines are linear combinations of the k_i and q_j. After introducing the Feynman parameters, the integral to be done is of the form.

$$\int_0^1 d\alpha_1 \cdots d\alpha_I\, \delta(1-\Sigma\alpha) \int \frac{d^4k_1 \dots d^4k_L}{D^I}$$

Where

$$D = \Sigma_{ij=1}^{L} A_{ij} k_i \cdot k_j + \Sigma_{i=1}^{L} B_i \cdot k_i + C$$

A is an $L \times L$ matrix that is linearly dependent on the Feynmann parameters. It is positive definite except at the endpoints of the Feynman parameter integrations. B is a vector with L four vector components. It is linear in the Feynman parameters and linear in the external momenta. C is a number, depending linearly on the Feynman parameters, and the external momenta^2 and the

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masses squared that appear in the propogators. It has a small positive imaginary part.

Now shift the k integrations to eliminate the terms linear in k.

$$k_i' = k_i + \frac{1}{2}\sum_j (A^{-1})_{ij} B_j \qquad d^4k_i = d^4k_i'$$

$$D = \sum_{ij=1}^{L} A_{ij}\, k_i'\cdot k_j' + C'$$

where $C' = C - \frac{1}{4}\sum_{ij} B_i A^{-1}_{ij} B_j$

C is still linear in external momenta^2 and the masses squared, but now it has some awful dependence on the Feynman parameters because of A_{ij}^{-1}. It still has a small positive imaginary part.

Now diagonalize A_{ij} with an orthogonal transformation on the set of four vectors k_i'

$$k_i' = O_{ij}\, k_i'' \qquad \det O = 1$$

$$\prod_{i=1}^{L} d^4k_i' = \prod_{i=1}^{L} d^4k_i''$$

$$D = \sum_{ij=1}^{L} A_{ij}\, O_{ik}\, k_k''\cdot O_{jl}\, k_l'' + C'$$

$$= \sum_{ij=1}^{L} (O^T A O)_{kl}\, k_k''\cdot k_l'' + C'$$

$$= \sum_{i=1}^{L} a_i\, k_i''\cdot k_i'' + C'$$

where

$$(O^T A O)_{kl} = \delta_{kl}\, a_l$$

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Finally we'll make a transformation to eliminate the a_i

$$k_i'' = \frac{1}{\sqrt{a_i}}\, k_i'''$$

$$\prod_{i=1}^{L} d^4k_i'' = \prod_{i=1}^{L}\left(\frac{1}{\sqrt{a_i}}\right)^4 d^4k_i'''$$

$$= (\det A)^{-2} \prod_{i=1}^{L} d^4k_i'''$$

Th integral to be done has been reduced to

$$\int_0^1 d\alpha_1 \cdots d\alpha_I\, \delta(1-\Sigma\alpha)(\det A)^{-2} \int \frac{d^4k_1''' \cdots d^4k_L'''}{D^I}$$

where $D = \sum_{i=1}^{L} k_i'''\cdot k_i''' + C'$

Now we can perform Wick rotations on each of the $k_i'''^{\,0}$ variables independently to get

$$\int_0^1 d\alpha_1 \cdots d\alpha_I\, \delta(1-\Sigma\alpha)(\det A)^{-2}\, i^n \int \frac{d^4k_{1E}\cdots d^4k_{LE}}{D^I}$$

$$d^4k_i''' = i\, d^4k_{iE} \qquad k_{i4} = -ik_i'''^{\,0}$$

$$D = -\sum_{i=1}^{L} k_E^2 + C'$$

This is one big spherically symmetric integral in 4L dimensions! Easily done with only a slight generalization of our integral table. We have reduced a general graph to an awful integral over Feynman parameters; this actually is progress. NOTE THAT YOU DON'T HAVE TO DIAGONALIZE A WHEN APPLYING THIS FORMULA. ALL YOU NEED IN THE END IS det A.

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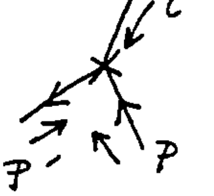
The definition of g in Model 3

Renormalization condition (6), the committee definition of g, has not been stated or turned into an equation among Green's functions. The statement is needed to fix the counterterm in $\mathcal{L}$

$$\mathcal{L} = \cdots \quad -F\psi^{*\prime}\psi'\phi' + \cdots$$

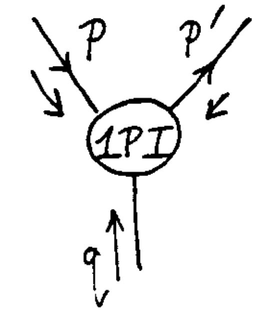
Which has Feynman rule



$-iF(2\pi)^4$
$\times\,\delta^{(4)}(p+p'+q)$

Model 3 does not exist in the real world, so no committee has actually gotten together to define g. We'll play committee.

Define



$$= \underset{\sim}{-i}\,\Gamma(p^2, p'^2, q^2)$$

The $-i$ is a sensible convention, put there so that at lowest order $\Gamma = g$

Why can we consider Γ to be a function of p^2, p'^2 and q^2?

Γ is a Lorentz invariant, so it must be a function of Lorentz invariants only. There are only two independent momenta, $q = -p - p'$, so the only Lorentz invariants are p^2, p'^2 and $p\cdot p'$.

However $p\cdot p'$ can be traded in for q^2.

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So here is our committee definition of g: ²

$$g \equiv \Gamma'(\underline{p}^2, \underline{p}'^2, \underline{q}^2)$$

The bars mean some specific point in momentum² space.

This is a reasonable if not obvious generalization of the types of conditions we used to determine A, B, C, D and E. The proof of the iterative determination of F is identical.

While all points $\underline{p}^2, \underline{p}'^2, \underline{q}^2$ are equally good as far as determining F is concerned, there is one that is more equal than others. It might well be the one the committee picks, because as we will show, it has some experimental significance. The point is

$$\underline{p}^2 = \underline{p}'^2 = m^2 \qquad \underline{q}^2 = \mu^2$$

To actually find a trio of four vectors satisfying these conditions, as well as

$p + p' + q = 0$, you have to make some of their components complex. This point is not kinematically accessible. One can show in general however, that the domain of analyticity of Γ', considered as a function of three complex variables, is sufficiently large to define the analytic continuation of Γ' from any of its physically accessible regions to this point.

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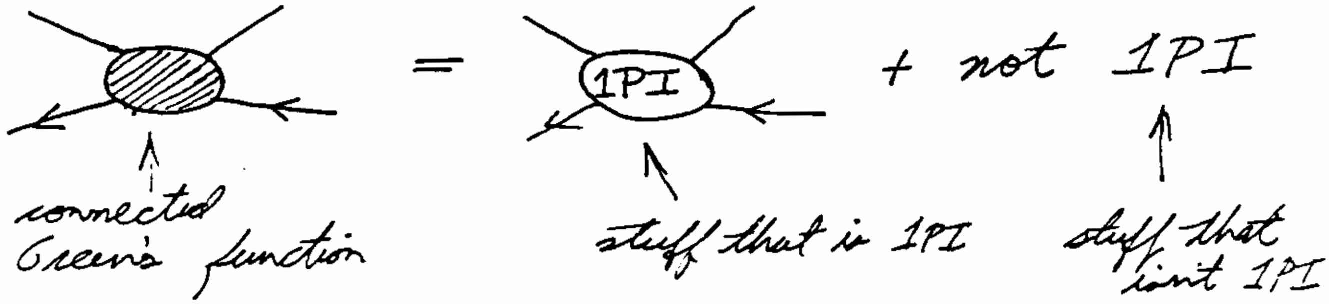
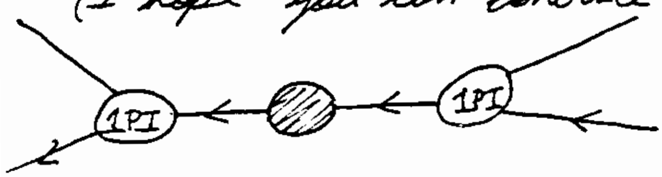
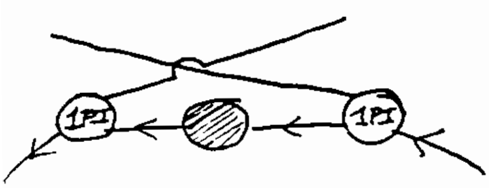
What is this point's experimental significance?[3]
Look at the process $\phi + N \to \phi + N$.
Diagramatically,



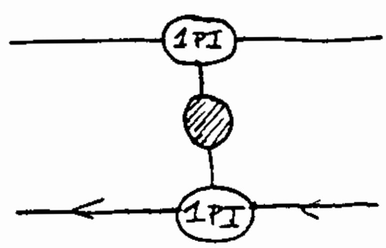
We can say more about the stuff that isn't 1PI! By definition, there is some line in the graph which can be cut and the graph falls into two pieces. If I ignore interactions on the external legs, the cutting of the internal line separates the graph into two pieces each having two external lines. The $\frac{1}{2}\binom{4}{2} = 3$ possibilities look like s t and u channel graphs. If cutting the internal line separates the incoming meson and nucleon from the outgoing ones the graph must be a contribution to (I hope you can convince yourself)



the other two possibilities are



and



Using our definitions, the first graph (on mass shell) is

$$-i\Gamma(s, m^2, \mu^2)\, D'(p_s^2)\left(-i\Gamma(m^2, s, \mu^2)\right)$$

This has a pole at $s=m^2$ because the full nucleon propagator has a pole there, and because of our renormalization conditions, we can write down what the residue of that pole is. $s=m^2$ we have

$$-ig\frac{i}{s-m^2}(-ig) + \text{analytic stuff at } s=m^2$$

Because of our definitions, the residue of the pole of these graphs is $-ig^2$.

Now what about the other two graphs. They look like they have poles at $u=m^2$ and $t=\mu^2$, but we don't expect them to have a pole at $s=m^2$. Furthermore, the graph

probably has all sorts of cuts, but it is unlikely that it has a pole at $s=m^2$ because there is no propagator on the inside of the graph that carries the whole incoming momentum.

To summarize  $= \dfrac{-ig^2}{s-m^2}$ + plus analytic, or at least no pole, near $s=m^2$

Experimentally, the residue of this pole can be measured by looking at $\phi+N$ scattering in the physical region, and extrapolating down to $s=m^2$. You just measure the s wave scattering. When this was done they found pole-like behavior with $g\simeq 13.5$.

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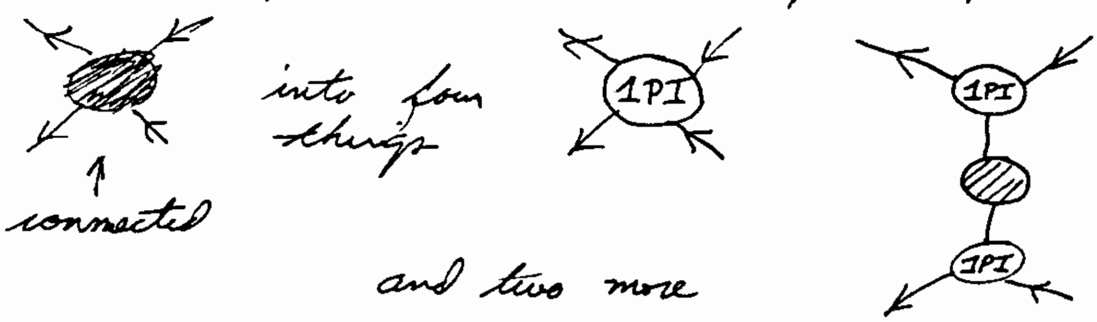
actually, they weren't very good at making pion beams back when they did this, so they measured the pole in $\gamma + p \to p + \pi$, which measures e.g. When $g = 13.5$ was determined this way, it put the last nail in the coffin for the attempts to consider the strong interactions perturbatively with the pion and nucleons as fundamental particles.

Consider the process $N+N \to N+N$. By similar arguments, we can split up



the thing to note is that this decomposition leads us to expect

$$= (-ig)^2 \frac{i}{t-\mu^2} + \text{stuff with no pole at } t=\mu^2$$

Since $t=\mu^2$ is unphysical, to measure the effect of this pole, you again have to extrapolate.

Our simple model states that the residue of this pole is the same, $-ig^2$. When they did this experiment with $p+p \to p+p$, after doing some work to eliminate electromagnetic effects, they got agreement (to within 10%). Furthermore, the fit showed that the location of the pole was at $t = m_\pi^2$. (They only fit the high partial waves, where they felt justified comparing with p.t.)

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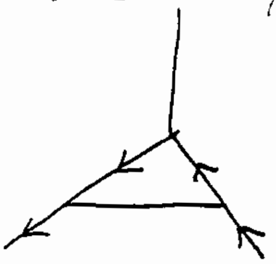
Renormalization vs. Infinities

The $O(g^3)$ correction to Γ' in model 3



is finite. The counterterm

(3)

is needed only to make the theory agree with the committee definition of g. To see that the graph is finite, look at its high momentum behavior. Without even combining denominators you can see that at high q the integral looks like $\int \frac{d^4q}{q^6}$. This extreme convergence is peculiar to model 3 (and other models where all the couplings have positive mass dimension we will later see).

Consider a model with a four scalar field interaction

$$\mathcal{L} = \cdots + g\phi^4 + \cdots$$

ϕ^4 could be ABCD or $(\psi^*\psi)^2$.

Look at the lowest order correction to a propogator:

whoops, there is but it is a constant and is really easy to cancel

after combining denominators, you see that the integral looks like $\int \frac{d^8q}{q^6}$ at high q.

2 loops

quadratic divergence

three propogators

Fortunately there are renormalization counterterms —×— to cancel this infinity.

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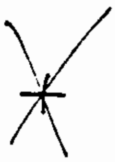
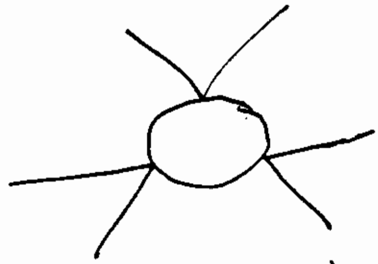
What about other graphs in this theory?
There is

 $\sim \int \frac{d^4q}{q^4}$ log divergent

There is a committee definition of g, and a ϕ^4 counterterm which can cancel off the log divergence

 What about 

This is finite, which is good. We would need a ϕ^6 counterterm to cancel this graph's divergence if it weren't convergent. That would require another committee definition, say for $3 \to 3$ scattering at some momentum.

← This is a more stringent def'n than is often used.

Definition A Lagrangian is renormalizable only if all the counterterms required to remove infinities from Green's functions are terms of the same type as those present in the original Lagrangian.

Suppose a theory has a ϕ^5 (ABCDE) interaction. Then

 is log divergent, and you would need a ϕ^6 counterterm to cancel it. Since the theory did not originally contain a ϕ^6 interaction, we say ϕ^5 theory is not renormalizable.

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It seems fairly clear that to correct this defect, you just add a ϕ^6 term to your Lagrangian, then your ϕ^6 counterterm will be of the same type as the interaction term in the original Lagrangian. But then, there is

to worry about. These are also log divergent and they require ϕ^7 and ϕ^8 counterterms to cancel them. So $g\phi^5 + h\phi^6$ is not a renormalizable interaction either. You can see that adding $j\phi^7$ and $k\phi^8$ to the Lagrangian is not going to help.

This shows that any polynomial interaction of degree higher than 4 is not renormalizable. We have not shown that polynomials of degree 4 or less are renormalizable, but what we have found so far suggests it.

Now these theories with an infinite series of interactions are disgusting because they contain an infinite number of independently adjustable parameters. Unless you make some additional statement, you cannot make any predictions. One possibility is to hunt for some relationship among the terms in the infinite series. Perhaps

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^2 - \tfrac{\mu^2}{2}\phi^2 - \lambda\cos\alpha\phi \text{ is renormalizable}$$

Perhaps $S = \int d^4x\sqrt{-g}\,R$, the Einstein-Hilbert action is renormalizable. Suffice it to say that noone has ever been able to construct a renormalizable non-polynomial interaction that is not equivalent to free field theory.

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Unstable particles.

Let's look at Model 3 in the regime $\mu > 2m$.
In that case, $\pi_f(k^2)$, is not real at the subtraction point! You can see this by looking at π_f, or you can look at the nonperturbative formula

$$\text{Im}\, \pi'(k^2) = -\pi \frac{\sigma(k^2)}{|D'(k^2)|^2} \qquad \text{Im}[-i D'(k^2)] = -\pi \sigma(k^2)$$

$\sigma(k^2) \neq 0$ when $k^2 = \mu^2 > 4m^2$ so $\text{Im}\,\pi' \neq 0$.
Our subtraction, which says

$$\pi'(\mu^2) = 0 \quad \text{and} \quad \left.\frac{d\pi'}{dk^2}\right|_{k^2=\mu^2} = 0$$

would be causing us to subtract imaginary terms from the Lagrangian. This is unacceptable because a non-Hermitian Hamiltonian is unacceptable. One road is to just say, for $\mu > 2m$, the meson is unstable, I have no business calculating meson-meson scattering or nucleon-meson scattering, or anything else involving an external meson, so just drop all renormalization conditions and subtractions related to the meson.

This road is not ideal for two reasons. The definition of the theory does not change in any smooth way as μ increases beyond $2m$, and we lose the bonus of renormalization, the elimination of infinities. We will modify our subtraction procedure for $\mu > 2m$ so that it still removes ∞'s, and is continuously related

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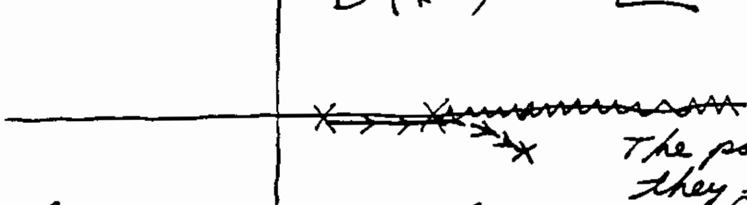
to the subtractions made for $\mu<2m$, but so that we do not make imaginary subtractions. Our modified procedure is to quite a degree ad hoc, but we will see that it is useful. For $\mu>2m$, demand

$$\mathrm{Re}\ \Pi'(\mu^2)=0$$

$$\mathrm{Re}\ \left.\frac{d\Pi'}{dk^2}\right|_{k^2=\mu^2}=0$$

We will see that with these renormalization conditions for $\mu>2m$, and the usual ones for $\mu<2m$, that as you increase μ, the pole in $D'(k^2)$ moves up the real axis until it touches the branch cut, and then it moves onto the second sheet

$D'(k^2)$ $\quad \underline{\lfloor k^2}$



The poles can run, but they can't hide.?

What do I mean by second sheet? The value of $D'(k^2)$ for $k^2>4m^2$ (and real) is obtained by taking the limit from positive imaginary k^2 down onto the real axis, according to the $i\epsilon$ prescription. The value of $D'(k^2)$ for $\mathrm{Im}\,k^2<0$ is defined by that integral expression for $D'(k^2)$ in terms of $\sigma(a^2)$. $D'(k^2)$ has a discontinuity across the cut. What we get when we <u>analytically continue</u> $D'(k^2)$ to $\mathrm{Im}\,k^2<0$ from its values for $\mathrm{Im}\,k^2>0$, to get a function that is continuous along the old cut, is called $D'(k^2)$ on the second sheet.

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A couple other ways of saying this:

The branch point is fixed, the value along the real axis is physical and can't be changed, but within those restrictions, we can move the cut leading from the branch point to ∞ around any way we like.

The second sheet is what you get from peering down from above the cut. In some sense this is much closer to the physical region, because it is not separated by a discontinuity.

In model 3, for $\mu > 2m$, we will now compute $(-iD')^{-1}$ to $O(g^2)$ when $k^2-\mu^2$ is order g^2.

$$[-iD'(k^2)]^{-1} = k^2-\mu^2-\Pi'(k^2)$$

a formula we will use is

$$\operatorname{Im}\Pi'(k^2) = |D'(k^2)|^{-2}(-\pi)\,\sigma(k^2)$$

(3.H...)

$$= -\frac{1}{2}|D'(k^2)|^{-2}\sum_{|n\rangle\neq|0\rangle,|p\rangle}|\langle n|\phi'(0)|0\rangle|^2(2\pi)^4\delta^{(4)}(k-P_n)$$

($|p\rangle$: one meson)

using the def'n of σ.

Now let's work on $[-iD'(k^2)]^{-1}$

$$[-iD'(k^2)]^{-1} = k^2-\mu^2-\underbrace{\Pi'(\mu^2)}_{\text{this is } O(g^2)}-\underbrace{(k^2-\mu^2)}_{\text{this is } O(g^2)}\left.\frac{d\Pi'}{dk^2}\right|_{k^2=\mu^2} + O(g^4)$$

$$= k^2-\mu^2-\underbrace{\operatorname{Re}\Pi'(\mu^2)}_{0}-i\operatorname{Im}\Pi'(\mu^2)+O(g^4)$$

0 by our convenient renormalization condition.

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Now using that formula for $\mathrm{Im}\,\Pi'(k^2)$, [12]

$$[-iD'(k^2)]^{-1} = k^2-\mu^2 + \frac{i}{2}\left.\frac{\sum_{n\neq|0\rangle,|k\rangle} |\langle n|\phi'(0)|0\rangle|^2 (2\pi)^4\delta^{(4)}(k-P_n)}{|D'(k^2)|^2}\right|_{k^2=\mu^2} + O(g^4)$$

$$= k^2-\mu^2 + i\mu\Gamma + O(g^4)$$

$$= k^2 - \left(\mu - \frac{i\Gamma}{2}\right)^2 + O(g^4)$$

The nice thing I have noticed in the next to last step is that what multiplies i in the first expression is $\mu\Gamma$ when $k^2=\mu^2$. Compare with Nov. 4, p.1. The $|D'|^{-2}$ serves to exactly eliminate the external propogator you would get in relating $\langle n|\phi'(0)|0\rangle$ to $\langle n|(S-1)|k\rangle$ $\propto i\mathcal{A}$.

This does not prove that Γ is a lifetime! That Γ was an inverse lifetime in the theory with a turning on and off function does not suffice to show that it is a lifetime in our full blown scattering theory.

To summarize what we have found so far, we have found that in model 3 with $\mu>2m$, and some ad hoc renormalization conditions, in the small g limit, there is a pole in $D'(k^2)$ at $k^2=\left(\mu-\frac{i\Gamma}{2}\right)^2$ on the second sheet. $D'(k^2)$ is still analytic on the cut complex plane. In a sense, this pole is close to the physical region. Our perturbative analysis shows that as $g\to 0$, $\Gamma\to 0$, and the actual value of $D'(k^2)$ along the real axis should be more and more dominated by the presence of this pole when $k^2 \approx \mu^2$.

Nov. 25
13

What we have done so far has depended on perturbation theory in model 3, although the way Γ appeared, it is clear how a perturbative calculation would go in other models. What we will do next does not depend on perturbation theory in the coupling constant, or on any model.

Our only assumption now will be that

$$D'(k^2) = \frac{i}{k^2-\mu^2+\mu i\Gamma} + \text{small terms}$$

for some range of the real axis near $k^2=\mu^2$. That is, the pole on the second sheet dominates the behavior of $D'(k^2)$ near $k^2=\mu^2$.

We will now do two thought experiments and show that this behavior is what experimentalists are talking about when they say they have discovered an unstable particle.

Our first thought experiment is to blast the vacuum at $\vec{x}=t=0$. A theorist blasts the vacuum by turning on a source

$$\mathcal{L} \to \mathcal{L} + \rho(\vec{x},t)\,\phi'(\vec{x},t) \qquad \rho(\vec{x},t) = \lambda\,\delta^{(4)}(x)$$

An experimentalist blasts the vacuum at $\vec{x}=t=0$ by crashing two protons together at the origin of coordinates.

The amplitude that you'll get any momentum eigenstate $|n\rangle$ is proportional to

$$\lambda\langle n|\phi'(0)|0\rangle + O(\lambda^2)$$

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The probability of having momentum k in the final state is proportional to

$$\lambda^2 \sum_{|n\rangle} |\langle n|\phi'(0)|0\rangle|^2 (2\pi)^4 \delta^{(4)}(P_n - k) + O(\lambda^3)$$

$$= 2\pi\lambda^2 \sigma(k^2)\theta(k^0) + O(\lambda^3)$$

I don't have to specify $|n\rangle \neq |0\rangle, |k\rangle$ one meson (as long as I stay away from $k=0$ so $\delta^{(4)}(P_n - k) = 0$ when $|n\rangle = |0\rangle$) since there are no physical one meson states to emerge from blasting the vacuum when "the meson is unstable."

$\sigma(k^2)$ is in turn proportional to $-\mathrm{Im}\,\pi'$

$$-(\mathrm{Im}\,\pi'(k^2))|D'(k^2)|^2$$

So the probability of finding momentum k, not equal to zero, $k^0 > 0$ is proportional to

$$-\lambda^2 \mathrm{Im}\,\pi'(k^2)|D'(k^2)|^2 + O(\lambda^3)$$

Finally using the form of D' which is assumed to dominate near $k^2 = \mu^2$ $(\pi'(k^2) = -\mu i \Gamma)$ we have

$$\frac{\lambda^2 \mu \Gamma}{(k^2-\mu^2)^2 + \mu^2\Gamma^2} + O(\lambda^3)$$

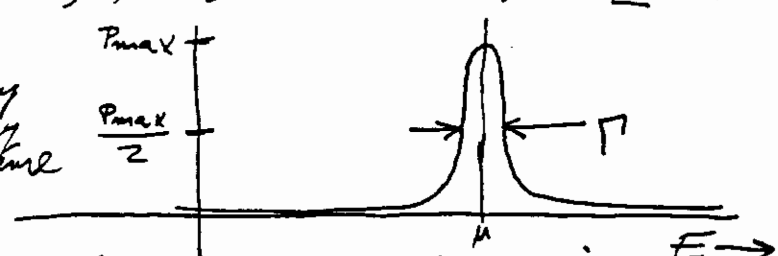
We can look at the center of mass energy of the decay products. That is, we can think of this probability distribution as a function of E, the decay products' COM energy, instead of as a function of k^2. The probability of finding a COM energy E in the decay products is proportional to (drop the $O(\lambda^3)$)

$$\frac{\mu\Gamma}{(E^2-\mu^2)^2+\mu^2\Gamma^2} = \frac{\mu\Gamma}{(E-\mu)^2(E+\mu)^2+\mu^2\Gamma^2}$$

Approximation preserves the character of the function if $\Gamma \ll \mu$

$$\approx \frac{\mu\Gamma}{(E-\mu)^2(2\mu)^2+\mu^2\Gamma^2} = \frac{\mu\Gamma}{(4\mu^2)\left[(E-\mu)^2+\frac{\Gamma^2}{4}\right]}$$

Probability of finding energy E in the decay products COM frame



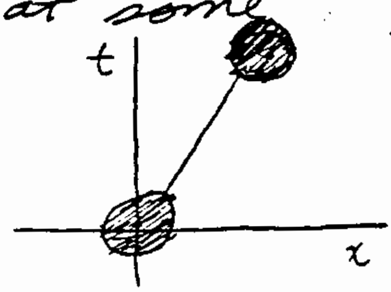
This is called a Breit-Wigner or Lorentzian line shape, and it is familiar from QM. Γ is the full width at half maximum, or decay width. As Γ gets smaller, the peak gets narrower and higher.

So we have shown that μ and Γ, which locate the pole on the second sheet of $D'(k^2)$ are the mass and decay width respectively that an experimenter reports when she says she has found an unstable particle.

Experimenters have another way of measuring Γ, which is purported to be equivalent. They use a clock, and the average lifetime is Γ^{-1}. We will now do a second thought experiment to show that this second way of determining Γ is equivalent.

Dec. 2
1/22

We have explained "width" in the phrase "decay width." With a second thought experiment we'll explain "decay." In thought experiment 2, we'll produce an unstable particle near the origin and detect it a long ways away at some region near y. The region of production cannot be too sharply localized as we are going to make states only with $k^2 \simeq \mu^2$.

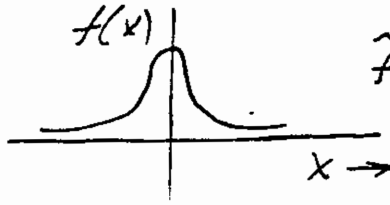


We'll make the initial state by hitting the vacuum with

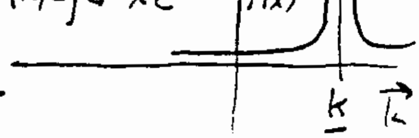
$\int d^4x\, f(x)\, \phi'(x)$ $f(x)$ is fairly well localized in position space and its Fourier transform is fairly well localized in momentum space about a momentum $\underline{k}$.

Initial state is

$$\int d^4x\, f(x) \phi'(x) |0\rangle$$



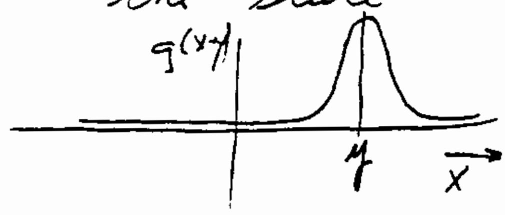
$\tilde{f}(k) = \int d^4x\, e^{ik\cdot x} f(x)$

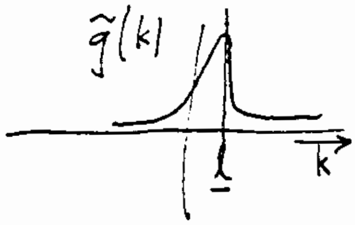


$\underline{k}^2 \simeq \mu^2$

I'll detect the particle by finding the amplitude that this state becomes the state

$$\int d^4x\, g(x-y) \phi'(x) |0\rangle$$





$g(x-y)$ is concentrated around y, $\tilde{g}(k)$ is concentrated around $\underline{l}$. The amplitude, parameterized by y is

$$A(y) = \langle 0 | \int d^4x'\, g^*(x'-y)\, \phi'(x') \int d^4x\, f(x) \phi'(x) | 0 \rangle$$

Dec. 2

2

If the point y is far later in time than the origin, we can make this a time ordered vacuum expectation value with negligible error.

$$A(y) = \int d^4x\, d^4x' \quad g^*(x'-y) f(x) \langle 0 | T(\phi'(x')\phi'(x)) | 0\rangle$$

$$= \int d^4x\, d^4x' \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4}\, e^{ik\cdot(x'-y)} \tilde{g}^*(k)\, e^{-ik'\cdot x} \tilde{f}(k')$$
$$\langle 0 | T(\phi'(x')\phi'(x)) | 0\rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4}\, e^{-ik\cdot y}\, \tilde{g}^*(k) \tilde{f}(k') \underbrace{\int d^4x\, d^4x'\, e^{ik\cdot x'} e^{-ik'\cdot x}\, G'(x',x)}_{\tilde{G}'(-k,\,k') \;\equiv (2\pi)^4 \delta^{(4)}(k-k') D'(k^2)}$$

Compare this F.T. convention with the one for f on the previous page

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot y}\, \tilde{g}^*(k) \tilde{f}(k) D'(k^2) \leftarrow \text{This integral gives zero, unless } \underline{k} \approx \underline{l}$$

Recall that $\tilde{g}(k)$ is concentrated around $\underline{l}$ and $\tilde{f}(k)$ is concentrated around $\underline{k}$, with $\underline{k}^2 \approx \mu^2$. Assume that $D'(k^2)$ is dominated by a stable or unstable particle pole at $k^2 \approx \mu^2$ and that $\tilde{f}(k)$ is sufficiently tightly concentrated around $\underline{k}$ that we can make the approximation

$$A(y) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot y}\, \tilde{g}^*(k) \tilde{f}(k) \frac{i}{k^2-\mu^2+i\mu\Gamma}$$

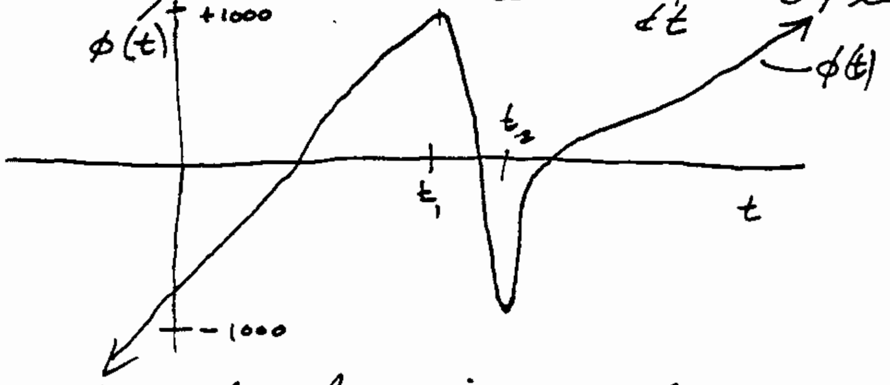
(The stable case is handled by taking the limit $\Gamma \to 0+$)

We want to analyze this for large y which is difficult because the phase of the exponential is varying rapidly as k changes. Furthermore as k^2 increases through μ^2, the phase of the propagator changes rapidly. There is a method for handling these kinds of integrals.

DEC. 2
3

Method of stationary phase.

Given $I = \int dt\, e^{i\phi(t)} f(t)$, where ϕ is a rapidly varying function of t except for a few points where $\frac{d\phi}{dt} = 0$, call them t_i



See Whittaker and Watson Modern Analysis

Then the integral gives nothing almost everywhere, except at those points where the phase stops wildly varying for a moment at those points t_i, the contribution can be approximated by

$$I = \sum_i e^{i\phi(t_i)} f(t_i) \underbrace{\int dt\, e^{\frac{i}{2}\phi''(t_i)(t-t_i)^2}}_{\sqrt{\frac{2\pi}{|d^2\phi/dt^2|\,|_{t_i}}}\, e^{i\pi/4}}$$

We will rewrite $\frac{i}{k^2-\mu^2+i\mu\Gamma}$ as an exponential so we can use this method. Unfortunately we introduce another integral, but it can also be done by the method of stationary phase.

$$\frac{i}{k^2-\mu^2+i\mu\Gamma} = \int_0^\infty \frac{ds}{2\mu}\, e^{i\frac{s}{2\mu}(k^2-\mu^2+i\mu\Gamma)}$$

Assume the variation in phase of $\tilde{g}^*(k)\tilde{f}(k)$ is slow compared to the variation in the exponential, or take $g=f$. DEC. 2

$$A(y)=\int_0^\infty d\left(\frac{s}{2\mu}\right)\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot y+i\frac{s}{2\mu}(k^2-\mu^2+i\mu\Gamma)}\,\tilde{g}^*(k)\tilde{f}(k) \quad 4$$

To do the k integration by stationary phase, set

$$0=\frac{\partial}{\partial k_\mu}\left(-k\cdot y+\frac{s}{2\mu}(k^2-\mu^2)\right)=-y^\mu+\frac{s}{\mu}k^\mu$$

Thus the only stationary phase point in the k integrations is at $k_0=\frac{\mu}{s}y \qquad \left|\frac{d^2\phi}{dk^{\mu 2}}\right|_{k_0}=\frac{s}{\mu}$

$$A(y)\simeq-\int_0^\infty\frac{ds}{2\mu}\frac{1}{(2\pi)^2}\left(\frac{\mu}{s}\right)^2\tilde{g}^*\left(\frac{\mu}{s}y\right)\tilde{f}\left(\frac{\mu}{s}y\right)e^{i\frac{s}{2}(-\mu+i\Gamma)}\,e^{-i\frac{\mu}{2s}y^2}$$

No sum

To do the s integration by stationary phase, set

$$0=\frac{\partial}{\partial s}\left(-\frac{s}{2}\mu-\frac{\mu}{2s}y^2\right) \qquad s=\sqrt{y^2}$$

Note that there is no stationary phase point if y^2 is spacelike. As $y\to\infty$, $y^2<0$, there is no probability that a particle will be detected. We have recovered causality.

Call $\sqrt{y^2}$ "s_0", then $\left|\frac{d^2\phi}{ds^2}\right|_{s_0}=\frac{\mu}{s_0}$, and

$$A(y)=-e^{i\pi/4}\sqrt{\frac{2\pi s_0}{\mu}}\frac{1}{2\mu}\frac{1}{(2\pi)^2}\left(\frac{\mu}{s_0}\right)^2\tilde{g}^*\left(\frac{\mu}{s_0}y\right)\tilde{f}\left(\frac{\mu}{s_0}y\right)e^{-i\mu s_0}\,e^{-\frac{\Gamma s_0}{2}}$$

These factors can be understood. Suppose you classically propagate a stable particle with velocity

$V_\mu=\frac{k_\mu}{\mu}$. In a proper time s, it will arrive at a point $y_\mu=V_\mu s$, where $V_\mu^2=1\Rightarrow s=\sqrt{y^2}$.

This is just classical kinematics, but you see we have ecovered it in the limit of large y from quantum field theory. The conditions of stationary phase are the equations of classical kinematics.

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The factor $e^{-i\mu S_0}$ is just e^{-iEt} of quantum mechanics that has come out in a Lorentz invariant generalization. There is a factor $S_0^{-3/2}$. That is there because if you wait long enough, every packet: because of an initial uncertainty in velocity is spreading out in all directions linearly with time. In 3-D this means that the probability density at the center of the packet goes down like $\frac{1}{t^3}$. So the amplitude falls like $\frac{1}{t^{3/2}}$. The Lorentz invariant generalization of this is that the amplitude at the center of the packet falls like $\frac{1}{(\text{proper time})^{3/2}}$.

Finally there is the unstable case. We have $e^{-\Gamma S_0/2}$ in the amplitude, which means that the probability has a factor $e^{-\Gamma S_0}$. They are indeed decaying and again we have gotten the Lorentz invariant generalization of $e^{-\Gamma t}$. Γ is the decay rate per unit proper time.

¿¿ This talk of "correct generalization" must be made more precise. It should be possible to do the computation in the frame where $y = (\sqrt{y^2}, \vec{0})$. However our stationary phase computations are not justified in that case ??

"WHERE IT BEGINS AGAIN"

If we had been proceeding logically, starting from first principles, making the most general statements about relativistic quantum field theory we could and, only after exhausting those, made simplifying assumptions and approximations, we would have begun the course by listing all possible field transformation laws, then we would have constructed all possible quadratic Lagrangians, that is all possible combinations that are at most quadratic in the fields and transform as scalars under the Lorentz group. Then we would do canonical quantization and in the process discard many of the Lagrangians because of one or another inconsistency, like the Hamiltonian not being bounded below. At this point we would have all possible free particle theories, and we could start adding interactions, higher order polynomials, to the Lagrangian. The actual order we have been doing this course, is to spend a lot of time studying relativistic invariants made up of the simplest kind of fields, scalar fields. Under Lorentz transformations a set of scalar fields transform like

$$\phi^a(x) \qquad a = 1, \ldots, n \qquad\qquad \Lambda \in SO(3,1)$$

$$\Lambda: \ \phi^a(x) \longrightarrow \phi^a(\Lambda^{-1}x)$$

The only Lorentz scalars you can construct have derivatives, 0, 2, 4, ... of them, which act on the scalars and are completely contracted with $g^{\mu\nu}$ or $\epsilon^{\mu\nu\lambda\sigma}$.

$$\Box\phi_1 \quad \partial_\mu\phi_2\,\partial^\mu\phi_3 \qquad\qquad \epsilon^{\mu\nu\lambda\sigma}\ \partial_\mu\phi_1\,\partial_\nu\phi_2\,\partial_\lambda\phi_3\,\partial_\sigma\phi_4$$

The list of possible quadratic Lagrangians is pretty short, and we have gone a long ways toward exploring them. In fact we have even gone a long ways toward studying the total list of interacting scalar fields since the renormalization vs. infinities arguments pretty well rule out Lagrangians that are more than quartic in the fields.
We haven't exhausted the study of scalars, but we are now going to go on to

DISCOVERING ALL POSSIBLE LORENTZ TRANSFORMATION LAWS OF FIELDS

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We'll phrase the analysis in a quantum language, but the all that we are about to do can be carried through classically. Assume we have a finite number of fields,

$$\phi^a(x) \qquad a = 1, \dots, N$$

Let Λ denote an abstract element of $SO(3,1)$, the part of the Lorentz group connected to the identity. More concretely, Λ can also be thought of as a 4×4 matrix, that preserves the metric, $\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} = g^{\mu\nu}$, is proper, $\det \Lambda = 1$, and is orthochronous, $\Lambda^0{}_0 > 0$. For each Λ there is a unitary transformation

$$U(\Lambda)^\dagger \phi^a(x) U(\Lambda) = D^a{}_b(\Lambda)\, \phi^b(\Lambda^{-1}x) \qquad (\Sigma_b \text{ implied})$$

For each Λ there is some $N \times N$ matrix $D^a{}_b(\Lambda)$ that gives a linear relationship between the complete set of commuting observables at $\Lambda^{-1}x$ and those at x, for all x. If we think of the D's as matrices and the ϕ's as column vectors, we can write

$$U(\Lambda)^\dagger \phi(x) U(\Lambda) = D(\Lambda)\phi(\Lambda^{-1}x)$$

D's are $N\times N$ matrices
"The dimension of D is N"

a property of the U's reflects itself in the D's
It only takes a couple of lines to prove this

$$U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2) \Longrightarrow \underline{D(\Lambda_1\Lambda_2) = D(\Lambda_1)D(\Lambda_2)}$$

also $U(1) = 1 \Rightarrow \underline{D(1) = 1}$ and from these two properties of the D's $\underline{D(\Lambda^{-1}) = D(\Lambda)^{-1}}$. It seems that the D matrices obey all the properties of the group, and you might think from any set of D's you could reconstruct the group. You can't though. Many elements of the group can map into a single D matrix, that is it is possible that $D(\Lambda) = D(\Lambda')$ while $\Lambda \neq \Lambda'$. The trivial prototypical example is $D(\Lambda) = 1$ for all Λ. (all fields are scalar)
A set of D's that obey the group laws is called a representation. If $D(\Lambda) = D(\Lambda') \Rightarrow \Lambda = \Lambda'$, the representation is "faithful".

A person who is tired of group theory is tired of life.

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An additional complication that we are only going to consider in a very cavalier way is the possibility which is impossible to rule out in quantum mechanics that

$$U(\Lambda_1)U(\Lambda_2) \neq U(\Lambda_1\Lambda_2), \quad U(\Lambda_1)U(\Lambda_2) = U(\Lambda_1\Lambda_2)e^{i\phi(\Lambda_1,\Lambda_2)}$$

The product law is not exactly true in general. It only need be true up to a phase. It turns out that for SO(3) and SO(3,1), the phases can be removed except in representations called spinor representations, where a rotation by π about any axis $\vec{e}$ followed by another rotation by π about $\vec{e}$ gives

$$U(\vec{e}\,\pi)\,U(\vec{e}\,\pi) = -\mathbb{1}$$

notation for the unitary operator that rotates around $\vec{e}$ by π

Two rotations by π are physically equivalent to no rotation at all, so you would expect to have gotten $\mathbb{1}$. (If you want to study this, the only good reference I know of is Bargmann, V., "On Unitary Ray Representations of Continuous Groups," Annals of Mathematics, Vol. 59, (1954) p. 1, and it is in the basement of Cabot. I am not recommending this however. We will get all the right results with much less effort by being cavalier and lucky.) The possibility that the product of the unitary operators only the unitary operator of the product up to a phase reflects itself identically in the composition law for the D's. Another thing to note about the D's is that they are not in general unitary. Try as you may, you can not use the unitarity of the U's to prove unitarity of the D's.

Our task of finding all possible Lorentz transformation laws of fields has been reduced to the task of making a catalog of all finite dimensional representations of SO(3,1).

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Shortening the catalog of finite dimensional representations of SO(3,1)

Suppose I have a representation $D(\Lambda)$. I can make a new representation of $SO(3,1)$, that obeys all three conditions, by defining

$$D'(\Lambda) = S\, D(\Lambda)\, S^{-1}$$

where S does not vary with Λ, it is some definite invertible matrix. Equivalent to this though is just a redefinition of the basis fields. If the fields transform as $\Lambda: \phi(x) \to \phi(\Lambda^{-1}x)$, the new basis $\phi'(x) = S\phi(x)$ transforms as $\Lambda: \phi'(x) \to D'(\Lambda)\,\phi'(\Lambda^{-1}x)$. This is not worth listing as a new kind of field theory. Define two representations D and D' to be equivalent if there exists an invertible S such that

$$D'(\Lambda) = S\, D(\Lambda)\, S^{-1} \quad \text{for all } \Lambda$$

and write $D' \sim D$. If there is no such S, D and D' are inequivalent $D' \not\sim D$. Our catalog will only conclude inequivalent representations of $SO(3,1)$ one representative from each "equivalence class."

I'll give a useful example. The Lorentz transformation of parity is not in $SO(3,1)$, because it is not connected to the identity, it has determinant -1. (You can think of parity, P, abstractly or as a 4×4 matrix.) For every $\Lambda \in SO(3,1)$, I can obtain another element of $SO(3,1)$ $\Lambda_P \equiv P\Lambda P$ $(P = P^{-1})$. This association is one to one, and it preserves the group multiplication law $\Lambda_P \Lambda'_P = (\Lambda\Lambda')_P$. These properties make it an "automorphism". With this automorphism of $SO(3,1)$, I can construct a new representation of the group from any given representation. Starting with a representation D, define

$$D_P(\Lambda) = D(\Lambda_P) \quad \text{this new rep obeys all three conditions}$$

It may or may not be true that $D_P \sim D$.

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To make the example more concrete, lets look at what the parity automorphism does to one of the representations of $SO(3,1)$ we all know and love, say the two index tensor.

$$\Lambda: T^{\mu\nu} \to \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}$$ Transformation law of a tensor

We read off $\quad D^{\mu\nu}{}_{\alpha\beta}(\Lambda) = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta$

The D's in the representation induced by the parity automorphism are

$$D_P{}^{\mu\nu}{}_{\alpha\beta}(\Lambda) = D^{\mu\nu}{}_{\alpha\beta}(\Lambda_P) = (P\Lambda P)^\mu{}_\alpha (P\Lambda P)^\nu{}_\beta$$

Now we expect that we have not constructed an inequivalent representation of $SO(3,1)$ this way. After all what parity is really doing is just turning $T^{00} \to T^{00}$, $T^{ij} \to T^{ij}$ and $T^{i0} \to -T^{i0}$, $T^{0j} \to -T^{0j}$. If it is interpretable as a change of basis, we must be able to find the similarity transformation relating the two representations. Lets massage the expression for D_P until we find it.

$$D_P{}^{\mu\nu}{}_{\alpha\beta}(\Lambda) = (P\Lambda P)^\mu{}_\alpha (P\Lambda P)^\nu{}_\beta = P^\mu{}_\sigma \Lambda^\sigma{}_\tau P^\tau{}_\alpha \, P^\nu{}_\varphi \Lambda^\varphi{}_\psi P^\psi{}_\beta$$

$$= P^\mu{}_\sigma P^\nu{}_\varphi \; \Lambda^\sigma{}_\tau \Lambda^\varphi{}_\psi \; P^\tau{}_\alpha P^\psi{}_\beta \qquad S^{\mu\nu}{}_{\sigma\varphi} \equiv P^\mu{}_\sigma P^\nu{}_\varphi$$

$$= S^{\mu\nu}{}_{\sigma\varphi} \, D(\Lambda)^{\sigma\varphi}{}_{\tau\psi} \, S^{-1\,\tau\psi}{}_{\alpha\beta} \quad \text{i.e.} \quad D_P(\Lambda) = S\,D(\Lambda)\,S^{-1} \checkmark$$

(You can do the vector case. I did the two index tensor because it is a little less trivial)

Shortening the catalog of finite dimensional inequivalent representations of $SO(3,1)$

Suppose someone has two theories, one with a set of fields

$$\phi_{1a} \qquad a = 1, \dots, N_1$$

transforming as $\quad \Lambda: \; \phi_1(x) \to D^{(1)}(\Lambda)\,\phi_1(\Lambda^{-1}x)$
and the other with a set of fields

$\phi_{2a} \quad a = 1, \dots, N_2$ transforming as $\Lambda: \phi_2(x) \to D^{(2)}(\Lambda)\,\phi_2(\Lambda^{-1}x)$

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Now this person comes to you and says, I have a new theory with N_1+N_2 fields, which he has assembled into a vector ϕ, and they transform like

$$\Lambda: \phi(x) \to D(\Lambda)\phi(\Lambda^{-1}x) \qquad D(\Lambda) = \begin{pmatrix} D^{(1)}(\Lambda) & 0 \\ 0 & D^{(2)}(\Lambda) \end{pmatrix}$$

For example, theory 1 could contain a vector and theory 2 could contain a scalar. The new theory would have be five dimensional. This is hardly a big breakthrough. It is such a simple extension of what was previously known that it is not worth including in our catalog. Define this representation D to be the "direct sum" of $D^{(1)}$ and $D^{(2)}$, $D = D^{(1)} \oplus D^{(2)}$. It has dimension N_1+N_2. The D's are in block diagonal form. Call a representation "reducible" if it is equivalent to a direct sum, otherwise, call it "irreducible."

Our task is to build the remarkably shorter catalog, the catalog of finite dimensional inequivalent irreducible representations of $SO(3,1)$.

By a wonderful fluke, peculiar to living in 3+1 dimensions, the representations of $SO(3,1)$ can be rapidly obtained from the representations of $SO(3)$. You know all about the representations of $SO(3)$ from undergraduate QM, so we will be able to wrap the catalog up by the end of next lecture. If we lived in 9+1 dimensions, there would be no quick reduction of the problem of finding the representations of $SO(9,1)$ to the problem of finding the representations of $SO(9)$, which 9+1 dimensional students solve as undergraduates. We will review the representations of $SO(3)$ just enough to refresh your memory. They are carefully constructed in a few pages in a way that generalizes to other groups beginning on page 16 of Howard Georgi *Lie algebras in Particle Physics*. Actually what is constructed there are the representations of the Lie algebra of $SO(3)$ rather than the representations of the Lie Group but you'll see in the next 4 pp. that that is what we want.

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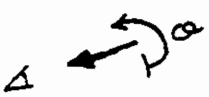
THE FINITE DIMENSIONAL INEQUIVALENT IRREDUCIBLE REPRESENTATIONS OF SO(3) 12

An element R of $SO(3)$ can be thought of abstractly or as a 3×3 matrix. It is specified by given an axis of rotation, and an angle of rotation about the axis, $\vec{e}$ and θ. Let's standardize the vector that defines the axis by taking it to be a unit vector. The product $\vec{e}\theta$ defines the rotation completely, $R(\vec{e}\theta)$. Its length gives the amount of rotation in the counterclockwise direction when looking down toward the tail of the vector $\vec{e}$



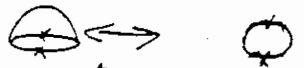
If we let θ take on any value from 0 to 2π we have included twice every element of $SO(3)$. The reason being is that a rotation about $\vec{e}$ by an angle θ is exactly the same as a rotation about $-\vec{e}$ by an angle $2\pi-\theta$. So we'll restrict $\theta\in[0,\pi]$. This still includes twice the rotations by π because

$$R(\vec{e}\pi) = R(-\vec{e}\pi)$$

not the sphere S^2, the ball B^3

The group $SO(3)$ is topologically like the ball in three space of radius π, except antipodal points on the surface of the ball are identified.

Just so you have some impressive jargon at your disposal, the ball just described, with the identification, is topologically like the the projective 3 sphere. I can explain that in one lower dimension where I can visualize it. The projective 2 sphere is S^2 with each pair of antipodal points identified. Each point on the whole bottom half of the sphere below the equator has a point it is identified with in the half of the sphere above the equator. Chuck the whole bottom half of the sphere leaving the top half and the equator. Each point on the equator is still identified with one other point on the equator. But it is clear that



x's represents a pair of identified points

Just flatten the sphere out into the disk. and this disk is the "ball" in two space, with antipodal points on the ball identified.

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a method of infinitesimal analysis

ike $\frac{d}{d\theta} R(\vec{e}\theta)\Big|_{\theta=0} \equiv -i\,\vec{e}\cdot\vec{J}$

This expression defines $\vec{J}$, a set of three 3×3 matrices if you think of R as a 3×3 matrix, and something more abstract if you think of R more abstractly. The three matrices are called the Lie algebra of SO(3). There are three because SO(3) is a three parameter group. (SO(n) is an $\frac{n(n-1)}{2}$ parameter group) How do you see that this derivative is linear in $\vec{e}$? Recall the picture of SO(3) as a ball. Assume all the differentiability you desire. What this derivative is is a directional derivative in the direction $\vec{e}$ at the center of the ball and $-i\vec{J}$ is the gradient.

Someplace in this analysis we have to put in the properties of SO(3). Rather than making mathematical statements about 3×3 matrices, we'll put in two properties physically that are enough to specify the group.

$$\text{①}\quad R(\vec{e}\theta)R(\vec{e}\theta') = R(e[\theta+\theta'])$$
$$\text{②}\quad R'^{-1}R(\vec{e}\theta)R' = R(R'^{-1}\vec{e}\theta)$$

These are physically motivated statements about properties of rotations

Property ① is obvious. How to see property ②? A way of characterizing the axis of rotation is to say it is the axis such that any vector parallel to this axis is unchanged by the rotation. $R'^{-1}\vec{e}$ is unchanged by the RHS of ②. It is also unchanged by the LHS because R' turns it into $\vec{e}$, which is unchanged by $R(\vec{e}\theta)$ and R'^{-1} turns it back into $R'^{-1}\vec{e}$. Thus the LHS is a rotation by θ about $R'^{-1}\vec{e}$. We won't actually use all the information contained in ① and ②. We will only use them in infinitesimal form, that is, we'll take derivatives with respect to θ and θ'. This loss of information decreases the restrictions on the form of the representations, and is the reason we pick up representations up to a phase, spinors, even though our formalism hasn't explicitly included them.

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Applying the infinitesimal analysis to a representation

Take $\frac{d}{d\theta} D(R(\vec{e}\,\theta))\Big|_{\theta=0} \equiv -i\,\vec{e}\cdot\vec{J}$ ① — The $\vec{J}$'s the "generators" of this representation

This is a very concrete equation. The D's are some representation of dimension N and the three $\vec{J}$'s are some $N\times N$ matrices. ① can be made into a statement about representations.

$$D(R(\vec{e}\,\theta'))\,D(R(\vec{e}\,\theta)) = D(R(\vec{e}\,[\theta+\theta']))$$

Take $\frac{d}{d\theta'}$, and set $\theta=0'$ to get (on the RHS $\frac{d}{d\theta}=\frac{d}{d\theta'}$)

$$-i\,\vec{e}\cdot\vec{J}\,D(R(\vec{e}\,\theta)) = \frac{d}{d\theta} D(R(\vec{e}\,\theta))$$

This is a simple differential equation. The solution with the boundary condition $D(R(\vec{e}\,0)) = \mathbb{1}$ is

$$D(R(\vec{e}\,\theta)) = e^{-i\vec{e}\cdot\vec{J}\,\theta}$$

The $\vec{J}$'s "generate the representation"

Now we can transfer our definitions about inequivalence and irreducibility to the generators. If the rep D is generated by $\vec{J}$ and the representation D' by $\vec{J}'$ and $D\sim D'$ that is $D(R) = S\,D'(R)\,S^{-1}$ for some S and all R then

$$S\,e^{-i\vec{e}\cdot\vec{J}'\theta}\,S^{-1} = e^{-i\vec{e}\cdot\vec{J}\,\theta} \iff S\vec{J}'S^{-1} = \vec{J},\ \text{i.e. } \vec{J}\sim\vec{J}'$$

Equivalence of two reps is the same as equivalence of their generators. What about irreducibility? If a representation is reducible, then it is equivalent to a representation that has block diagonal form for all rotations.

So $\vec{e}\cdot\vec{J} = i\frac{d}{d\theta} D(R(\vec{e}\,\theta))\Big|_{\theta=0}$ has block diagonal form.

Reducibility of a rep is the same as reducibility of its generators. I can even phrase this a little more strongly. If

$D(R) = \begin{pmatrix} D^{(1)}(R) & 0 \\ 0 & D^{(2)}(R) \end{pmatrix}$ then $\vec{J} = \begin{pmatrix} \vec{J}^{(1)} & 0 \\ 0 & \vec{J}^{(2)} \end{pmatrix}$ and if $D\sim D^{(1)}\oplus D^{(2)}$ then $\vec{J}\sim\vec{J}^{(1)}\oplus\vec{J}^{(2)}$

The task of finding inequivalent irreducible finite dimensional representations D of $SO(3)$ has been reduced to the task of finding inequivalent irreducible sets of 3 matrices $\vec{J}$, whose properties we are elucidating. DEC. 2 15

As a statement about representations, (2) is

$$D(R'^{-1})\,D(R(\vec{e}\theta))\,D(R') = D(R(R'^{-1}\vec{e}\,\theta))$$

Take $i\frac{d}{d\theta}$ at $\theta=0$ to get

$$D(R'^{-1})\,\vec{e}\cdot\vec{J}\,D(R') = (R'^{-1}\vec{e})\cdot\vec{J} = \vec{e}\cdot R'\vec{J}$$

rotation matrices preserve scalar products

Dropping the primes, and using the fact that $\vec{e}$ is an arbitrary unit vector this says $D(R)\vec{J}$

$$D(R^{-1})\,\vec{J}\,D(R) = R\vec{J}$$

Which is the statement that the generators of rotations, the 3 $\vec{J}$'s (N×N matrices) transform like a vector.

You can go further by writing R parameterizing R by $\vec{e}$ and θ (it shouldn't cause confusion to use these variables again). What we have found is

$$e^{i\vec{J}\cdot\vec{e}\theta}\,\vec{J}\,e^{-i\vec{e}\cdot\vec{J}\theta} = \vec{J} + \theta\,\vec{e}\times\vec{J} + O(\theta^2)$$

Where on the RH I have used a physical property of the rotation group, that for small θ, a rotation matrix acting on a vector changes it by

$$\theta\vec{e}\times\vec{V} + O(\theta^2)$$

(use the RH rule to make sure this agrees with the convention and picture on page 12). Take $-i\frac{d}{d\theta}$ of this equation to get

$$[\vec{J}\cdot\vec{e},\vec{J}] = -i\,\vec{e}\times\vec{J}$$

Take $\vec{e}=\hat{e}_x$ and look at y component to get

$[J_x, J_y] = i J_z$. also can get $[J_i, J_j] = i\epsilon_{ijk}J_k$

The generators $\vec{J}$ form a representation of the Lie algebra of the group. They satisfy the same commutation relns.

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Facts about finite dimensional inequivalent irreducible representations of the Lie algebra of the rotation group.

A complete set of them is the

$\vec{J}^{(s)}$ $\quad s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ $\quad$ "spin s"

$\vec{J}^{(0)} = \vec{0} \qquad \vec{J}^{(\frac{1}{2})} = \frac{\vec{\sigma}}{2} \qquad (J_i^{(1)})^j{}_k = -i\epsilon_{ijk}$, etc.

no sum

$J_z^{(s)}|m\rangle = m|m\rangle \qquad m = -s, -s+1, -s+2, \dots, s-2, s-1, s$

in a usual basis for the $2s+1$ dimensional vector space the $\vec{J}^{(s)}$ act on.

The $\vec{J}^{(s)}$ are hermitian. Every representation is equivalent to a hermitian representation.

Facts about finite dimensional inequivalent irreducible representations up to a phase of the rotation group, $D^{(s)}(R(\vec{e}\theta)) = e^{-i\vec{e}\cdot\vec{J}^{(s)}\theta}$

(1) The reps of the Lie algebra just listed not only generate the reps of the rotation group, they generate the reps up to a phase. The integer s are representations. The half integer s are reps up to a phase. More specifically, they are double valued

$$D^{(s)}(R(2\pi\vec{e})) = (-1)^{2s}\,\mathbb{1}$$

(2) $\quad \dim D^{(s)} = 2s+1$ $\qquad D^{(s)}(R^{-1}) = [D^{(s)}(R)]^{-1} = [D^{(s)}(R)]^{\dagger}$

(3) The hermiticity of the $\vec{J}^{(s)}$ implies the $D^{(s)}$ are unitary. Every representation of the rotation group is equivalent to a unitary representation of the rotation group. Of course in dumb bases, like $\vec{i}, \vec{j}, 7\vec{k}$ for the space $D^{(1)}$ acts on, the $D^{(1)}$'s preserve $x^2+y^2+\frac{1}{49}z^2$, and they are not unitary.

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(4) If I have any representation of any group G, $g \in G$, $D^a{}_b(g)$, I can define a new representation,

$$g \in G, \quad D^{*a}{}_b(g) = \left(D^a{}_b(g)\right)^* \quad \text{(no matrix transpose)}$$

This new representation has the same dimension as the original representation. It may or may not be equivalent to the representation you obtained it from. If D is irreducible, the new rep, D^* is irreducible. In $SO(3)$ since there is only one inequivalent irreducible representation of a given dimension, a rep must be equivalent to its associated complex conjugate rep. Furthermore

$$D^{(s)} \sim D^{(s)*} \quad \text{and} \quad D^{(s)}(R(\vec{e}\theta)) = e^{-i\vec{e}\cdot\vec{J}\theta}$$

implies $\vec{J}^{(s)} \sim -\vec{J}^{(s)*}$. The $-$ sign is present because of the i in the exponential.

(5) Direct product of representations. On page 16, a new notation, the notation of bras and linear operators, ($J_z^{(s)}|m\rangle = m|m\rangle$) appeared out of the blue. It is intuitively clear that a matrix of numbers acting on a column vector, can be reinterpreted as a linear operator acting on a vector space, but I would like to make the connection precise.

no sum implied

Suppose I have an n dimensional representation, that is, an $n \times n$ matrix for every element of a group, $D^i{}_j(g)$. Now let me define a linear operator, $D(g)$, (I am sorry this notation has also been used for the matrix) although we shall see why the ambiguity is small, which will act on an n dimensional vector space, with an orthonormal basis $|i\rangle$ $i = 1, \ldots, n$. $D(g)$ is defined by

$$D(g)|i\rangle = \sum_j |j\rangle\langle j|D(g)|i\rangle = \sum_j |j\rangle D^j{}_i(g)$$

The intermediate step was motivational, the definition is

$$D(g)|i\rangle \equiv \sum_j |j\rangle D^j{}_i(g)$$

From the definition, and the fact that the basis is orthonormal, you can recover the matrix $D^j{}_i(g)$ from the abstract linear operator.

$$\langle j|D(g)|i\rangle = \langle j|\left(\sum_k |k\rangle \quad D^k{}_i(g)\right)$$

$$= \sum_k \delta_{jk} D^k{}_i(g) = D^j{}_i(g)$$

Let's see what the composition law

$$\sum_j D^i{}_j(g) D^j{}_k(g') = D^i{}_k(gg')$$

which must be satisfied by a representation, implies about our new abstract linear operators.

$$D(g)D(g')|i\rangle = D(g)\left(\sum_j |j\rangle D^j{}_i(g')\right)$$

$$= \sum_{j,k} |k\rangle D^k_j(g) D^j{}_i(g')$$

$$= \sum_k |k\rangle D^k{}_i(gg')$$

$$= D(gg')|i\rangle \qquad \therefore \quad D(g)D(g') = D(gg')$$

It is nice to see this work out, because we also write the composition law of the matrices as

$$D(g)D(g') = D(gg')$$

Now we see that is true for the matrices and the abstract linear operators. <u>There was some danger</u> this was not going to work out. We might have gotten $D(g)D(g') = D(g'g)$ for the operator composition law.

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In this spiffy notation, I'll define the tensor product of two representations. First to define tensor product space. If I have two vector spaces

V_1 with a basis $|i\rangle_1$ $i=1,\dots,d_1$

and V_2 " " " $|j\rangle_2$ $j=1,\dots,d_2$

I define a vector space $V_1 \times V_2$ with a basis

$$|i,j\rangle \equiv |i\rangle_1 \otimes |j\rangle_2$$

The dimension of $V_1 \times V_2$ is $d_1 \cdot d_2$.

Given a linear operator on V_1, A, and a linear operator on V_2, B, I can define a linear operator on $V_1 \times V_2$ called $A \otimes B$ by

$$(A \otimes B)\,|i,j\rangle = \left(A|i\rangle_1\right) \otimes \left(B|j\rangle_2\right)$$

or $$\langle i',j'|\,A \otimes B\,|i,j\rangle = {}_1\langle i'|A|i\rangle_1\,{}_2\langle j'|B|j\rangle_2$$

Now if we have a representation $D^{(1)}$ acting on V_1 and $D^{(2)}$ acting on V_2, we can define a new representation denoted $D^{(1)} \otimes D^{(2)}$ acting on $V_1 \times V_2$:

$$\left(D^{(1)} \otimes D^{(2)}\right)(g) = D^{(1)}(g) \otimes D^{(2)}(g)$$

The new representation may or may not be reducible.

If the two representations are the same, that is $d=d_1=d_2$ $V=V_1=V_2$ and $D=D^{(1)}=D^{(2)}$, we can show that $D \otimes D$ is reducible (except in the case $d=1$). The trick is that in this case it makes to sense to talk about

$$\frac{1}{\sqrt{2}}\left(|i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle\right) \text{ and } \frac{1}{\sqrt{2}}\left(|i\rangle \otimes |j\rangle - |j\rangle \otimes |i\rangle\right)$$

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(or in another notation $\frac{1}{\sqrt{2}}(|i,j\rangle + |j,i\rangle)$ and $\frac{1}{\sqrt{2}}(|i,j\rangle - |j,i\rangle)$)

You can think of this as a new basis. The number of basis elements of the first type is $\frac{d(d+1)}{2}$, the number of the second is $\frac{d(d-1)}{2}$.

You can get fancy by defining projection operators

$$P_+(|i\rangle \otimes |j\rangle) \equiv \frac{1}{2}\left(|i\rangle \otimes |j\rangle + |j\rangle \otimes |i\rangle\right)$$

and

$$P_-(|i\rangle \otimes |j\rangle) = \frac{1}{2}\left(|i\rangle \otimes |j\rangle - |j\rangle \otimes |i\rangle\right)$$

The $d(d+1)/2$ elements of the symmetric part of the basis satisfy $P_+|sym\rangle = |sym\rangle$ $P_-|sym\rangle = 0$

The $\frac{1}{2}$ is put in so that $P_+^2 = P_+$ and $P_-^2 = P_-$

also $P_+ + P_- = \mathbb{1}$ $P_+ P_- = 0$

With these projection operators, I'll show that $D \otimes D$ is reducible. What I need to show is that

$$(D \otimes D)(g)\, P_\pm = P_\pm (D \otimes D)(g)$$

It's just a matter of using the definitions (somehow math proofs are always just a matter of using the definitions, although it is usually beyond me to do it)

$$(D \otimes D)(g)\, P_\pm \; |i\rangle \otimes |j\rangle$$

$$= \frac{1}{2}(D \otimes D)(g) \; (|i\rangle \otimes |j\rangle \pm |j\rangle \otimes |i\rangle)$$

$$= \frac{1}{2}\left(D(g)|i\rangle \otimes D(g)|j\rangle \pm D(g)|j\rangle \otimes D(g)|i\rangle\right)$$

$$= P_\pm \; D(g)|i\rangle \otimes D(g)|j\rangle = P_\pm (D \otimes D)(g) \;\; |i\rangle \otimes |j\rangle$$

<u>That does it</u>. It may not be obvious to you that this shows that the representation is reducible, since our definition of reducibility was in terms of matrices, so I'll make the connection precise. and I'll try to phrase the connection so that you can see a representation is reducible whenever you have a set of projection operators like P_+ and P_-, commuting with it.

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If you have a set of projection operators

$P_i, \quad i=1,\dots,m \qquad P_i^2 = P_i \qquad P_i P_j = 0, \; i \neq j$

$\sum_i P_i = \mathbb{1}$ then I can choose the basis of the vector space they act on so that it breaks up into bases for various subspaces that are either annihilated or unaffected by the P_i.

(In the example of importance, there are $\frac{n(n+1)}{2}$ basis vectors unaffected by P_+ and annihilated by P_-, while the other $\frac{n(n-1)}{2}$ basis vectors unaffected by P_- and annihilated by P_+)

I'll write the basis vectors as $|i,\alpha\rangle$ where $\alpha = 1,\dots,d_i$, where d_i is the dimension of the ith subspace

nothing to do with direct products → just a way of labelling the basis

$P_i |i,\alpha\rangle = |i,\alpha\rangle \qquad P_i |j,\alpha\rangle = 0 \quad j \neq i$

The big assumption about these projection operators is that they commute with the representation operators. Let's call the rep D.

$$P_i D(g) = D(g) P_i \quad \text{for all } i \text{ and } g$$

Let's look at the matrix associated with $D(g)$ in this basis and see what we can show about it.

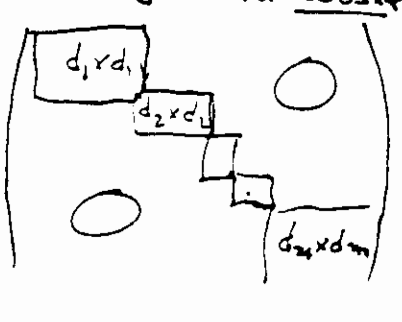
$$D^{i\alpha}{}_{j\beta}(g) = \langle i,\alpha | D(g) | j,\beta\rangle$$

$$= \langle i,\alpha | D(g) \textstyle\sum_k P_k | j,\beta\rangle = \langle i,\alpha | D(g) P_j | j,\beta\rangle$$

$$= \langle i,\alpha | P_j D(g) | j,\beta\rangle \propto \delta_{ji}$$

The proportionality constant depends g, i, j, α and β, but that doesn't matter.

This is the statement that in this basis, the matrix $D(g)$ is block diagonal.



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In the example of importance, by finding P_+ and P_- that commute with $D\otimes D$, we have shown that $D\otimes D$ is equivalent to

$$[(D\otimes D)^{i\alpha}_{j\beta}(g)] = \left(\begin{array}{c|c} \text{Some } \frac{n(n+1)}{2} \times \frac{n(n+1)}{2} \text{ matrix} & 0 \\ \hline 0 & \frac{n(n-1)}{2} \times \frac{n(n-1)}{2} \end{array}\right)$$

These two blocks may or may not be further reducible. If the $\frac{n(n+1)}{2}$ dim block is reducible into m irreducible components, $D^{(1)}, \ldots, D^{(m)}$, each of these representations is said to be in the symmetric part of the tensor product. If the $\frac{n(n-1)}{2}$ dimension block is reducible into m' irreducible components, $D'^{(1)}, \ldots, D'^{(m')}$, each of these is said to be in the antisymmetric part of the tensor product.

So after a five page rambling explanation of tensor products, I'll finally state the fifth fact about the rotation group. Tensoring two irreducible reps together

$$D^{(s_1)} \otimes D^{(s_2)} \sim D^{(s_1+s_2)} \oplus D^{(s_1+s_2-1)} \oplus D^{(s_1+s_2-2)} \oplus \cdots \oplus D^{(|s_1-s_2|)}$$

$$\equiv \bigoplus \sum_{s=|s_1-s_2|}^{s_1+s_2} D^{(s)}$$

Tensoring two identical reps together

$$D^{(s)} \otimes D^{(s)} = D^{(2s)} \oplus D^{(2s-1)} \oplus D^{(2s-2)} \oplus \cdots \oplus D^{(0)}$$

is in symmetric part

is in antisymmetric part

symmetric if s is an integer, antisymmetric if s is a $\frac{1}{2}$ integer

They just alternate.

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Parameterizing the connected homogeneous Lorentz group

The rotation group was parameterized by a direction $\vec{e}$ and an angle θ. We'll show that $SO(3,1)$ can be parameterized by $\vec{e}$, θ, and another direction and "angle", by showing that any Lorentz transformation can be decomposed into a rotation and a boost.

A Lorentz transformation is called a rotation if it takes

$$t \to t \quad \text{and} \quad \vec{x} \to R\vec{x}$$

we'll denote such a Lorentz transformation by R, and you'll have to understand from context when $R \in SO(3,1)$ and when R is a 3×3 orthogonal matrix.

A boost in the x direction by an "angle", velocity parameter, ϕ, has takes

$$t \to t\cosh\phi + x\sinh\phi$$

$$x \to t\sinh\phi + x\cosh\phi \qquad \begin{matrix} y \to y \\ z \to z \end{matrix}$$

we'll denote such a L.T. $A(e_x\phi)$, restricting $0 \le \phi < \infty$ to avoid parameterizing each boost more than one way. In general

$$A(\vec{e}\phi):\ t \to t\cosh\phi + \vec{e}\cdot\vec{x}\sinh\phi$$

$$\vec{x} \to \vec{e}\,t\sinh\phi + \vec{x} + \vec{e}\,(\cosh\phi - 1)\,\vec{e}\cdot\vec{x}$$

This generalization is forced upon you by an equation near the bottom of p.3

To go along with the formula

$$R(\vec{e}\theta')R(\vec{e}\theta) = R(\vec{e}[\theta+\theta'])$$

we also have the formula (which can be verified with a little algebra) (the algebra involves using formulas for $\sinh(\phi_1+\phi_2)$ and $\cosh(\phi_1+\phi_2)$)

$$A(e\phi')A(\vec{e}\phi) = A(\vec{e}[\phi+\phi'])$$

This is why the velocity parameter is such a useful parameter for boosts, it just adds.

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Now to prove that any Lorentz tranformation can be uniquely decomposed into a rotation followed by a boost. The proof is by construction and the construction is unambiguous, it has no freedom, which implies uniqueness.

Starting with a general Lorentz transformation Λ consider its action on the vector $e_0 \equiv (1, \vec{0})$. Since this vector has time component >0 and since all L.T. connected to the identity preserve this when the vector is timelike, we must have

$$\Lambda: e_0 \to (\gamma, \alpha\vec{e})$$

where α, γ real and greater than zero but otherwise unknown and $\vec{e}$ is some unit vector. The key thing about Lorentz tranformations is that they leave the length of a vector unchanged, so we know there is a restriction on γ and α:

$$\gamma^2 - \alpha^2 = 1. \text{ This and } \gamma > 0 \Rightarrow \gamma = \sqrt{1+\alpha^2}$$

Let's rename $\alpha = \sinh\phi$, $\phi > 0$, then $\gamma = \cosh\phi$ and the most general thing that e_0 can transform into under a Lorentz tranformation is

$$\Lambda: e_0 \to (\cosh\phi, \vec{e}\sinh\phi)$$

This determination of an "angle" ϕ and a direction $\vec{e}$ allows me to (uniquely) read off the boost that will bring e_0 back to rest, it is

$$A^{-1}(\vec{e}\phi)$$

$$A^{-1}(\vec{e}\phi)\Lambda: e_0 \to e_0$$

This means this product is some rotation, call it R.

$$A^{-1}(\vec{e}\phi)\Lambda = R \qquad \therefore \qquad \Lambda = R A(\vec{e}\phi)$$

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Just like when we were working with SO(3), formulas like the show that rotations and boosts can be written as exponentials. In a rep D it implies

last 2 on page 1

$$\frac{d}{d\theta} D(R(\vec{e}\theta))\Big|_{\theta=0} \equiv -i\vec{L}\cdot\vec{e}, \qquad D(R(\vec{e}\theta)) = e^{-i\vec{L}\cdot\vec{e}\theta}$$

$$\frac{d}{d\phi} D(A(\vec{e}\phi))\Big|_{\phi=0} \equiv -i\vec{M}\cdot\vec{e}, \qquad D(A(\vec{e}\phi)) = e^{-i\vec{M}\cdot\vec{e}\phi}$$

Just as for SO(3), if you know the inequivalent irreducible reps of $\vec{L}$ and $\vec{M}$, the generators of SO(3,1), you know the inequivalent irreducible reps up to a phase of SO(3,1). $\vec{L}$ is playing the exact same role as $\vec{J}$ did in our discussion of SO(3), in fact we would have reused $\vec{J}$ if it weren't that it is conventionally used for something else. So from the properties of rotations, we have

$$[L_i, L_j] = i\epsilon_{ijk} L_k \qquad (\Sigma_k \text{ implied})$$

(that came from $R'^{-1} R(\vec{e}\theta) R' = R(R'^{-1}\vec{e}\theta)$)

A property of the Lorentz group is that

$$R^{-1} A(\vec{e}\phi) R = A(R^{-1}\vec{e}\phi)$$

← can be used to get the general boost from a boost in the z direction

(you can convince yourself that both sides are boosts by ϕ along $R^{-1}\vec{e}$) Out of this (take $i\frac{d}{d\phi}$) comes the statement (applied to some rep D)

these are just like the SO(3) arguments so I haven't written them out in detail

$$D(R^{-1})\,\vec{e}\cdot\vec{M}\,D(R) = R^{-1}\vec{e}\cdot\vec{M} \quad \text{which implies}$$

$$D(R^{-1})\vec{M} \quad D(R) = R\vec{M}$$

which implies after a little more work

$$[L_i, M_j] = i\epsilon_{ijk} M_k$$

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Using $D(R)^{-1} M_i D(R) = \underbrace{R_{ij}}_{\substack{\text{3x3 orthogonal} \\ \text{rotation matrix}}} M_j$ (Σ_j implied)

It is easy to see that

$$D(R)^{-1} M_i M_j D(R) = R_{ik} R_{jl} M_i M_j$$

that is $M_i M_j$ transforms like a two index tensor under rotations. Therefore

$[M_i, M_j]$ is a two index antisymmetric tensor under rotations. If the Lie algebra of $SO(3,1)$ is going to close, the commutator of two boost generators must be a linear combination of a boost generator and a rotation generator. The most general thing I can make that is a two index antisymmetric tensor that is linear in the boost and rotation generators, which transform like vectors, is

$$\alpha \epsilon_{ijk} M_k + \beta \epsilon_{ijk} L_k$$

Therefore it must be that

$$[M_i, M_j] = i \epsilon_{ijk} [\alpha M_k + \beta L_k]$$

Still more evasive reasoning shows that $\alpha = 0$. Using the Parity automorphism introduced on December 2, page 9, we get a new representation from the one we were working with by defining $D_P(\Lambda) = D(\Lambda_P)$. Now for a rotation, $PRP = R$ so $D_P(R) = D(R)$. The generators of rotations in D_P are the same as those in D. However for $P\, A(\vec{e}\phi) P = A(-\vec{e}\phi)$ so the generator of boosts are minus the generator of boosts in D. This can be summarized, $L_i \to L_i$ $M_i \to -M_i$. The commutation relations have to still work under this transformation. $[L_i, L_j] = i\epsilon_{ijk} L_k$ is OK. $[L_i, -M_j] = i\epsilon_{ijk}(-M_k)$ is OK but $[-M_i, -M_j] = i\epsilon_{ijk}[\alpha(-M_k) + \beta L_k]$ is OK only if $\alpha = 0$.

with a fair amount of work — The − sign would not be present if this were so(4) instead of so(3,1)

In fact, you can check from the definitions, $[M_i, M_j] = -i\epsilon_{ijk}L_k$ i.e. $\beta=-1$ DEC. 4 5

From the commutation relations of the Lie algebra of any group, there is a general method called the method of highest weight. Fortunately, a miracle occurs, and we will not have to go through that method.

Define
$$\vec{J}^{(\pm)} = \frac{1}{2}\left(\vec{L} \pm i\vec{M}\right)$$
no i would be here if it were so(4) we were studying

$\vec{L} = \vec{J}^{(+)} + \vec{J}^{(-)}$

$-i\vec{M} = -(\vec{J}^{(+)} - \vec{J}^{(-)})$ This sign is obvious, see p.15

You can verify from the commutators of the $\vec{L}$'s and $\vec{M}$'s that
$$[J_i^{(\pm)}, J_j^{(\pm)}] = i\epsilon_{ijk} J^{(\pm)}{}_k$$
and
$$[J_i^{(\pm)}, J_j^{(\mp)}] = 0$$

my brain is better equipped to think about matrices than abstract linear operators, but you can think about it either way.

The $\vec{J}^{(+)}$'s and $\vec{J}^{(-)}$'s form two independent, commuting, SO(3) algebras. What we want to find is matrices that have these commutation relations—we want a complete set of inequivalent irreducible ones.

Here are some: For any s_1, s_2, $s_1 = 0, \frac{1}{2}, 1, \ldots$ $s_2 = 0, \frac{1}{2}, 1, \ldots$

take $J^{(+)}{}_i = J_i^{(s_1)} \otimes Id_{2s_2+1}$

$(2s_1+1)\times(2s_1+1)$ dimensional identity — $(2s_2+1)\times(2s_2+1)$ dimensional identity

and $J^{(-)}{}_i = Id_{2s_1+1} \otimes J_i^{(s_2)}$

The $J^{(s)}{}_i$ are our friends from the last lecture. Let me show that this satisfies one of the comm. relns. you can check the others

$$[J_i^{(+)}, J_j^{(+)}] = [J_i^{(s_1)} \otimes Id_{2s_2+1}, J_j^{(s_1)} \otimes Id_{2s_2+1}]$$
$$= J_i^{(s_1)} \otimes Id_{2s_2+1}\, J_j^{(s_1)} \otimes Id_{2s_2+1} - i \leftrightarrow j$$
$$= J_i^{(s_1)} J_j^{(s_1)} \otimes Id_{2s_2+1} - i \leftrightarrow j = [J_i^{(s_1)}, J_j^{(s_1)}] \otimes Id_{2s_2+1} = i\epsilon_{ijk} J_k^{(s_1)} \otimes Id_{2s_2+1}$$
$= i\epsilon_{ijk} J_k^{(+)}$

If you can't prove this from $(A\otimes B)^+ = A^+ \otimes B^+$ please feel free to come ask me to elaborate.

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Now you can solve for the $\vec{L}$'s and $\vec{M}$'s in terms of the $\vec{J}^{(\pm)}$.

$$\vec{L} = \vec{J}^{(+)} + \vec{J}^{(-)} = \vec{J}_1^{(s_1)} \otimes Id_{2s_2+1} + Id_{2s_1+1} \otimes \vec{J}^{(s_2)}$$

Because the $\vec{J}^{(s)}$ are hermitian, $\vec{L}$ is hermitian.

$$\vec{M} = \frac{1}{i}\left(\vec{J}^{(+)} - \vec{J}^{(-)}\right)$$ is antihermitian (because of the $\frac{1}{i}$)

The commutation relations among the $\vec{J}^{(+)}$ and $\vec{J}^{(-)}$ (which were derived by using the definitions in terms of $\vec{L}$ and $\vec{M}$), give us back the correct commutation relations for $\vec{L}$ and $\vec{M}$

Exponentiating $\vec{L}$ and $\vec{M}$ for any choice of s_1 and s_2 gives a representation called $D^{(s_1, s_2)}$

I'll define $\Lambda(\vec{e}\theta, \vec{f}\phi) = R(\vec{e}\theta)A(\vec{f}\phi)$

$$D^{(s_1,s_2)}(\Lambda(\vec{e}\theta, \vec{f}\phi))$$
$$= e^{-i\vec{L}\cdot\vec{e}\theta}\, e^{-i\vec{M}\cdot\vec{f}\phi}$$
$$= e^{-i(\vec{J}^{(+)}+\vec{J}^{(-)})\cdot\vec{e}\theta}\, e^{-(\vec{J}^{(+)}-\vec{J}^{(-)})\cdot\vec{f}\phi}$$

only for $\vec{e}\,//\,\vec{f}$

$$= e^{-i\vec{J}^{(+)}(\vec{e}\theta - i\vec{f}\phi)}\, e^{-i\vec{J}^{(-)}(\vec{e}\theta + i\vec{f}\phi)}$$

This has a simpler form for a pure boost or a pure rotation.

$$D^{(s_1, s_2)}(A(\vec{e}\phi)) = e^{-\left(\vec{J}^{(+)} - \vec{J}^{(-)}\right)\cdot\vec{e}\phi}$$
$$D^{(s_1, s_2)}(R(\vec{e}\theta)) = e^{-i\left(\vec{J}^{(+)} + \vec{J}^{(-)}\right)\cdot\vec{e}\theta}$$

It turns out that these are only representations up to a phase of $SO(3,1)$ if s_1+s_2 is a half integer. The claim is that these are a complete set of inequivalent irreducible reps up to a phase of the Lorentz group.

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I would like to explain the distinction between $D^{(s_1,s_2)}$, a representation of $SO(3,1)$, and $D^{(s_1)} \otimes D^{(s_2)}$, the direct product of two representations of $SO(3)$, which is in general reducible.

$$\left(D^{(s_1)} \otimes D^{(s_2)}\right)\left(R(\vec{e}\theta)\right) = e^{-i\vec{J}^{(s_1)}\cdot\vec{e}\theta} \otimes e^{-i\vec{J}^{(s_2)}\cdot\vec{e}\theta}$$

This is a rep of $SO(3)$, it has three generators, and they are given by taking

$$i\frac{d}{d\theta}\left(D^{(s_1)} \otimes D^{(s_2)}\right)\left(R(\vec{e}\theta)\right)\Big|_{\theta=0}$$

$$= \left(\vec{J}^{(s_1)} \otimes Id_{2s_2+1} + Id_{2s_1+1} \otimes \vec{J}^{(s_2)}\right)\cdot\vec{e}$$

These three generators, (one for each direction $\vec{e}$ can point) bear some resemblance to the six generators of $SO(3,1)$:

$$\vec{J}^{(+)} = \vec{J}^{(s_1)} \otimes Id_{2s_2+1} \quad \text{and} \quad \vec{J}^{(-)} = Id_{2s_1+1} \otimes \vec{J}^{(s_2)}$$

and if you restrict yourself to elements of $SO(3,1)$ such that the coefficient of $J_i^{(+)}$ is the same as the coefficient of $J_i^{(-)}$, that is, the rotations, you really have got the same thing. But, the coefficient of $J^{(+)}$ is independent of that of $J^{(-)}$. $D^{(s_1)} \otimes D^{(s_2)}$ is a reducible rep of $SO(3)$, $D^{(s_1,s_2)}$ is an irreducible rep of $SO(3,1)$.

A standard basis for the representation $D^{(s_1,s_2)}$ is the basis

$|m_+, m_-\rangle$ $\quad m_+ = -s_1, -s_1+1, \ldots, s_1-1, s_1$
$\quad m_- = -s_2, -s_2+1, \ldots, s_2-1, s_2$

which simultaneously diagonalize the two commuting hermitian operators $J_z^{(+)}$ and $J_z^{(-)}$:

$$J_z^{(\pm)}|m_+, m_-\rangle = m_\pm |m_+, m_-\rangle$$

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It would have been possible to carry out the analysis in a more Lorentz invariant fashion by defining (according to Ramond, p. 10)

$$J^{ij} = \epsilon_{ijk} L_k \qquad J^{00} = J^{ii}_{\ \text{no sum}} = 0$$
$$J^{0i} = -J^{i0} = -M_i$$

the commutator relations would have been unified into

$$[J^{\mu\nu}, J^{\alpha\beta}] = i g^{\nu\alpha} J^{\mu\beta} - i g^{\mu\alpha} J^{\nu\beta} - i g^{\nu\beta} J^{\mu\alpha} + i g^{\mu\beta} J^{\nu\alpha}$$

a Lorentz transformation would be parameterized by an antisymmetric matrix $\epsilon_{\mu\nu}$.

$$D(\Lambda(\epsilon_{\mu\nu})) = e^{-i\epsilon_{\mu\nu} J^{\mu\nu}}$$

This approach buys you nothing but elegance, at the expense of clarity.

Another thing worth noting is that a parallel analysis can be made of SO(4). It is kind of nice to look through the calculations and see where i's are changed to $-i$'s and where $g^{\mu\nu}$ becomes $\delta_{\mu\nu}$.

Carefully spelling out and proving any of the statements on pp. 6-11 would be instructive. I feel I have put enough of the outlines here that I could pursue the proofs to my satisfaction. I would enjoy making some of these statements more concrete. If people come ask me about them it will force me to do so.

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Facts about $D^{(s_1, s_2)}$ summarized

(1) The dimension of $D^{(s_1,s_2)}$ is $(2s_1+1)(2s_2+1)$

(2) When s_1+s_2 is a half integer we have a representation up to a phase

$$D(R(\vec{e}\pi))\,D(R(\vec{e}\pi)) = -1$$

(3) For $R \in SO(3,1)$, a rotation $D^{(s_1,s_2)}(R)$ is unitary, however for $A \in SO(3,1)$, a boost,

$D^{(s_1,s_2)}(A)$ is not. In fact it is hermitian.

This can be seen from the hermiticity of $\vec{J}^{(+)}$ and $\vec{J}^{(-)}$.

This agrees with some general theorems of group theory, worth noting.

The finite dimensional representations of a compact group are always equivalent to unitary representations, the generators can always be chosen to be hermitian in some basis.

$SO(3,1)$ is not a compact group. The range of boosts is infinite: $0 \le \phi < \infty$.

The unitary representations of a non-compact group are always infinite dimensional. The finite dimensional ones are never unitary.

although the $\vec{J}^{(\pm)}$ we have found are hermetian. Their coefficients, when Λ is not a rotation, in the exponential are not purely imaginary.

One can consider ∞ dimensional unitary reps. These would presumably describe an ∞ number of particle types. To agree with reality, infinitely many particles would some how have to be hidden....

The sketch of fact (4) can be made more concrete by giving a name S to the matrix that satisfies $S\vec{J}^{(s)*}S = -\vec{J}^{(s)}$ and using it to display the similarity transformation between $D^{(s_1,s_2)*}$ and $D^{(s_1,s_2)}$. DEC. 4

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(4) What rep do we get by taking the complex conjugate of $D^{(s_1,s_2)}$.

Since $\vec{J}^{(\pm)} \sim -\vec{J}^{(\pm)*}$ (follows from their expressions in terms of $\vec{J}^{(s_1)}$ and $\vec{J}^{(s_2)}$ and the properties of $\vec{J}^{(s)}$, Dec. 2 p. 17), the representation of a rotation is equivalent

$$D^{(s_1,s_2)*}(R(\vec{e}\,\theta)) = e^{+i(\vec{J}^{(+)}+\vec{J}^{(-)})^*\cdot\vec{e}\,\theta}$$
$$\sim e^{-i(\vec{J}^{(+)}+\vec{J}^{(-)})\cdot\vec{e}\,\theta} = D^{(s_1,s_2)}(R(\vec{e}\,\theta))$$

The representation of a boost however is screwed up in a way that cannot be undone by some equivalence.

$$D^{(s_1,s_2)*}(A(\vec{e}\,\phi)) = e^{-(\vec{J}^{(+)}-\vec{J}^{(-)})^*\cdot\vec{e}\,\phi}$$
$$\sim e^{-(\vec{J}^{(-)}-\vec{J}^{(+)})\cdot\vec{e}\,\phi}$$

The roles of $\vec{J}^{(-)}$ and $\vec{J}^{(+)}$ have been exchanged. This is true for both the rotations, $J^{(+)} \leftrightarrow J^{(-)}$ does nothing to them, and the boosts. We can identify the new rep we have made by noting that the exchange of $J^{(+)} \leftrightarrow J^{(-)}$ is just like the exchange of s_1 and s_2. That is

$$D^{(s_1,s_2)*}(\Lambda) \sim D^{(s_2,s_1)}(\Lambda)$$

If you believe that $D^{(s_1,s_2)}$ is not equivalent to $D^{(s_2,s_1)}$ (unless $s_2 = s_1$), this shows that $D^{(s_1,s_2)*}$ is not equivalent to $D^{(s_1,s_2)}$ (unless $s_1 = s_2$).

(5) The effect of parity on a rep was already discussed (p. 4). $L_i \to L_i$ $M_i \to -M_i$ is also interpretable as $J^{(+)} \leftrightarrow J^{(-)}$, so

$$D_P^{(s_1,s_2)} \sim D^{(s_2,s_1)}$$

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(6) $D^{(s_1', s_2')} \otimes D^{(s_1'', s_2'')}$

$$= \oplus \sum_{s_1=|s_1'-s_1''|}^{s_1'+s_1''} \sum_{s_2=|s_2'-s_2''|}^{s_2'+s_2''} D^{(s_1, s_2)}$$

If $s_1 = s_1' = s_1''$ and $s_2 = s_2' = s_2''$ then in the tensor product

$D^{(2s,2s)}$ is symmetric $D^{(2s,2s-1)}$ is antisymmetric
$D^{(2s,2s-2)}$ is symmetric, $D^{(2s,2s-3)}$ is A, etc.
$D^{(2s-1,2s-1)}$ is symm $D^{(2s-1,2s-2)}$ is A
$D^{(2s-2,2s-2)}$ is S, etc.

The proof of these statements would probably be instructive to construct. I expect with some thought they can be derived from the analogous statements about SO(3) in short order.

(7) If you understood page 7, this fact about $D^{(s_1,s_2)}$ will be easy to understand.

Any time you have a representation of a group (up to a phase), you have a representation of any of its subgroups (up to a phase).

Even if the representation of the group is irreducible, the representation of the subgroup may be reducible.

By restricting ourselves to the rotations, a subgroup of SO(3,1), we get a representation of SO(3) which is in general reducible. In fact we are down to the case where the coefficients of $\vec{J}^{(+)}$ and $\vec{J}^{(-)}$ are identical, and for those coefficients, we have exactly the same possible representation matrices as if we had taken $D^{(s_1)} \otimes D^{(s_2)}$. So $D^{(s_1,s_2)}$ "subduces" a rep of SO(3) which is

$$D^{(s_1)} \otimes D^{(s_2)} \sim \oplus \sum_{s=|s_1-s_2|}^{s_1+s_2} D^{(s)}$$

(s,0) and (0,s) are irreducible reps of SO(3,1) and SO(3)

$\left\{ \begin{array}{l} \text{Single valued} \\ \text{Double valued} \end{array} \right\}$ if $s^+ + s^-$ is $\left\{ \begin{array}{l} \text{integer} \\ \text{half-odd-integer} \end{array} \right\}$

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Where is the vector?

If the vector representation which we all know and love, is irreducible (I've never been able to reduce it), it must be equivalent to one of the four dimensional representations we have constructed. The only irreducible four dimensional reps on our list are

$$D^{(3/2,0)} \qquad D^{(0,3/2)} \qquad \text{and} \qquad D^{(1/2,\,1/2)}$$

$$(2\cdot\tfrac{3}{2}+1)\cdot(0+1) \qquad (2\cdot 0+1)(2\cdot\tfrac{3}{2}+1) \qquad (2\cdot\tfrac{1}{2}+1)\cdot(2\cdot\tfrac{1}{2}+1)$$

$$4 \qquad\qquad 4 \qquad\qquad 4$$

(there are only three ways to factor 4)

$D^{(3/2,0)}$ cannot be the vector for any of several reasons. (1) The complex conjugate of $D^{(3/2,0)}$ is $D^{(0,3/2)}$, which is inequivalent to $D^{(3/2,0)}$. However the vector is equivalent to its complex conjugate. Manifestly, because in the usual basis, the matrices that transform a vector are purely real. (2) A similar argument applies by considering the effect of parity. (3) If you restrict yourself to the rotation subgroup of $SO(3,1)$, the four vector representation is reducible into a three vector and a rotational scalar. $D^{(3/2,0)}$ however remains irreducible under this restriction. It is a spinor with spin 3/2 under rotations.

Identical arguments rule out $D^{(0,3/2)}$.

$D^{(1/2,1/2)}$ must be the vector. It looks funny, but it must be right. It certainly can't be ruled out along the lines of the above three arguments. Later we'll make a vector out of
$D^{(1/2,0)} \otimes D^{(0,1/2)}$, which is equivalent to $D^{(1/2,1/2)}$

What about rank 2 tensors

Usually we get a rank 2 tensor by taking the tensor product of two vectors. If A^μ and B^ν transform like vectors, $A^\mu B^\nu$ transforms like a tensor. If $D^{(1/2,1/2)}$ is a vector, then $D^{(1/2,1/2)} \otimes D^{(1/2,1/2)}$ must be a tensor. Now

$$D^{(1/2,1/2)} \otimes D^{(1/2,1/2)} \sim \underset{\substack{S \\ \text{dim } 9}}{D^{(1,1)}} \oplus \underset{\substack{A \\ \text{dim } 3}}{D^{(0,1)}} \oplus \underset{\substack{A \\ \text{dim } 3}}{D^{(1,0)}} \oplus \underset{\substack{S \\ \text{dim } 1}}{D^{(0,0)}}$$

There must be a way to reduce the tensor $T^{\mu\nu}$ into 9, 3, 3, and 1 dimensional subspaces that transform independently under $SO(3,1)$. We can rewrite

$$T^{\mu\nu} = \frac{1}{2}\left(T^{\mu\nu} + T^{\nu\mu}\right) + \frac{1}{2}\left(T^{\mu\nu} - T^{\nu\mu}\right)$$

$$\equiv \underset{\substack{\uparrow \\ 10 \text{ dim}}}{S^{\mu\nu}} + \underset{\substack{\uparrow \\ 6 \text{ dim}}}{A^{\mu\nu}}$$

and from our general arguments about symmetric and antisymmetric tensor products these two subspaces must transform independently. The 10 dim symmetric subspace must contain a 9 dim and 1 dim subspace which transform independently. Indeed, the linear combination

$g_{\mu\nu} S^{\mu\nu}$ is a Lorentz invariant,

so that 1 dimensional subspace transforms independently of the other 9 components of $S^{\mu\nu}$ which are in

$S^{\mu\nu} - \frac{1}{4} g_{\mu\nu} S^{\mu\nu}$ the

"traceless" part of S. This is $D^{(1,1)}$, the symmetric traceless tensor. In general, $D^{(n/2, n/2)}$ is the symmetric traceless tensor of rank n.

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Just for completeness, lets find the two irreducible parts of $A^{\mu\nu}$. Given any antisymmetric two index tensor, you can define a new antisymmetric tensor by

$$A^D{}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}A^{\alpha\beta} \qquad \epsilon_{0123} = +1$$

$A^D{}_{01} = A^{23}$ ← the $\frac{1}{2}$ was inserted to avoid $A^D{}_{01} = 2A^{23}$

$A^{D\,01} = -A^{23}, \qquad A^{D\,23} = A^{01}$

The cute thing about this operation is that

$A^{DD\mu\nu} = -A^{\mu\nu}$

The square of the dualing operation is -1 and the eigenvalues must be $\pm i$. The dualing operation also commutes with Lorentz transformations (that have $\det \Lambda = +1$). Thus, the Lorentz transformations transform the subspaces of each eigenvalue of the dualing operation independently.

$$A^{\mu\nu} = \frac{1}{2}\left(A^{\mu\nu} + i A^{D\mu\nu}\right) + \frac{1}{2}\left(A^{\mu\nu} - i A^{D\mu\nu}\right)$$
$$\equiv A^{(+)\mu\nu} + A^{(-)\mu\nu}$$

$$A^{(\pm)D\mu\nu} = \frac{1}{2}\left[A^{D\mu\nu} \pm i A^{DD\mu\nu}\right]$$
$$= \frac{1}{2}\left[A^{D\mu\nu} \mp i A^{\mu\nu}\right] = \mp i A^{(\pm)\mu\nu}$$

The six dimensional antisymmetric part of the tensor product has been broken into two 3 dimensional parts, which must be $D^{(1,0)}$ and $D^{(0,1)}$. It is no surprise that this splitting required taking complex combinations because $D^{(1,0)}$ and $D^{(0,1)}$ are not equivalent to their complex conjugates.

Spinors

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This is the example we are really interested in. We'll be using, for the rest of the course, fields transforming like $D^{(1/2,0)}$ and $D^{(0,1/2)}$. In fact, this is really the only application of the last couple lectures that is going to be used. (The scalar and vector reps you (presumably) already understood.) So if you get a good handle on how to manipulate these two representations, you don't really have to understand all the representation theory that has gone before.

We'll study $D^{(0,1/2)}$. Using the formulas on Dec. 4 page 5 with $s_1=0$, $s_2=\frac{1}{2}$, and using

$$\vec{J}^{(0)}=\vec{0}\ ,\qquad \vec{J}^{(\frac{1}{2})}=\frac{\vec{\sigma}}{2}$$

We have,

$$\vec{J}^{(+)}=\vec{0}\qquad\qquad \vec{J}^{(-)}=\frac{\vec{\sigma}}{2}$$

In this representation, a rotation is given by

$$D(R(\vec{e}\,\theta))=e^{-i\left(\vec{J}^{(+)}+\vec{J}^{(-)}\right)\cdot\vec{e}\,\theta}$$
$$=e^{-i\vec{\sigma}\cdot\vec{e}\,\theta/2}$$

a boost is represented by

$$D(A(\vec{e}\,\phi))=e^{-(\vec{J}^{(+)}-\vec{J}^{(-)})\cdot\vec{e}\,\phi}=e^{+\vec{\sigma}\cdot\vec{e}\,\phi/2}$$

because of this plus sign, let's call a field that transforms under $D^{(0,1/2)}$ u_+.

It better be. Up to equivalences, there is only one way for an irreducible rep. of a given dimension to transform under rotations.

In the $D^{(1/2,0)}$ rep, a rotation is represented the same way, but a boost has a minus sign in the exponential, so the field transforming as $D^{(1/2,0)}$ will be called u_-.

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Because $D^{(0,1/2)*} \sim D^{(1/2,0)}$, u_+^* must transform like $D^{(1/2,0)}$. If we want to think of u_+^* as a row vector, we'll write $u_+^\dagger$.
Since

$$D^{(1/2,0)} \otimes D^{(0,1/2)} \sim D^{(1/2,1/2)}$$

The product of u and $u^\dagger$ must be a four vector, and if we can find the right combinations, we can make that explicit.

An easy way to find the right combinations is to recall the transformations of spinors under the subgroup of rotations. Both u_+ and $u_+^\dagger$ transform as rotational spinors, and a rotational scalar can be made by taking

$u_+^\dagger u_+$ and a rotational vector is $u_+^\dagger \vec{\sigma} u_+$

The four vector must be $(u_+^\dagger u_+, \alpha u_+^\dagger \vec{\sigma} u_+)$.

α is unknown from this argument, but we can find it by looking at a boost along e_z with rapidity ϕ.

$$A(\vec{e}_z\phi): \quad u_+ \to e^{+\sigma_z\phi/2} u_+$$

$$\text{and} \quad u_+^\dagger \to u_+^\dagger (e^{\sigma_z\phi/2})^\dagger$$

$$= u_+^\dagger e^{\sigma_z\phi/2}$$

so $u_+^\dagger u_+ \to u_+^\dagger e^{\sigma_z\phi} u_+$

$$= u_+^\dagger (\cosh\phi + \sigma_z \sinh\phi) u_+$$

$$= \cosh\phi \, u_+^\dagger u_+ + \sinh\phi \, u_+^\dagger \sigma_z u_+.$$

If we take $\alpha = 1$, i.e. $V^0 = u_+^\dagger u_+$ $\vec{V} = u_+^\dagger \vec{\sigma} u_+$, this says

$$V^0 \to \cosh\phi \, V^0 + \sinh\phi \, V_z$$

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V^3 if you want to keep your indices up.

Let's see what V_z and V_x (or V_y) transform into.

$$V_z = u_+^\dagger \sigma_z u_+ \to u_+^\dagger e^{\sigma_z\phi/2}\sigma_z e^{\sigma_z\phi/2} u_+$$
$$= u_+^\dagger(\sigma_z \cosh\phi + \sinh\phi)u_+$$
$$= \cosh\phi\, V_z + \sinh\phi\, V^0$$

$$V_x = u_+^\dagger \sigma_x u_+ \to u_+^\dagger e^{\sigma_z\phi/2}\sigma_x e^{\sigma_z\phi/2} u_+$$

But using $e^{\sigma_z\phi/2} = \cosh\frac{\phi}{2} + \sinh\frac{\phi}{2}\sigma_z$

and $\sigma_z\sigma_x = -\sigma_x\sigma_z$, we see that we have found $V_x \to u_+^\dagger e^{+\sigma_z\phi/2} e^{-\sigma_z\phi/2}\sigma_x u_+$

$= V_x$ similarly $V_y \to V_y$

If you had gone through this procedure for u_-, you would have found that the vector is

$$W^\mu = (u_-^\dagger u_-, -u_-^\dagger \vec{\sigma} u_-)$$

It looks like the two component fields we have found have a fighting chance of describing spin $1/2$ particles.

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Promote the two component objects $u_\pm$ into two component functions of space time, spinor fields.

Criteria for a free theory made up of a u_+ field

$$\mathcal{L}(u_+, u_+^\dagger, \partial_\mu u_+, \partial_\mu u_+^\dagger)$$

(i) $S = \int d^4x\, \mathcal{L}$ had better be real, $S = S^*$.

(ii) $\mathcal{L}$ must be a Lorentz scalar, but not necessarily parity invariant.

(iii) $\mathcal{L}$ bilinear in the fields so we get a linear equation of motion and a free field theory.

(iv) No more than two derivatives in $\mathcal{L}$. If we can't construct anything of this type we'll go to three four or more derivatives.

(v) Want the theory to have a conserved charge $u_+ \to e^{i\theta} u_+ \quad u_+^\dagger \to e^{-i\theta} u_+^\dagger$

(Rules out Majorana neutrinos) because all known spin 1/2 particles in the world do carry some conserved quantum number, like baryon number.

Property (v) with property three forces us to have one u_+ and one $u_+^\dagger$ factor in each term. Now the product $u_+ u_+^\dagger$ is a four vector, which is neither a scalar itself, nor can a scalar be built from it with an even number of derivatives. However, a scalar can be built from it and one derivative. So (ii) and (iv) imply

$$\mathcal{L} \propto (u_+^\dagger \partial_0 u_+ + u_+^\dagger \vec{\sigma}\cdot\vec{\nabla} u_+)$$

the coefficient of proportionality must be purely imaginary to satisfy (i). proof involves a parts integration By rescaling u_+ and $u_+^\dagger$ we have

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$$\mathcal{L} = \pm i\left[u_+^\dagger \partial_0 u_+ + u_+^\dagger \vec{\sigma}\cdot\vec{\nabla} u_+\right]$$

(For u_- we would have arrived at

$$\mathcal{L} = \pm i\left[u_-^\dagger \partial_0 u_- - u_-^\dagger \vec{\sigma}\cdot\vec{\nabla} u_-\right].)$$

This is called the Weyl Lagrangian. Let's derive the equation of motion by varying w.r.t. $u_+^\dagger$

$$\partial_0 u_+ + \vec{\sigma}\cdot\vec{\nabla} u_+ = 0$$

We can see that any solution of this equation is a solution of the Klein-Gordon equation by acting on it with $\partial_0 - \vec{\sigma}\cdot\vec{\nabla}$ to get.

$$\left(\partial_0^2 - \nabla^2\right) u_+ = 0$$

an identity matrix has been supressed. To derive this you need to use equality of mixed partials and $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}$ a 2x2 identity is again supressed here

So all the solutions of the Weyl equation satisfy the wave equation, and they must be of the form

$$u_+(x) = u_+ e^{-ik\cdot x} \qquad k^2 = 0 \quad (\text{no need yet for } k^0 > 0)$$

some constant two component column vector.

{For u_- we would have gotten $\partial_0 u_- - \sigma\cdot\vec{\nabla} u_- = 0$, which leads to $u_-(x) = u_- e^{-ik\cdot x}$ }

Now we plug these potential solutions back into the ~~eu~~ equation to get the condition on the constant spinors

$$(k^0 - \vec{\sigma}\cdot\vec{k})\, u_+ = 0$$

2×2 identity understood

Lets take $\vec{k} = k^0 e_z$. This then says

$$(1-\sigma_z)\, u_+ = 0 \qquad u_+ = \begin{pmatrix}1\\0\end{pmatrix}$$

(at this point we would have gotten $u_- = \begin{pmatrix}0\\1\end{pmatrix}$)

In general u_+ is the eigenstate of $\dfrac{\vec{\sigma}\cdot\vec{k}}{k^0}$, which since $|\vec{k}| = k^0$

with eigenvalue 1

has the interpretation of being the spin along the direction of motion (at least for $k^0>0$).

Unlike the normal theory of spinors, for a given k^μ, we only have one solution, one direction of spin. This would not be possible if these particles weren't massless. If they were massive, you could always boost to their rest frame, turn their spin around and then boost back, and you'd have a particle with spin pointed the opposite direction. The spin of a massless particle is usual referred to as helicity, the component of angular momentum along the direction of motion. Spin is usually reserved for massive particles. It is the angular momentum in the rest frame.

Because $D^{(0,1/2)} = D^{(1/2,0)}$ which is inequivalent to $D^{(0,1/2)}$ there^P is no parity transformation in this theory. More physically, we have found ~~to~~ a theory with a single u_+ spinor has only one helicity. Since parity reverses linear momentum, but leaves angular momentum unaffected, a theory with particles of only one helicity can't be parity invariant.

Some guesses about the quantum field u_+ and the particles it will annihilate and create. DEC. 9 4

The solution of the Weyl equation going like $e^{-ik\cdot x}$ with $k^0 > 0$, will probably multiply an annihilation operator in the expansion of the quantum field u_+. By the known transformation properties of the solution of the field equation we can obtain the transformation properties of the states it annihilates. We expect

$$\langle 0|\, u_+(x)\, |k\rangle \propto e^{-ik\cdot x} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$k^2 = 0,\ k^0 > 0 \quad k_x = k_y = 0,\ k_z = k^0$$

or

$$\langle 0|\, u_+(0)\, |k\rangle \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We are going to find the J_z value of $|k\rangle$:

$$J_z |k\rangle = \lambda |k\rangle, \quad \text{find } \lambda.$$

In the quantum theory

$$U(R(\vec{e}_z\theta)) = e^{-iJ_z\theta}$$

and thus $U(R(\vec{e}_z\theta))\,|k\rangle = e^{-i\lambda\theta}\,|k\rangle$,

$$U(R(\vec{e}_z\theta))\,|0\rangle = |0\rangle.$$

On the other hand

$$U^\dagger(R(\vec{e}_z\theta))\, u_+(0)\, U(R(\vec{e}_z\theta)) = D^{(0,1/2)}(R(\vec{e}_z\theta))\, u_+(0)$$

and this eqn. can be used by taking the $\langle 0|$ —— $|k\rangle$ matrix element. We get

$$e^{-i\lambda\theta}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \propto e^{-i\sigma_z\theta/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-i\theta/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

i.e. $\lambda = \frac{1}{2}$. The annihilation operator multiplying $u_+ e^{-ik\cdot x}$, $k_0 > 0$, will annihilate particles with helicity ½ along the direction of motion.

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when Weyl came up with this theory, which describes neutrinos, the world thought the world was parity invariant, and his theory was dismissed quickly. People thought he was just playing with irrelevant mathematics.

Creation operators in the expansion of $u_+(x)$ will create $\lambda = -\frac{1}{2}$ particles. The field always changes the helicity by the same amount. The field $u_+^\dagger$ will annihilate particles with $\lambda = -\frac{1}{2}$ and create particles with $\lambda = +\frac{1}{2}$.

The annihilation operators in the u_- field annihilate particles with helicity $-\frac{1}{2}$, like neutrinos are observed to have, and the creation ops create particles with helicity $+\frac{1}{2}$, the antineutrinos. Conventionally such a field is called "left handed." By the right hand rule, a "right handed" field annihilates "right handed" particles, that is, ones whose angular momentum is along the direction of motion (agrees with I&Z, p.88).

Instead of canonically quantizing this theory now, we are going to move on and find a Lagrangian for massive particles that can include parity. The representation the fields are in must be equivalent to the representation obtained by parity. We could look at the irreducible reps of this type, $D^{(n/2, n/2)}$, but when you restrict to rotations, these reps contain do not contain spinors, in fact as already stated, they turn out to be rank n symmetric traceless tensors under $SO(3,1)$. The simplest parity invariant reducible rep with spinors is $D^{(1/2,0)} \oplus D^{(0,1/2)}$

a set of two complex doublets u_+ and u_-.

We'll restrict the possible Lagrangians by a set of conditions like those on page 1.

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$\mathcal{L}(u_+, u_-, u_+^\dagger, u_-^\dagger, \partial_\mu u_+, \dots)$ must be

(i) Bilinear

(ii) real, at least $S = S^*$

(iii) No more than one ∂_μ derivative

(iv) $\mathcal{L}$ is a Lorentz scalar

(v) The Lagrangian has a $U(1)$ symmetry under which u_+ and u_- transform the same way:

$$u_\pm \longrightarrow u_\pm\ e^{i\theta} u_\pm \qquad u_\pm^\dagger \longrightarrow e^{-i\theta} u_\pm^\dagger$$

(vi) The theory has a parity operation

$$u_+(\vec{x}, t) \longrightarrow a\ u_-(-\vec{x}, t)$$
$$u_-(\vec{x}, t) \longrightarrow b\ u_+(-\vec{x}, t)$$

under which $\mathcal{L}$ is invariant.

$u_+^\dagger u_-$

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$D^{(\frac{1}{2},0)}(\Lambda) \otimes D^{(\frac{1}{2},0)} \sim D^{(1,0)}(\Lambda) \oplus D^{(0,0)}(\Lambda) \propto u_+^\dagger u_-$?

after rescaling u_+ and u_-, the only Lagrangian satisfying the first five conditions is

$$\mathcal{L} = \pm\Big[i u_+^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ + i\underset{\epsilon = \pm 1}{\epsilon}\, u_-^\dagger(\partial_0 - \sigma\cdot\vec{\nabla})u_- - m u_+^\dagger u_- - m^* u_-^\dagger u_+ \Big]$$

By adjusting the relative phase of u_- and u_+, we can always take m to be real and nonnegative.

Let's find out if we can define a parity of the type on page 6.

$$u_+^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ \to u_-^\dagger(-\vec{x},t)\, a^*(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\, a\, u_-(-\vec{x},t)$$

$$= |a|^2 u_-^\dagger(-\vec{x},t)\left(\partial_0 - \vec{\sigma}\cdot\frac{\partial}{\partial(-\vec{x})}\right)u_-(-\vec{x},t)$$

If this term in the parity transformed Lagrangian is going to equal the term in the original Lagrangian

$$i\epsilon\, u_-^\dagger(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})u_-$$

we must have $\epsilon = +1$ and $|a|^2 = 1$.

By considering the effect of parity on the other terms, you also get

$|b|^2 = 1$ and $ab^* = 1$

the conditions on a and b can be summarized by saying $a = b = e^{i\lambda}$

$$P: u_\pm(\vec{x},t) \to e^{i\lambda} u_\mp(-\vec{x},t)$$

Now if a theory has an internal symmetry (and this one has an internal $U(1)$ symmetry), then I can redefine parity to be the old parity composed with any element of the internal symmetry group, and I'll have just as good a definition of parity. In this case, compose parity with the symmetry

$$u_\pm(x) \to e^{-i\lambda} u_\pm(x)$$

Then the new parity has the effect

$$P: \; u_\pm(\vec{x}, t) \to u_\mp(-\vec{x}, t)$$

The Lagrangian we have found is the Dirac Lagrangian, although it doesn't look like it yet. Let's derive the equations of motion. By varying $u_+^\dagger$, you get

$$i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ = m u_- \quad \text{and by varying } u_-^\dagger$$

$$i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})u_- = m u_+$$

This is Dirac's equation although it doesn't look like it yet. The solutions of the Dirac equation are solutions of the Klein-Gordon equation. To see this, multiply the first equation by $-i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$ and use the second equation to get

$$(\partial_0^2 - \vec{\nabla}^2)u_+ = -m^2 u_+$$

or $(\Box + m^2)u_+ = 0$. This verifies that the thing we have been calling m actually is a mass, and not $m/2$ or $m^{3/5}$ or whatever.

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We are now going to modify the equations in order to make them more obscure and sophisticated looking. Actually, we'll be building a machinery which will speed up calculations. Assemble the two two-component fields into a single four component one.

$$\psi \equiv \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$ (this is not the only way to do this, see below.)

Remember a 2x2 identity matrix is supressed here

Then $$\mathcal{L} = \pm\left[i u_+^\dagger(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})u_+ + i u_-^\dagger(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})u_- - m u_+^\dagger u_- - m u_-^\dagger u_+ \right]$$

can be rewritten as

$$\mathcal{L} = \pm\left[i\left(\psi^\dagger \partial_0 \psi + \psi^\dagger \vec{\alpha}\cdot\vec{\nabla}\psi\right) - m\psi^\dagger \beta \psi \right]$$

where $\vec{\alpha}$'s and β are $\underline{4 \times 4}$ hermitian matrices

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

each entry represents a 2x2 block

In this spiffy notation we can also write the effect of parity as

$$P: \psi(\vec{x}, t) \rightarrow \beta\,\psi(-\vec{x}, t)$$

The equation of motion obtained by varying $\psi^\dagger$ is

$$i(\partial_0 + \vec{\alpha}\cdot\vec{\nabla})\psi = \beta m \psi$$ This is the Dirac equation (1929)

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A pleasant surprise is that in this notation we can also give the effect of a Lorentz boost without defining a whole bunch more matrices. The equations

$$\Lambda: \quad u_+ \rightarrow e^{\vec{\sigma}\cdot\vec{e}\phi/2}\, u_+$$
$$u_- \rightarrow e^{-\vec{\sigma}\cdot\vec{e}\phi/2}\, u_-$$

can be assembled into

$$\Lambda: \quad \psi \longrightarrow e^{\vec{\alpha}\cdot\vec{e}\,\phi/2}\,\psi$$

The generator of boosts (what is dotted into $-i\vec{e}\phi$ in the exponential) is

$$\vec{M} = \frac{i\vec{\alpha}}{2}$$

We can get the generators of rotations quickly by using

$$[M_i, M_j] = -i\epsilon_{ijk} L_k$$

$$\vec{L} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

I said $\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$ is not the only way of making a four component spinor out of two two-component spinors. Another way is

$$\psi = \begin{pmatrix} u_+ + u_- \\ u_+ - u_- \end{pmatrix} \frac{1}{\sqrt{2}}$$

If we had done that, the $\vec{\alpha}$'s and β would have come out differently. In fact this second way is the way Dirac originally did it.

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To summarize, in the first basis, called the

WEYL BASIS (usually called the Weyl representation, but the terminology is incorrect)

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

$$\vec{M} = \frac{i\vec{\alpha}}{2} \qquad [M_i, M_j] = -i\epsilon_{ijk} L_k \Rightarrow \vec{L} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

DIRAC (OR STANDARD) BASIS

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \psi = \frac{1}{\sqrt{2}}\begin{pmatrix} u_+ + u_- \\ u_+ - u_- \end{pmatrix}$$

$$\vec{M} = \frac{i\vec{\alpha}}{2} \qquad \vec{L} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

In either basis, the Dirac Lagrangian is

$$\mathcal{L} = \pm\left[i\psi^\dagger\left(\partial_0 + \vec{\alpha}\cdot\vec{\nabla}\right)\psi - m\psi^\dagger\beta\psi\right]$$

4×4 identity matrix suppressed

The Dirac basis is the standard basis because the solutions of the Dirac equation become especially simple in the nonrelativistic limit in this basis.

There are extensive discussions of the solutions of the Dirac equation in the literature. All you have to do is ignore all references to holes and negative energy solutions. Just because something was understood in a poor way 50 years ago, doesn't mean you have to learn it that way today.

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12

PLANE WAVE SOLUTIONS OF THE DIRAC EQUATION

We are going to look for solutions of the form

$$\psi = u_{\vec{p}}\, e^{-ip\cdot x} \quad \text{or} \quad \psi = v_{\vec{p}}\, e^{ip\cdot x}$$

where $u_{\vec{p}}$ and $v_{\vec{p}}$ are space-time independent four component spinors. Of course, to have a chance of satisfying the Dirac equation, a proposed solution must satisfy the Klein-Gordon equation, so $p^2 = m^2$, or

$$p^0 \equiv \sqrt{\vec{p}^{\,2} + m^2} \qquad p \text{ is a forward pointing vector on the mass shell}$$

(The other solution, $p^0 = -\sqrt{\vec{p}^{\,2}+m^2}$, is taken care of in the way we have busted up the problem into two cases: one with $e^{-ip\cdot x}$ dependence and one with $e^{+ip\cdot x}$ dependence.)

If you plug the first type of solution,

$$\psi = u_{\vec{p}}\, e^{-ip\cdot x}$$

(called "positive frequency", a convention that goes all the way back to $H\psi = i\hbar\frac{\partial\psi}{\partial t} \to \psi_n \sim e^{-i\omega_n t}$)

into the Dirac equation,

$$i(\partial_0 + \vec{\alpha}\cdot\vec{\nabla})\psi = \beta m \psi$$

you get

$$(p^0 - \vec{\alpha}\cdot\vec{p})\, u_{\vec{p}} = \beta m\, u_{\vec{p}}$$

Lets look at a special case: $\vec{p} = 0$, $p^0 = m$. Then the equation says

$$\therefore\; u_{\vec{0}} = \beta\, u_{\vec{0}}$$

each entry represents a 2x2 block

In the standard basis, $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and there are two linearly independent solutions to this equation, they are of the form

$$u_{\vec{0}} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

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I can choose the direction ~~my~~ two standard linearly independant solutions point in the subspace they are allowed in, as well as their total normalization, any way I like. A convenient choice is

$$u_{\vec{0}}^{(1)} = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_{\vec{0}}^{(2)} = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Whatever basis you choose, make it satisfy,

$$u_{\vec{0}}^{(r)\dagger} u_{\vec{0}}^{(s)} = 2m\,\delta_{rs}$$

and $u_{\vec{0}}^{(r)\dagger} \vec{\alpha}\, u_{\vec{0}}^{(s)} = 0$

the second condition follows fairly easily from the form of $\vec{\alpha}$. In the standard basis it connects the upper two components with the lower two.

By arguments like those used for the solutions of the Weyl equation, we expect that in the expansion of the quantum field ψ, an annihilation operator that annihilates $J_z = +\frac{1}{2}$ electrons with momentum p will multiply

$$u_{\vec{p}}^{(1)} e^{-ip\cdot x}$$

and an annihilation operator that annihilates electrons with $J_z = -\frac{1}{2}$ will multiply

$$u_{\vec{p}}^{(2)} e^{-ip\cdot x}$$

The $2m$ in the normalization of the the u's is there to agree with conventional normalization for relativistic states: reduces to $2m$ when $\vec{p}=0$

$$\langle p' | p \rangle = (2\pi)^3 (2\omega_{\vec{p}})\, \delta^{(3)}(\vec{p}-\vec{p}\,')$$

Good for neutrinos or extremely high energy physics

In many equations it will allow us to take a smooth $m \to 0$ limit.

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2

The Dirac Lagrangian was constructed to be a Lorentz scalar, so you expect that the equations of motion derived from it are Lorentz invariant in the sense that given one solution of the Dirac equation, I ought to get another solution by Lorentz transforming it. So while we could just go ahead and solve the Dirac equation for arbitrary $\vec{p}$ (it's just the problem of finding the eigenvalues and eigenspinors of some $\vec{p}$ dependent 4x4 matrix), we will flaunt Lorentz invariance by getting solutions with momentum $\vec{p}$ by boosting those with momentum $\vec{0}$.

$$u_{\vec{p}}^{(r)} = e^{\vec{\alpha}\cdot\vec{e}\,\phi/2}\, u_{\vec{0}}^{(r)} \quad \text{where } \vec{e} = \frac{\vec{p}}{|\vec{p}|}$$

$$\text{and } \sinh\phi = \frac{|\vec{p}|}{m} \quad (\cosh\phi = \tfrac{E}{m})$$

You can mechanically verify that the conditions

$$u_{\vec{p}}^{(r)\dagger} u_{\vec{p}}^{(s)} = 2p^0 \delta_{rs} \qquad u_{\vec{p}}^{(r)\dagger} \vec{\alpha}\, u_{\vec{p}}^{(s)} = 2\vec{p}\,\delta_{rs}$$

are satisfied, but it is actually not necessary.

$\left(u_{\vec{p}}^{(r)\dagger} u_{\vec{p}}^{(s)},\ u_{\vec{p}}^{(r)\dagger} \vec{\alpha}\, u_{\vec{p}}^{(s)}\right)$ is a four vector, and you know it is $(2m\delta_{rs}, \vec{0})$ when $\vec{p}=0$: that determines it completely for arbitrary $\vec{p}$. (This proof is sweet because it is basis independent)

Using $(\vec{\alpha}\cdot\vec{e})^2 = 1$, you can rewrite

$$u_{\vec{p}}^{(r)} = \left[\cosh\frac{\phi}{2} + \vec{\alpha}\cdot\vec{e}\,\sinh\frac{\phi}{2}\right] u_{\vec{0}}^{(r)}$$

4x4 identity supressed

Using $\cosh\frac{\phi}{2} = \sqrt{\frac{1+\cosh\phi}{2}}$ and $\sinh\frac{\phi}{2} = \sqrt{\frac{\cosh\phi - 1}{2}}$ and $\cosh\phi = \frac{E}{m}$, you can rewrite

$$u_{\vec p}^{(r)} = \left[\sqrt{\frac{E+m}{2m}} + \sqrt{\frac{E-m}{2m}}\,\vec\alpha\cdot\vec e\right] u_{\vec 0}^{(r)}$$

In the standard basis, with $\vec p$ (hence $\vec e$) pointing in the z direction, this is

$$u_{\vec p}^{(1)} = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ \sqrt{E-m} \\ 0 \end{pmatrix} \qquad u_{\vec p}^{(2)} = \begin{pmatrix} 0 \\ \sqrt{E+m} \\ 0 \\ -\sqrt{E-m} \end{pmatrix}$$

The normalization is already proving useful. Thanks to that factor of $\sqrt{2m}$, this doesn't blow up when $m \to 0$.

A similar set of relations holds for the $v_{\vec p}^{(r)}$.

$$v_{\vec 0}^{(1)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_{\vec 0}^{(2)} = \sqrt{2m}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$v_{\vec p}^{(r)} = e^{\vec\alpha\cdot\vec e\,\phi/2}\, v_{\vec 0}^{(r)}$$

$$\left(v_{\vec p}^{(r)\dagger} v_{\vec p}^{(s)},\ v_{\vec p}^{(r)\dagger}\vec\alpha\, v_{\vec p}^{(s)}\right) = \left(2p^0\delta_{rs},\ 2\vec p\,\delta_{rs}\right)$$

In the standard basis, with $\vec p$ in the z direction,

$$v_{\vec p}^{(1)} = \begin{pmatrix} \sqrt{E-m} \\ 0 \\ \sqrt{E+m} \\ 0 \end{pmatrix} \qquad v_{\vec p}^{(2)} = \begin{pmatrix} 0 \\ -\sqrt{E-m} \\ 0 \\ \sqrt{E+m} \end{pmatrix}$$

The $\vec{\alpha}$'s and β satisfy a simple algebra.

$$\alpha_i^2 = 1 \qquad \{\alpha_i, \alpha_j\} = 0 \qquad i \neq j$$

$$\beta^2 = 1 \qquad \{\beta, \alpha_i\} = 0$$

Every $\vec{\alpha}$ and β squares to 1 and anticommutes with all three others.

<u>A famous theorem due to Pauli</u>

Any set of 4 4×4 matrices obeying these equations is equivalent to any other set.

The theorem says "everything is in here." Anything we get by manipulating a set of 4×4 matrices satisfying this algebra can be obtained by manipulating the algebra.

"Anything" means any equation that is unaffected by a similarity transformation, or any result that is basis independent, like a cross section summed over final spin states and averaged over initial ones.

An example of a statement that is not basis independent is

$$\beta^\dagger = \beta$$

This is true in the Weyl or standard basis, but in general, just because β is hermitian, it does not follow that

$$S^{-1} \beta S \quad \text{is hermitian.}$$

It is true, though if S is unitary. Sometimes we'll restrict ourselves to bases that are related by a unitary S. They are called "unitarily equivalent."

Coleman's proof uses something we already believe (but in fact takes some effort to prove): That up to equivalences we have found all the finite dimensional reps of the Lorentz group.

Start by constructing a representation of the Lorentz group from the $\vec{\alpha}$'s and β.

Let $M_i \equiv i\frac{\alpha_i}{2}$ Define L_k by $[M_i, M_j] = -i\epsilon_{ijk}L_k$

$$\left(L_k = \frac{i}{2}\epsilon_{ijk}[M_i, M_j]\right)$$

Using the algebra the α's are supposed to obey, it is easy to show that

$$[L_i, M_j] = i\epsilon_{ijk}M_k \text{ and } [L_i, L_j] = i\epsilon_{ijk}L_k$$

Thus we have defined a representation of the Lorentz group. Furthermore, it is a four dimensional representation, and the rotation generators square to $\frac{1}{4}$, $L_i^2 = \frac{1}{4}$. Thus it must be made of just spin $\frac{1}{2}$ reps when you restrict this four dimensional rep of the Lorentz group to the rotation subgroup. What we have made must be equivalent to

$$D^{(1/2,0)} \oplus D^{(0,1/2)},\ D^{(0,1/2)} \oplus D^{(0,1/2)} \text{ or } D^{(1/2,0)} \oplus D^{(1/2,0)}$$

But now the existence of β can be used to rule out the second two possibilities. Note that $\beta^2 = 1 \Leftrightarrow \beta = \beta^{-1}$ so using the algebra

$\beta^{-1}\vec{\alpha}\beta = \beta\vec{\alpha}\beta = -\vec{\alpha}$. For our generators this implies

$$\beta^{-1}\vec{M}\beta = -\vec{M} \quad \text{and} \quad \beta^{-1}\vec{L}\beta = \vec{L}$$

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Recall the stuff about parity: Given a representation of the Lorentz group, D, I can define a new rep D_P by

$$D_P(\Lambda) = D(\Lambda_P).$$

The generators of the rotations in this new rep were the same. The generators of boosts in D_P were minus the generators of boosts in D. In general the representation obtained from

$D^{(n/2,\, m/2)}$ was $D^{(m/2,\, n/2)}$

These are equivalent only if $m=n$. The parity transform of

$$D^{(1/2,0)} \oplus D^{(1/2,0)} \quad \text{is} \quad D^{(0,1/2)} \oplus D^{(0,1/2)}$$

and vice versa. These (reducible) reps are not equivalent to their parity transformed reps. But β, by the equations at the bottom of the previous page, is such an equivalence.* These reps cannot be candidates for what we have constructed.

We must have constructed the rep $D^{(1/2,0)} \oplus D^{(0,1/2)}$ (it is equivalent to its parity transform), and there is only one such rep up to equivalence transformations.

A jargony way of saying what we have found is

There is only one rep of $SO(3,1)$ plus parity that is four-dimensional and only contains spin $1/2$ particles.

A little bit of the proof remains to be done. We have shown the $\vec{\alpha}$'s are always equivalent, but it remains to be shown that β is equivalent by the same transformation.

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So suppose I have found similarity transformation that puts the $\vec{\alpha}$'s into standard form

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$$

Can I find a further similarity transformation that leaves the $\vec{\alpha}$'s unchanged, but brings β into standard form? If I could, this would show that any set of $\vec{\alpha}$'s and β is equivalent to any other, since they are all equivalent to a standard form.

With $\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$, the algebra

$$\{\vec{\alpha}, \beta\} = 0 \implies \beta = \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix}$$

this block is proportional to a 2×2 identity.

(This is fairly easy to show write $\beta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and find out what the conditions are on each of the 2×2 matrices A, B, C and D. No matrix anticommutes with all three Pauli matrices, and the only matrix that commutes with all three of them is the identity.)

The other condition on the β from the algebra is $\beta^2 = 1$, which implies

$$\lambda_1 \lambda_2 = 1 \qquad \text{or} \qquad \lambda = \lambda_1 = \lambda_2^{-1}$$

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So $\quad \vec{\alpha} = \begin{pmatrix} \sigma & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}$ and $\beta = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}$

and we need to find a similarity transformation that leaves the $\vec{\alpha}$'s unaffected, but puts β into the standard form

$$\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The similarity transformation is $\quad S = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$

$$S^{-1}\beta S = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \lambda^{-1} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

a lot of people including some Nobel laureates did lengthy calculations using explicit representations of the Dirac algebra in the 1930's. This must have been unnecessary since the whole thing is tied up in the commutation relations.

From now on, we are going to assume we are in a basis where

$$\vec{\alpha} = \vec{\alpha}^{\dagger} \qquad \text{and} \qquad \beta = \beta^{\dagger}.$$

A basis which is obtained from this basis by a unitary transformation will also satisfy these relations. All popular representations satisfy these relations

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DIRAC ADJOINT, PAULI-FEYNMAN NOTATION 9
DUE TO PAULI

Since $\psi^\dagger \beta \psi$ is a Lorentz invariant ($-m\psi^\dagger\beta\psi$ appears in the Lagrangian), that is since under

$$\Lambda : \psi \to D(\Lambda)\psi \qquad \psi^\dagger\beta\psi \to \psi^\dagger\beta\psi$$

We are going to define a new adjoint

$$\bar\psi = \psi^\dagger\beta$$

This new adjoint has every property you'd like an adjoint to have except that $\bar\psi\psi$ is not always greater than zero.

Then we can write, oh so slickly

$$\Lambda : \bar\psi\psi \to \bar\psi\psi \qquad (\text{or } \Lambda : \bar\chi\psi \to \bar\chi\psi \text{ for two Dirac spinors})$$

[The situation is a lot like SO(3,1). The usual inner product between two four vectors

$$y^T x = \sum_\mu y^\mu x^\mu$$

is not a Lorentz invariant. The combination that is is

$$y^T g x$$
$$g = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

So we define a "new transpose", $y^T g$ and call it a covariant vector, and put its indices down. Then we can write, oh so slickly,

$$\Lambda : \sum_\mu y_\mu x^\mu \to \sum_\mu y_\mu x^\mu .$$

As a statement about the 4×4 matrices Λ this is

$$\Lambda^T g \Lambda = g \quad \text{or} \quad g\Lambda^T g \Lambda = 1 \;]$$

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The definition of the adjoint of an operator is obtained from the definition of the adjoint of a vector by

$$\phi^\dagger A^\dagger \psi \equiv (\psi^\dagger A \phi)^*$$

That tells you an arbitrary matrix element of $A^\dagger$. Similarly, the Dirac adjoint of an operator is obtained by

$$\bar{\phi} \bar{A} \psi \equiv (\bar{\psi} A \phi)^*$$

It is the work of a moment to show that

$$\bar{A} = \beta A^\dagger \beta$$

all your favorite equations for the adjoint of an operator follow for the Dirac adjoint, and the proofs are the same since it all comes from the definition of the adjoint.

$$\overline{\alpha A + \beta B} = \alpha^* \bar{A} + \beta^* \bar{B}$$

$$\overline{AB} = \bar{B}\bar{A}$$

$$\overline{A\psi} = \bar{\psi}\bar{A}$$ This last relation implies that $\Lambda: \bar{\psi}\psi \to \bar{\psi}\,\overline{D(\Lambda)}\,D(\Lambda)\psi$ but this equals $\bar{\psi}\psi$ so $\overline{D(\Lambda)}\,D(\Lambda) = 1$ is the statement about the 4x4 matrices $D(\Lambda)$. They aren't unitary, but they are "Dirac unitary".

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Remember that

$V^\mu = (\chi^+\psi, \chi^+\vec{\alpha}\psi)$ transforms like a four-vector.

Here comes some more notation to make this look slick too.
We can rewrite V^μ as

$$V^\mu = (\bar{\chi}\,\beta\psi, \bar{\chi}\,\beta\vec{\alpha}\psi) = \bar{\chi}\gamma^\mu\psi$$

Where $\gamma^\mu = (\beta, \beta\vec{\alpha})$

These are the famous Dirac γ matrices.

With a slight abuse of language, we can say that the γ matrices transform like a vector. Of course, they don't transform at all. What is meant by this, no more, no less, is that

$$\overline{D(\Lambda)}\,\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu .$$

We can also say that the product $\gamma^\mu\gamma^\nu$ transforms like a tensor. The proof is:

$$\overline{D(\Lambda)}\,\gamma^\mu\gamma^\nu D(\Lambda) = \overline{D(\Lambda)}\,\gamma^\mu \underbrace{D(\Lambda)\overline{D(\Lambda)}}_{\text{fancy way of inserting } 1}\gamma^\nu D(\Lambda)$$

$$= \Lambda^\mu{}_\sigma \gamma^\sigma \Lambda^\nu{}_\tau \gamma^\tau$$

$$= \Lambda^\mu{}_\sigma \Lambda^\nu{}_\tau \gamma^\sigma\gamma^\tau .$$

This is the transformation law for a two index tensor.

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If in some math book you start reading about Clifford algebras, it is a special case of them we are studying. more generally, $\mu,\nu = 1,\dots,N/2$ and any # of diagonal components of $g^{\mu\nu}$ can be -1.

The anti-commutation relations for the γ matrices follow from those for the $\vec{\alpha}$'s and β. They can all be summed up in

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$ which you should check.

also, in the special class of bases we have restricted ourselves to,

$$\gamma^{0\dagger} = \gamma^0 \qquad \gamma^{i\dagger} = -\gamma^i$$

an elegant way of summarizing these four relations is

$$\overline{\gamma^\mu} = \gamma^\mu$$

Which you could check, but here is a high-powered proof instead.

In the same sense as on the previous page, both sides of this equation transform like a four-vector (The RHS we have already accepted this for;

$$\overline{D(\Lambda)}\,\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\,\gamma^\nu.$$

The LHS transforms the same way as you can see by taking the bar of this equation to get

$$\overline{D(\Lambda)}\,\overline{\gamma^\mu}\,D(\Lambda) = \Lambda^\mu{}_\nu\,\overline{\gamma^\nu})$$

This matrix is unaffected. It is real, and it is not transposed, because we are transposing only in the spinor indices. maybe I should just say, for any given μ, this is just a set of 4 real coefficients.

So $\overline{\gamma^\mu} = \gamma^\mu$ is Lorentz a covariant equation. To see if it is correct we only have to check one of its components, say $\mu=0$. For $\mu=0$ it reduces to

$$\gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0 \quad \checkmark.$$

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Now that we have these Dirac γ matrices, we can rewrite the Dirac Lagrangian as

$$\mathcal{L} = \pm\left[i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi\right]$$ the eqn of motion is $i\gamma^\mu\partial_\mu\Psi - m\Psi = 0$ (from varying $\bar{\Psi}$).

We can make this look even more sophisticated and obscure by introducing a super compact notation due to Feynman. Let

$$\not{a} \equiv a_\mu\gamma^\mu \quad (= a^\mu\gamma_\mu = a\cdot\gamma)$$

The algebra of the γ matrices can be summarized in this notation as

$$\{\not{a}, \not{b}\} = 2a\cdot b \qquad (\not{a}^2 = a^2)$$

The Dirac Lagrangian and eqn. of motion are

$$\mathcal{L} = \pm\ \bar{\Psi}(i\not{\partial} - m)\Psi \qquad (i\not{\partial} - m)\Psi = 0$$

The proof that each component of every solution of the Dirac equation satisfies the Klein-Gordon equation is

$$(i\not{\partial} - m)\Psi = 0 \Rightarrow (-i\not{\partial} - m)(i\not{\partial} - m)\Psi = 0$$

$$\Rightarrow (\Box + m^2)\Psi = 0$$

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Parity and γ_5

$$P: \psi(\vec{x},t) \to \beta\psi(-\vec{x},t)$$

β could also be written γ^0

$$\bar\psi(\vec{x},t) \to \overline{\beta\psi(-\vec{x},t)} = \bar\psi(-\vec{x},t)\bar\beta$$
$$= \bar\psi(-\vec{x},t)\beta$$

So
$$P: \bar\psi\psi(\vec{x},t) \to \bar\psi\psi(-\vec{x},t)$$

Not only is $\bar\psi\psi$ a scalar under the Lorentz transformations connected to the identity, it is a scalar under parity.

$$P: \bar\psi\gamma^\mu\psi \to \bar\psi\beta\gamma^\mu\beta\psi$$
$$= \begin{cases} \bar\psi\gamma^0\psi & \mu=0 \\ -\bar\psi\gamma^i\psi & \mu=i \end{cases}$$

This is how you expect a vector to transform under parity.

What about $\bar\psi\gamma^\mu\gamma^\nu\psi$. It's clear that the 00 component will be unaffected by parity, as will the ii components, while the oi components will go into minus themselves. This is a tensor under L.T. and parity. Actually we have obtained nothing new from the 00 and ii components because $\gamma^{\mu 2}=1$. The only new quantities we have are the antisymmetric parts. Define

$$\sigma^{\mu\nu} = \frac{1}{2i}[\gamma^\mu,\gamma^\nu] \qquad \overline{\sigma^{\mu\nu}} = \sigma^{\mu\nu}$$

$\bar\psi\sigma^{\mu\nu}\psi$ is an antisymmetric tensor under L.T. and parity.

$\bar\psi_A\psi_B$

$$(1\cdots4)\left(D^{(\frac{1}{2},0)}\oplus D^{(0,\frac{1}{2})}\right)\otimes\left(D^{(\frac{1}{2},0)}\oplus D^{(0,\frac{1}{2})}\right)$$

$$\sim \underbrace{D^{(\frac{1}{2},\frac{1}{2})}\oplus D^{(\frac{1}{2},\frac{1}{2})}}_{\text{vector}}\oplus\underbrace{D^{(0,1)}\oplus D^{(1,0)}}_{\text{tensor}}\oplus\underbrace{D^{(0,0)}\oplus D^{(0,0)}}_{\text{scalar}}$$

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We can't proceed on building tensors of higher and higher rank. In the product of two four component objects there are only 16 possible bilinears. So far we have found

$$\underset{\text{scalar}}{1}+\underset{\text{vector}}{4}+\underset{\text{antisymmetric tensor}}{6}=11 \quad \text{of them.}$$

Let's jump up to tensors of the fourth rank and see what we can make

$$\bar{\psi}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\psi$$

is a fourth rank tensor, but if any two of the indices are the same, this reduces to something we have already found. There is only one possibility if all four indices have to be different, it is

$$\gamma^0\gamma^1\gamma^2\gamma^3$$

It is conventional to define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (\equiv \gamma^5)$$

$$= \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta \qquad (\epsilon_{0123}=+1)$$

Unfortunately, because of the i, $\overline{\gamma_5}=-\gamma_5$, but
$\gamma_5^+=\gamma_5$ and $\gamma_5^2=\gamma_5\gamma_5^+=+1$

Except for $\overline{\gamma_5}=-\gamma_5$, γ_5 is a lot like a fifth γ matrix, that is

$$\{\gamma_5,\gamma^\mu\}=0$$

We'll write $i\gamma_5$, because $\overline{i\gamma_5}=i\gamma_5$

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Now $\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$ transforms like a scalar under L.T., but goes into minus itself under parity, that is,

$$P: \bar{\Psi} i \gamma_5 \psi \longrightarrow -\bar{\Psi} i \gamma_5 \psi \quad \text{is a pseudoscalar}$$

$\bar{\Psi} i\gamma_5 \psi$ is hermitian, it can appear in a Lagrangian with a real coefficient.

$\bar{\Psi}\gamma^\mu\gamma_5\psi$ is also hermitian and

$$P: \bar{\Psi}\gamma^\mu\gamma_5\psi \longrightarrow \begin{cases} -\bar{\Psi}\gamma^0\gamma_5\psi & \mu=0 \\ \bar{\Psi}\gamma^i\gamma_5\psi & \mu=i \end{cases}$$

an axial vector

So now we have found a total of 16 bilinears transforming in distinct ways under parity and L.T.

S 1 scalar
P 1 pseudoscalar
V 4 vector
A 4 axial vector
T 6 antisymmetric tensor

Any other bilinear we might construct must be expressible in terms of these.

We could start building Lagrangians with interactions like (but we are going to proceed with canonical quantization)

$g\phi\bar{\Psi}i\gamma_5\psi$ (to conserve parity ϕ must be a pseudoscalar)

or $g\phi\bar{\Psi}\psi$ or $g\partial_\mu\phi\bar{\Psi}\gamma^\mu\psi$ (under parity ϕ would be a scalar)

or $g\phi\bar{\Psi}i\gamma_5\psi + h\phi\bar{\Psi}\psi$ (parity violating, no choice of parity is possible)

Some things that are very useful when deriving any basis independent result are <u>orthogonality</u> <u>and completeness</u> conditions for the $u^{(r)}$ and $v^{(r)}$

The $u_{\vec{p}}^{(s)}$ and $v_{\vec{p}}^{(r)}$ satisfy

$$(\not{p}-m)\,u_{\vec{p}}^{(r)}=0 \qquad (\not{p}+m)\,v_{\vec{p}}^{(r)}=0$$

Taking the bar of these <u>equations</u> we also have

$$\overline{u_{\vec{p}}^{(r)}}\,(\not{p}-m)=0 \qquad \overline{v_{\vec{p}}^{(r)}}\,(\not{p}+m)=0$$

ORTHOGONALITY CONDITIONS

We have already derived (see page 2)

$$\overline{u_{\vec{p}}^{(r)}}\gamma^\mu u_{\vec{p}}^{(s)} = 2p^\mu\delta_{rs} \text{ and } \overline{v_{\vec{p}}^{(r)}}\gamma^\mu v_{\vec{p}}^{(s)} = 2p^\mu\delta_{rs}$$

Now $\overline{v_{\vec{p}}^{(r)}}\gamma^\mu u_{\vec{p}}^{(s)}$ is also a four vector, and we can find what it is by looking at

$$\overline{v_{\vec{0}}^{(r)}}\gamma^0 u_{\vec{0}}^{(s)} = v_{\vec{0}}^{(r)+} u_0^{(s)} = 0$$

so $$\overline{v_{\vec{p}}^{(r)}}\gamma^\mu u_{\vec{p}}^{(s)} = \overline{u_{\vec{p}}^{(r)}}\gamma^\mu v_{\vec{p}}^{(s)} = 0$$

There are also the scalars

$$\overline{u_{\vec{p}}^{(r)}}u_{\vec{p}}^{(s)} \qquad \overline{v_{\vec{p}}^{(r)}}u_{\vec{p}}^{(s)} \quad \text{and} \quad \overline{v_{\vec{p}}^{(r)}}v_{\vec{p}}^{(s)}$$

Evaluating them for $\vec{p}=0$ is easy, and since they are scalar, that gives their value for general $\vec{p}$

u's are $\beta=+1$ eigenstates
v's $\beta=-1$

$$\overline{u_{\vec{p}}^{(r)}}u_{\vec{p}}^{(s)} = 2m\delta_{rs} \qquad \overline{v_{\vec{p}}^{(r)}}v_{\vec{p}}^{(s)} = -2m\delta_{rs}$$

$$\overline{u_{\vec{p}}^{(r)}}v_{\vec{p}}^{(s)} = \overline{v_{\vec{p}}^{(r)}}u_{\vec{p}}^{(s)} = 0$$

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COMPLETENESS RELATIONS

Suppose I have an orthogonal normalized basis for R^n,
$$\vec{e}^{(r)} \qquad r=1,\dots,n$$
Then $\quad \Sigma_n \vec{e}^{(r)} \vec{e}^{(r)T}$ is the identity matrix.
We are going to get the analog of this for our 4 solutions of the Dirac equation for any $\vec{p}$.

Define
$$A = \Sigma_r \, u_{\vec{p}}^{(r)} \overline{u_{\vec{p}}^{(r)}}$$
Let's see what A is by seeing what it does to a basis for our 4-dim spinor space.
$$A \, u_{\vec{p}}^{(s)} = \Sigma_r \, u_{\vec{p}}^{(r)} \, 2m \delta_{rs} = 2m \, u_{\vec{p}}^{(s)}$$
$$A \, v_{\vec{p}}^{(s)} = \Sigma_r \, u_{\vec{p}}^{(r)} \cdot 0 = 0$$

But we already know a matrix that has this effect on the basis, $\not{p}+m$, so
$$\Sigma_r \, u_{\vec{p}}^{(r)} \overline{u_{\vec{p}}^{(r)}} = \not{p}+m$$

Similarly,
$$\Sigma_r \, v_{\vec{p}}^{(r)} \overline{v_{\vec{p}}^{(r)}} = \not{p}-m$$

PHYSICS 253a

What Every 253a Student Needs to Know about The Dirac Equation

I have heard that some of you have had trouble keeping the Dirac equation in view through a cloud of SO(3,1) representation theory. This sheet has been prepared to help you. It contains all the results we have derived to date that we will need in the remainder of the course, without proofs.

1. Dirac Lagrangian, Dirac Equation, Dirac Matrices

The theory is defined by the Lagrange density,

$$\mathscr{L} = \psi^\dagger [i\partial_0 + i\vec{\alpha}\cdot\vec{\nabla} - \beta m]\psi \ .$$

Here ψ is a set of four complex fields, arranged in a column vector (a Dirac bispinor), and the α's and β are a set of four 4×4 Hermitian matrices (the Dirac matrices). The equation of motion (the Dirac equation) is

$$(i\partial_0 + i\vec{\alpha}\cdot\vec{\nabla} - \beta m)\psi = 0 \ .$$

The Dirac matrices obey the Dirac algebra,

$$\{\alpha_i,\alpha_j\} = 2\delta_{ij} \ , \quad \{\alpha_i,\beta\} = 0 \ , \quad \beta^2 = 1 \ .$$

Any set of 4×4 matrices obeying this algebra is equivalent to any other set. Two representations of the Dirac algebra that will be useful to us are the Weyl representation,

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \ , \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ,$$

and the standard representation,

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

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2. Space-Time Symmetries

The Dirac equation is invariant under both Lorentz transformations and parity.

Under a Lorentz transformation characterized by a 4×4 Lorentz matrix, Λ,

$$\Lambda: \psi(x) \to D(\Lambda)\psi(\Lambda^{-1}x) \quad ,$$

where the matrix D is defined from the α's by the following rules: For an acceleration by rapidity ϕ in a direction $\vec{e}$,

$$D\left(A(\vec{e}\phi)\right) = e^{\vec{\alpha}\cdot\vec{e}\phi/2} \quad .$$

$\alpha^i = \gamma^0\gamma^i$ (follows from $\gamma^0 = \beta$ $\gamma^i = \beta\alpha^i$ and $\beta^2 = 1$)

For a rotation by angle θ about an axis $\vec{e}$,

$$D\left(R(\vec{e}\theta)\right) = e^{-i\vec{L}\cdot\vec{e}\theta} \quad ,$$

where $\vec{L}$ is defined by

$$[\alpha_i, \alpha_j] = 4i\varepsilon_{ijk}L_k \quad .$$

$L_k = \frac{i}{4}\epsilon_{kij}\gamma^i\gamma^j$

In both the Weyl and standard representations

$$\vec{L} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad .$$

Under parity,

$$P: \psi(\vec{x},t) \to \beta\psi(-\vec{x},t) \quad .$$

3. Dirac Adjoint, γ Matrices

The Dirac adjoint of a Dirac bispinor is defined by

$$\bar{\psi} = \psi^{\dagger}\beta \quad ,$$

of a 4×4 matrix by

$$\bar{A} = \beta A^{\dagger}\beta \quad .$$

These obey the usual rules for adjoints, e.g.,

$$(\bar{\psi}A\phi)^{*} = \bar{\phi}\bar{A}\psi \quad .$$

The γ matrices are defined by

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i.$$

These are not all Hermitian,

$$\gamma^{\mu\dagger} = \gamma_\mu \equiv g_{\mu\nu}\gamma^\nu ,$$

but they are self-Dirac adjoint ("self-bar"),

$$\bar{\gamma}^\mu = \gamma^\mu .$$

The γ matrices obey the γ algebra,

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} .$$

They also obey

$$\bar{D}(\Lambda)\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu .$$

For any vector, a, we define

$$\not{a} = a_\mu \gamma^\mu .$$

It follows from the γ algebra that

$$\not{a}\not{b} + \not{b}\not{a} = 2a\cdot b .$$

In this notation, the Dirac Lagrange density is

$$\bar{\psi}(i\not{\partial}-m)\psi ,$$

and the Dirac equation is

$$(i\not{\partial}-m)\psi = 0 .$$

4. Bilinear Forms

There are sixteen linearly independent bilinear forms we can make from a Dirac bispinor and its adjoint. We can choose these sixteen to form the components of objects that transform in simple ways under the lorentz group and parity.

The scalar is

$$S = \bar{\psi}\psi .$$

The vector is

$$V^\mu = \bar{\psi}\gamma^\mu\psi .$$

The tensor is

$$T^{\mu\nu} = \bar{\psi}\sigma^{\mu\nu}\psi ,$$

where

$$\sigma^{\mu\nu} = \frac{1}{2i}\,[\gamma^\mu,\gamma^\nu] \ .$$

The <u>pseudoscalar</u> is

$$P = \bar\psi\, i\,\gamma_5\,\psi \ ,$$

where

$$\gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^5$$

The <u>axial vector</u> is

$$A^\mu = \bar\psi\,\gamma^\mu\,\gamma_5\,\psi \ .$$

γ_5 is in many ways "the fifth γ matrix". It obeys

$$(\gamma_5)^2 = 1, \qquad \gamma_5 = \gamma_5^\dagger = -\bar\gamma_5, \qquad \{\gamma_5,\gamma^\mu\} = 0 \ .$$

5. <u>Plane-Wave Solutions</u>

The positive-frequency solutions of the Dirac equation are of the form

$$\psi = u\,e^{-ip\cdot x} \ ,$$

where $p^2 = m^2$ and p^0 is positive. The negative-frequency solutions are of the form $\psi = v\,e^{ip\cdot x}$.

There are two positive-frequency and two negative-frequency solutions for each p. The Dirac equation implies that

$$(\not p - m)\,u = 0 = (\not p + m)\,v \ .$$

For a particle at rest, $p = (m,\vec 0)$, we can choose the two independent u's in the standard representation to be,

$$u^{(1)}_{\vec 0} = \begin{bmatrix}\sqrt{2m}\\0\\0\\0\end{bmatrix}, \qquad u^{(2)}_{\vec 0} = \begin{bmatrix}0\\\sqrt{2m}\\0\\0\end{bmatrix},$$

and the two independent v's to be

$$v^{(1)}_{\vec 0} = \begin{bmatrix}0\\0\\\sqrt{2m}\\0\end{bmatrix}, \qquad v^{(2)}_{\vec 0} = \begin{bmatrix}0\\0\\0\\\sqrt{2m}\end{bmatrix}.$$

We can construct the solutions for a moving particle, $u^{(r)}_{\vec p}$ and $v^{(r)}_{\vec p}$, by applying a Lorentz acceleration (see 2).

These solutions are normalized such that

$$\bar{u}^{(r)}_{\vec{p}}\, u^{(s)}_{\vec{p}} = 2m\,\delta^{rs} = -\bar{v}^{(r)}_{\vec{p}}\, v^{(s)}_{\vec{p}}\,, \qquad \bar{u}^{(r)}_{\vec{p}}\, v^{(s)}_{\vec{p}} = 0\ .$$

They obey the completeness relations

$$\sum_{r=1}^{2} u^{(r)}_{\vec{p}}\, \bar{u}^{(r)}_{\vec{p}} = \not{p} + m\ , \qquad \sum_{r=1}^{2} v^{(r)}_{\vec{p}}\, \bar{v}^{(r)}_{\vec{p}} = \not{p} - m\ .$$

Another way of expressing the normalization condition is

$$\bar{u}^{(r)}_{\vec{p}}\, \gamma^{\mu}\, u^{(s)}_{\vec{p}} = 2\delta^{rs}\, p^{\mu} = \bar{v}^{(r)}_{\vec{p}}\gamma^{\mu}\, v^{(s)}_{\vec{p}}\ .$$

This form has a smooth limit as m goes to zero.

Quantum Field Theory

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CANONICAL QUANTIZATION OF DIRAC LAGRANGIAN

$$\mathcal{L} = \pm\left[\psi^+(i\partial_0 + i\vec{\alpha}\,\vec{\nabla} - \beta m)\psi\right]$$

$\pi_A = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_A} = \pm i\psi_A^+$

$$\pi_\psi \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \pm i\psi^+ \quad (\psi, \psi^+ \text{ completely characterize system})$$

$$\mathcal{H} = \pm i\psi^+\partial_0\psi - \mathcal{L} = \pm\left[\psi^+(-i\vec{\alpha}\,\vec{\nabla} + \beta m)\psi\right] = \pm i\psi^+\partial_0\psi$$

USING E-L EQ.

$$\pm i\left[\psi_\alpha(\vec{x},t), \psi_\beta^+(\vec{y},t)\right] = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{\alpha\beta} \qquad \alpha,\beta = 1,2,3,4$$

SUPPRESS α, β INDICES

$$\left[\psi(\vec{x},t), \psi^+(\vec{y},t)\right] = \pm 1\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\left[\psi(\vec{x},t), \psi(\vec{y},t)\right] = 0 = \left[\psi^+(\vec{x},t), \psi^+(\vec{y},t)\right]$$

$$\left[\psi(\vec{x},t), \bar{\psi}(\vec{x},t)\right] = \pm\gamma^0\delta^3(\vec{x}-\vec{y})$$ EASILY SHOWN BY RE-INSERTING INDICES

$$\psi(x) = \sum_{r=1}^{2}\int d^3\vec{p}\,\frac{1}{(2\pi)^{3/2}}\cdot\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^{(r)}u_{\vec{p}}^{(r)}e^{-ip\cdot x} + c_{\vec{p}}^{(r)+}v_{\vec{p}}^{(r)}e^{ip\cdot x}\right]$$

$$\psi^+(x) = \sum_{r=1}^{2}\int d^3\vec{p}\,\frac{1}{(2\pi)^{3/2}}\cdot\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b_{\vec{p}}^{(r)+}u_{\vec{p}}^{(r)+}e^{ip\cdot x} + c_{\vec{p}}^{(r)}v_{\vec{p}}^{(r)+}e^{-ip\cdot x}\right]$$

$$\bar{\psi}(x) = \sum\int \quad '' \quad \left[\quad \bar{u}_{\vec{p}} \quad '' \quad \bar{v}_{\vec{p}} \quad ''\right]$$

<u>Ansatz</u>: $\left[b_{\vec{p}}^{(r)}, b_{\vec{p}'}^{(s)+}\right] = \delta^{rs}\delta^3(\vec{p}-\vec{p}')\,B$

TO AVOID DOING FOURIER INVERSION

$$\left[c_{\vec{p}}^{(r)+}, c_{\vec{p}'}^{(s)}\right] = \delta^{rs}\delta^3(\vec{p}-\vec{p}')\,C$$

$$[b,b] = [c,c] = [b,c] = 0 \Rightarrow [\psi,\psi] = [\psi^+,\psi^+] = 0$$
$$[\bar{\psi},\bar{\psi}] = 0$$

$$\left[\psi(\vec{x},t), \psi^+(\vec{y},t)\right] = \sum_r\int d^3\vec{p}\,\frac{1}{(2\pi)^3\,2E_{\vec{p}}}\left[B\,e^{-i\vec{p}(\vec{x}-\vec{y})}u_{\vec{p}}^{(r)}u_{\vec{p}}^{(r)+} + C\,e^{+i\vec{p}(\vec{x}-\vec{y})}v_{\vec{p}}^{(r)}v_{\vec{p}}^{(r)+}\right]$$

FORMULA ALSO TRUE WITH $+T$ REPLACED BY $\bar{\psi}$ and u^+, v^+ " $\bar{u}, \bar{v}$

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$$\sum_r u_{\vec p}^{(r)} u_{\vec p}^{(r)\dagger} = \sum_r u_{\vec p}^{(r)} \bar{u}_{\vec p}^{(r)} \beta = (\not{p}+m)\beta$$

$$=(E_{\vec p}\,\beta - \vec p\cdot\beta\vec\gamma + m)\beta = E_{\vec p} + \vec p\cdot\vec\alpha + \beta m$$

similiarly,

$$\sum_r v_{\vec p}^{(r)} v_{\vec p}^{(r)\dagger} = (\not{p}-m)\beta = E_{\vec p} + \vec p\cdot\vec\alpha - \beta m$$

If $B=C$, terms $\propto \vec\alpha, \vec\beta$ vanish, so that we may have integral proportional to 1

choose $B=C=\pm 1$, & we have canonical quantization relations:

$$[\psi(x,t), \psi^\dagger(y,t)] = \pm\int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)}$$

However, our expression for ψ has two annihilation or creation operators leading to problems with H:

$$H = \int d^3x\,\mathcal{H} = \pm\sum_{rs}\int\frac{d^3\vec p}{2E_{\vec p}}\Big(b_{\vec p}^{(r)\dagger} b_{\vec p}^{(s)} \underbrace{u_{\vec p}^{(r)\dagger} u_{\vec p}^{(s)}}^{\delta_{rs}\,2E_p} E_{\vec p}$$

$$- c_{\vec p}^{(r)} c_{\vec p}^{(s)\dagger} \underbrace{v_{\vec p}^{(r)\dagger} v_{\vec p}^{(s)}} E_{\vec p}\Big)$$

$$H = \pm\sum_r\int d^3\vec p\; E_{\vec p}\left(b_{\vec p}^{(r)\dagger} b_{\vec p}^{(r)} - c_{\vec p}^{(r)} c_{\vec p}^{(r)\dagger}\right) \qquad \delta_{rs}\,2E_{\vec p}$$

which is an unbounded below energy

for (+) sign c-type quanta carry negative energy

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5 TOPICS FOR THE REST OF THIS LECTURE

1) Canonical Anticommutation
2) Solves energy crisis
3) Fermi-Dirac statistics
4) Fields as observables
5) Classical Limit ...

1) $\begin{Bmatrix}\text{Fermi}\\ \text{Bose}\end{Bmatrix}$ p's and q's have $\begin{Bmatrix}\text{1/2 odd int.}\\ \text{int.}\end{Bmatrix}$ spin

At equal times Bose-Bose $[p^a, p^b] = [q^a, q^b] = 0$

$$[q^a, p^b] = i\delta^{ab}$$

B-F everything commutes

Fermi-Fermi $\{p^a, p^b\} = \{q^a, q^b\} = 0$

$$\{q^a, p^b\} = i\delta^{ab}$$

Hence, we make the changes

$$\{b^{(r)}_{\vec p}, b^{(s)+}_{\vec p}\} = \delta^{rs}\delta^3(\vec p - \vec p')\cdot B \quad \text{etc.}$$

$$\{\psi(\vec x, t), \psi(\vec q, t)\} = \pm\sum \int d^3\vec p \, \frac{1}{(2\pi)^3} \quad \ldots$$

IF WE ARE NOT CAREFUL WE WILL LOSE POSITIVE DEFINITENESS IN THE HILBERT SPACE

$$A \equiv \int d^3\vec p \sum_r f_r(\vec p)\, b^{(r)}_{\vec p}\,; \quad \{A, A^+\} = \pm\int \sum |f_r(\vec p)|^2$$

LOOK AT QUANTITY OF THE FORM

$$\langle\phi|\{A, A^+\}|\phi\rangle = \langle\phi|AA^+|\phi\rangle + \langle\phi|A^+A|\phi\rangle \geq 0$$

Hence we must choose + sign

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$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\psi$$

$$\{b_{\vec{p}}^{(r)}, b_{\vec{p}'}^{(s)+}\} = \{c_{\vec{p}}^{(r)}, c_{\vec{p}'}^{(s)+}\} = \delta^{rs}\,\delta^{(3)}(\vec{p}-\vec{p}')$$

all others zero

$$H = \sum_r \int d^3\vec{p}\; E_{\vec{p}}\left[b_{\vec{p}}^{(r)+} b_{\vec{p}}^{(r)} + c_{\vec{p}}^{(r)+} c_{\vec{p}}^{(r)}\right] - \cancel{\delta^3(0)}$$

which is bounded below

FROM ANTICOMMUTATION OF C and C^+

3) Consider PEDAGOGIC simplification of Hamiltonian for a moment

$$H = \sum_{\vec{p}} E_{\vec{p}}\, b_{\vec{p}}^+ b_{\vec{p}} \qquad \{b_{\vec{p}}^+, b_{\vec{p}}\} = \delta_{\vec{p},\vec{p}'} \quad \{b,b\} = \{b^+,b^+\} = 0$$

$$[AB,C] = A\{B,C\} - \{A,C\}B$$

$$[H, b_{\vec{p}}] = -E_{\vec{p}}\, b_{\vec{p}} \qquad [H, b_{\vec{p}}^+] = +E_{\vec{p}}\, b_{\vec{p}}^+$$

energy lowering

energy raising

Def. $b_{\vec{p}}|0\rangle = 0$ all $\vec{p}$ $\quad\langle 0|0\rangle = 1 \quad H|0\rangle = 0$

$$b_p^+|0\rangle = |\vec{p}\rangle \quad H|\vec{p}\rangle = E_{\vec{p}}|\vec{p}\rangle$$

$$\langle p'|p\rangle = \langle 0|b_{\vec{p}'}\, b_p^+|0\rangle = -\langle 0|b_{\vec{p}}^+\, b_{\vec{p}}|0\rangle + \delta_{\vec{p},\vec{p}'}$$

$$= \delta_{\vec{p}',\vec{p}}$$

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$$|\vec{p}_1, \vec{p}_2\rangle = b^+_{\vec{p}_1} b^+_{\vec{p}_2} |0\rangle = -|\vec{p}_2, \vec{p}_1\rangle$$

$$H|\vec{p}_1, \vec{p}_2\rangle = (E_{\vec{p}_1} + E_{\vec{p}_2})|\vec{p}_1, \vec{p}_2\rangle$$

Pauli Exclusion principle

$$b^+_{\vec{p}} |\vec{p}\rangle = (b^+_{\vec{p}})^2 |0\rangle = 0$$

(4) Observables made out of fermi fields

Recall Bose Fields

$$[\phi(x), \phi(y)]_{E.T.} = 0$$

and by Lorentz invariance this is true for all spacelike separated x and y.

With Fermi fields

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0$$

If $\psi(x)$ were an observable, we would have observables that did not commute at spacelike separation.

Observables can only be made of products with an even number of Fermi fields.

This is also necessary for just rotational properties. Under a rotation by 2π $\psi(x) \longrightarrow -\psi(x)$. No meter on any experimental apparatus ever gives a different reading when the experiment is rotated by 2π.

All observables are in single valued representations of the Lorentz group.

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5) Classical Limit ($\hbar \to 0$)

(a) Two classical limits physically

Take same physical situation.

N particles in a box all in the same energy and momentum eigenstate.

$E = N\hbar\omega$

(i) $\hbar \to 0$ $\quad$ N, E and $\vec{p}$ fixed.

$\omega, \vec{k} \to \infty$ $\quad$ wavelength $\to 0$

No diffraction. This is the classical particle limit.

(ii) $\hbar \to 0$ $\quad$ E, ω and $\vec{P}, \vec{k}$ fixed.

$N \to \infty$

Lots of wavy behavior, but lose quantum granularity.

For fermions we can only do the first limit because of the Pauli exclusion principle. There is no analog of the wave limit. There will never be a competing theory as there was for light with the corpuscular and wave theories.

(b) Classical limits formally.

$\hbar \to 0$ in canonical algebras.

Bose fields $\longrightarrow$ commuting quantities (numbers)

Fermi fields $\longrightarrow$ anticommuting quantities (Grassman variables)

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Working with classical fermi fields or Grassman variables

Never exchange the order of two terms in the classical field equations without a compensating minus sign or you would have no hope of the classical limit of the quantum theory agree with the classical theory even at order $\hbar^0$.

Derivation of the Euler-Lagrange equations.

$$dL = \underbrace{\frac{\partial L}{\partial \dot q^a}}_{\equiv p_a} d\dot q^a + \frac{\partial L}{\partial q^a} dq^a$$

If both the derivatives are kept to the same side of the differentials then I can integrate by parts in the action and get the usual E-L equations

$$\dot p_a = \frac{\partial L}{\partial q^a}$$

rather than something else. Define

$$H = p_a \dot q^a - L$$

not $H = \dot q^a p_a - L$, for example, which would give something different.

$$dH = dp_a \dot q^a - \dot p_a dq_a - dL$$

$$\frac{\partial H}{\partial p_a} = \dot q^a \qquad \frac{\partial H}{\partial q^a} = -\dot p_a$$

Try using the quantum relations $\dot p_a = -i[p_a, H]$ and $\dot q^a = -i[q^a, H]$ with the canonical anticommutation relation, and see if you can reproduce the Heisenberg equations of motion in the Dirac theory.

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Perturbation theory for spinors

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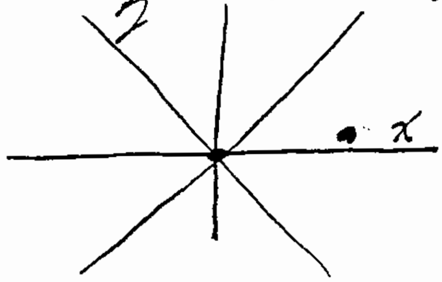
Because scalar fields commute at spacelike separations, the idea of time ordering is Lorentz invariant. That is,

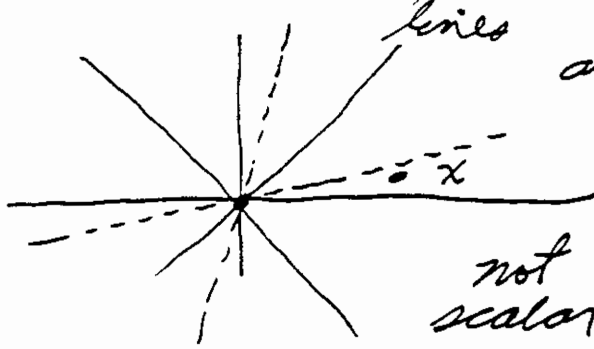


if $x^2 < 0$

$$T(\phi(0)\phi(x))$$

$$= \phi(x)\phi(0)$$

in the situation pictured, but in another frame, whose axes are represented by dotted lines x^0 is less than 0 and



$$T(\phi(0)\phi(x)) = \phi(0)\phi(x)$$

This ambiguity is not a problem for scalar fields since $[\phi(0), \phi(x)] = 0$ when $x^2 < 0$

When $x^2 > 0$ there is no ambiguity.

For spinor fields, this definition of time ordering is a failure. In one frame $T(\psi_\alpha(x)\bar\psi_\beta(0)) = \psi_\alpha(x)\bar\psi_\beta(0)$, and if $x^2 < 0$, in another frame it may be that

$$T(\psi_\alpha(x)\bar\psi_\beta(0)) = \bar\psi_\beta(0)\psi_\alpha(x) = -\psi_\alpha(x)\bar\psi_\beta(0) \quad \text{OH OH!}$$

The way to patch this up is to put an extra minus sign into the definition of the time ordered product whenever the number of permutations of Fermi fields required to turn a product into a time ordered product is odd.

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<u>assertion</u>: Most of the derivations we did, expressing S matrix elements in terms of physical vacuum expectation values of time ordered products of renormalized Heisenberg picture fields, and showing that

$$ {}_p\langle 0|T\left(\phi_H'(x_1)\cdots\phi_H'(x_n)\right)|0\rangle_p = \frac{\langle 0|T[\phi_I'(x_1)\cdots\phi_I'(x_n)e^{-i\int_{-\infty}^{\infty}d^4x\,\mathcal{H}_I}]|0\rangle}{\langle 0|T e^{-i\int_{-\infty}^{\infty}d^4x\,\mathcal{H}_I}|0\rangle}, $$

bare vacua

are unaffected by the fact that the fields may now be spinors and the time ordered product now has a $(-1)^p$ in it.

a way of seeing that this is probably true is to think about how we obtained $S = U_I(\infty,-\infty)$ in the formalism with the turning on and off function. about the only place we could have a problem is in obtaining the expression

$$ U_I(\infty,-\infty) = T e^{-i\int_{-\infty}^{\infty}d^4x\,\mathcal{H}_I}. $$

But you expect no problem there, because the Hamiltonian is quadratic in spinor fields, so you always move spinor fields around in pairs, and the permutation is then always even. The new minus sign in the time ordered product doesn't matter.

Once we have that big messy expression on the RHS above, we used Wick's theorem to turn the time ordered products into normal ordered products, and then wrote down wick diagrams to represent operators in the wick expansion, and Feynman diagrams to represent S matrix elements.

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<u>assertion:</u> Wick's theorem can be proven for spinor fields and with the extra minus sign in the time ordered product provided you also put an extra minus sign in the normal ordered product.

For example if A_1 and A_2 are Fermi fields

$$:A_1 A_2: = A_1^{(+)} A_2^{(+)} + A_1^{(-)} A_2^{(+)} + A_1^{(-)} A_2^{(-)} - A_2^{(-)} A_1^{(+)}$$

Note that $:A_1 A_2: = -:A_2 A_1:$ FERMI FIELDS ANTICOMMUTE INSIDE THE TIME ORDERED AND NORMAL ORDERED PRODUCTS

also $T(A_1 A_2) = -T(A_2 A_1)$. The contraction $\overline{A_1 A_2}$ is defined as usual to be the time ordered product minus the normal ordered product

$$\overline{A_1 A_2} = T(A_1 A_2) - :A_1 A_2:$$

The contraction is a c-number. Here is a proof by cases. Take $x_1^0 > x_2^0$. Then

$$T(A_1 A_2) = A_1 A_2 = A_1^{(+)} A_2^{(+)} + A_1^{(+)} A_2^{(-)} + A_1^{(-)} A_2^{(+)} + A_1^{(-)} A_2^{(-)}$$

and

$$\overline{A_1 A_2} = A_1^{(+)} A_2^{(-)} + A_2^{(-)} A_1^{(+)}$$

NOTATION $:\overline{A_1 A_2 A_3} A_4: \equiv -\overline{A_1 A_3} :A_2 A_4:$

$-$ sign for odd permutations is needed to prove Wick's thm.

$$= \{A_1^{(+)}, A_2^{(-)}\}$$

If we hadn't stuck that extra $-$ sign into the definition of the ordered product we would have gotten

$\overline{A_1 A_2} = [A_1^{(+)}, A_2^{(-)}]$ for $x_1^0 > x_2^0$, which is not a c#. Things are looking good though, $\{A_1^{(+)}, A_2^{(-)}\}$ is a c#. The case $x_2^0 > x_1^0$ clearly goes through too.

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Calculation of the contraction (propogator)

Since the contraction of two Fermi fields is a c-number, we can use the same trick for evaluating it as we did when we calculated the contraction of two scalar fields, i.e. take its vacuum expectation value.

$$\psi(x) = \sum_r \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\vec{p}}}} \left[b_{\vec{p}}^{(r)} u_{\vec{p}}^{(r)} e^{-ip\cdot x} + c_{\vec{p}}^{(r)\dagger} v_{\vec{p}}^{(r)} e^{ip\cdot x} \right]$$

$$\bar{\psi}(y) = \sum_{r'} \int \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2E_{\vec{p}'}}} \left[b_{\vec{p}'}^{(r')\dagger} \overline{u_{\vec{p}'}^{(r')}} e^{ip'\cdot y} + c_{\vec{p}'}^{(r')} \overline{v_{\vec{p}'}^{(r')}} e^{-ip'\cdot y} \right]$$

For $x^0 > y^0$

$$\overbrace{\psi(x)\bar{\psi}}(y) = \langle 0 | \overbrace{\psi(x)\bar{\psi}}(y) | 0 \rangle$$

$$= \langle 0 | \left[T(\psi(x)\bar{\psi}(y)) - \overset{0}{\cancel{:\psi(x)\bar{\psi}(y):}} \right] | 0 \rangle$$

$$= \langle 0 | \psi(x)\bar{\psi}(y) | 0 \rangle$$

$$= \sum_{rr'} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\vec{p}}}} \frac{d^3p'}{(2\pi)^{3/2}\sqrt{2E_{\vec{p}'}}} \overbrace{\langle 0 | b_{\vec{p}}^{(r)} b_{\vec{p}'}^{(r')\dagger} | 0 \rangle}^{\delta_{rr'}\delta^{(3)}(\vec{p}-\vec{p}')} e^{-ip\cdot x} e^{ip'\cdot y} u_{\vec{p}}^{(r)} \overline{u_{\vec{p}'}^{(r')}}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{-ip\cdot(x-y)} \underbrace{\sum_r u_{\vec{p}}^{(r)} \overline{u_{\vec{p}}^{(r)}}}_{\not{p}+m}$$

$$= (i\not{\partial}_x + m) \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

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For $y^0 > x^0$, you get

CRITICAL MINUS SIGN OUT FRONT IS THE ONE WE PUT INTO OUR TIME ORDERED PRODUCT FOR FERMI FIELDS

$$\psi(x)\bar{\psi}(y) = -\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{ip\cdot(x-y)} (\not{p}-m)$$

$$= (i\not{\partial}_x + m)\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} e^{ip\cdot(x-y)}$$

The nice thing about this result is that it is the same for x^0 less than or greater than y^0 (the sign of p in the exponential no longer matters once $\not{p}$ is turned into a derivative and pulled out). In either case

$$\psi(x)\bar{\psi}(y) = (i\not{\partial}_x + m)\ \underbrace{\phi(x)\phi(y)}_{\text{contraction for a scalar field of mass } m}$$

We've already massaged $\phi(x)\phi(y)$:

$$\phi(x)\phi(y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\epsilon}$$

So without further ado we can rewrite

$$\psi(x)\bar{\psi}(y) = (i\not{\partial}_x + m)\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2+i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

Both sides of this equation are 4×4 matrices. If you prefer,

$$\psi_\alpha(x)\bar{\psi}_\beta(y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}_{\alpha\beta} + m\,Id_{\alpha\beta})}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

3 COMMENTS ON PROPOGATOR DEC. 18

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1) The propogator $\frac{i(\not p + m)}{p^2 - m^2 + i\epsilon}$ is going to play the same role in the perturbation theory for Dirac fields as $\frac{i}{p^2-m^2+i\epsilon}$ played in the perturbation theory for scalar fields. Recall though that when you wrote down

$\xleftarrow{p}$ charged scalar, which stands for $\frac{i}{p^2-m^2+i\epsilon}$

it did not matter whether p was routed in the same direction or the opposite direction as charge flow, simply because

$$\frac{i}{(-p)^2 - m^2 + i\epsilon} = \frac{i}{p^2-m^2+i\epsilon}$$

Now however the propogator is not even in p.

$\xleftarrow{p}$ charged fermion, will stand for $\frac{i(\not p + m)}{p^2-m^2+i\epsilon}$

while

$\xrightarrow{p}$ charged fermion, will stand for $\frac{i(-\not p + m)}{p^2-m^2+i\epsilon}$

The propogator is a kind of projection operator.

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2) There is a more common way of writing the Fermion propogator, more common because it is shorter.

Rewrite $p^2 - m^2 + i\epsilon = (\not{p} - m + i\epsilon)(\not{p} + m - i\epsilon)$

(this still gives the right prescriptions for the location of poles as $\epsilon \to 0$, $m > 0$).

Then

adding this here does nothing to location of pole

$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} = \frac{i(\not{p}+m-i\epsilon)}{(\not{p}-m+i\epsilon)(\not{p}+m-i\epsilon)} = \frac{i}{\not{p}-m+i\epsilon}$$

There is no danger that these hoaky looking matrix manipulations are wrong because $\not{p}-m+i\epsilon$ commutes with $\not{p}+m-i\epsilon$

③ For Bosons the action is

$$S = \int d^4x \left(\partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi\right)$$

parts integration

$$= \int d^4x \ \psi^* \left(-\Box - m^2\right)\psi$$

In momentum space $-\Box - m^2$ becomes $p^2 - m^2$.
i.e. when acting on $e^{-ip\cdot x}$

For Fermions, the action is

$$S = \int d^4x \ \bar{\Psi}(i\not{\partial} - m)\psi$$

In momentum space $i\not{\partial} - m$ becomes $\not{p} - m$.

The fact that the charged Boson propogator came out to be $\frac{i}{p^2-m^2}$ while the charged Fermion propogator came out to be $\frac{i}{\not{p}-m}$ is at least an interesting coincidence.

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Feynman diagrams in

meson is a scalar   meson is a pseudoscalar

$\Gamma = 1 \text{ or } i\gamma_5$

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 - g\bar{\psi}\Gamma\psi\phi$$

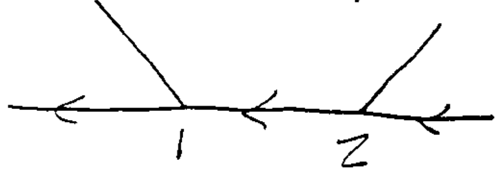
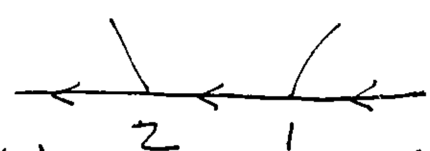
We'll "derive" the Feynman rules by looking at a couple of processes and hoping that the general process at general orders has an obvious generalization. Let's look at the order g^2 term in $Te^{-i\int d^4x\, \mathcal{H}_I}$, i.e.

$$\frac{(-ig)^2}{2!}\int d^4x_1\, d^4x_2\, T\left(\bar{\psi}\Gamma\psi\phi(x_1)\,\bar{\psi}\Gamma\psi\phi(x_2)\right).$$

This can contribute to many processes, let's look at how it contributes to $N+\phi \to N+\phi$. The relevant terms in the Wick expansion of the time ordered product are

$$\frac{(-ig)^2}{2!}\int d^4x_1\, d^4x_2\Big(:\bar{\psi}\Gamma\psi\phi(x_1)\,\bar{\psi}\Gamma\psi\phi(x_2): $$
$$+ :\bar{\psi}\Gamma\psi\phi(x_1)\,\bar{\psi}\Gamma\psi\phi(x_2): \Big)$$

The Wick diagrams for these two terms are

 and 

(they are of the same pattern)   respectively.

The picture for the second term looks identical to the first picture except for an exchange of the dummy variable $1 \leftrightarrow 2$. Is the second operator the same as the first? Well,

$$:\bar{\psi}\Gamma\psi\phi(x_1)\bar{\psi}\Gamma\psi\phi(x_2): \; = \; :\bar{\psi}\Gamma\psi\phi(x_2)\bar{\psi}\Gamma\psi\phi(x_1):$$

because interaction Hamiltonians are made out of fermion bilinears which commute inside the normal ordered product and this expression clearly differs from the one in the first term by an exchange of dummy indices $1 \leftrightarrow 2$.

I'll just look at how an electron or "nucleon" the first term contributes to $N+\phi \to N+\phi$ since the second one is identical, and serves only to cancel the $\frac{1}{2!}$.
We'll look at the matrix element of the operator between relativistically normalized states.

$$\langle p', r'; q' | (-ig)^2 \int d^4x_1 d^4x_2 : \overline{\Psi} \Gamma \Psi \phi(x_1) \overline{\Psi} \Gamma \Psi \phi(x_2) : | p, r; q \rangle$$

Notice that the incoming and outgoing electrons have to have their spin specified as well as their momentum; that is what the r does. [alternatively, you could just give an arbitrary spinor for the incoming and outgoing electrons, say u and u'.]
There is only one field in the normal ordered product that has an annihilation operator that can annihilate the incoming electron, $\Psi(x_2)$.

$$\langle 0 | \Psi(x_2) | p, r \rangle = \langle 0 | \sum_s \int \frac{d^3 l}{(2\pi)^3 2\omega_{\vec{l}}} b^{(s)}(l) u^{(s)}_{\vec{l}} e^{-il\cdot x_2} | p, r \rangle$$

$$= \sum_s \int \frac{d^3 l}{(2\pi)^3 2\omega_{\vec{l}}} u^{(s)}_{\vec{l}} e^{-il\cdot x_2} \underbrace{\langle 0 | b^{(s)}(l) | p, r \rangle}_{(2\pi)^3 2\omega_{\vec{p}} \delta^{(3)}(\vec{p}-\vec{l}) \delta_{sr}}$$

$$= u^{(r)}_{\vec{p}} e^{-ip\cdot x_2}$$

Similarly when $\overline{\Psi}(x_1)$ is used to create the outgoing electron it becomes

$$\bar{u}^{(r')}_{\vec{p}'} e^{+ip'\cdot x_1}$$

because that is the coefficient of $b^{(r')}(p')^\dagger$ in the expansion of $\overline{\Psi}(x_1)$. We also have derived an expression for $\Psi(x_1)\overline{\Psi}(x_2)$, which I'll use and our matrix element simplifies to

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$$(-ig)^2 \int d^4x_1 \, d^4x_2 \int \frac{d^4 l}{(2\pi)^4} e^{ip'\cdot x_1} e^{-ip\cdot x_2} e^{-il\cdot(x_1 - x_2)}$$
$$\times \bar{u}^{(r')}_{\vec{p}'} \Gamma \frac{i}{\not{l} - m + i\epsilon} \Gamma u^{(r)}_{\vec{p}} \langle q' | : \phi(x_1)\phi(x_2) : | q \rangle$$

There are two terms in $\langle q' | : \phi(x_1)\phi(x_2) : | q \rangle$,

$$e^{iq'\cdot x_1} e^{-iq\cdot x_2} + x_1 \leftrightarrow x_2$$

Let's first consider the first one for a second. The integral is

$$(-ig)^2 \int d^4x_1 \, d^4x_2 \int \frac{d^4 l}{(2\pi)^4} e^{i(p' - l + q')\cdot x_1} e^{i(-p + l - q)\cdot x_2}$$
$$\bar{u}^{(r')}_{\vec{p}'} \Gamma \frac{i}{\not{l} - m + i\epsilon} \Gamma u^{(r)}_{\vec{p}}$$

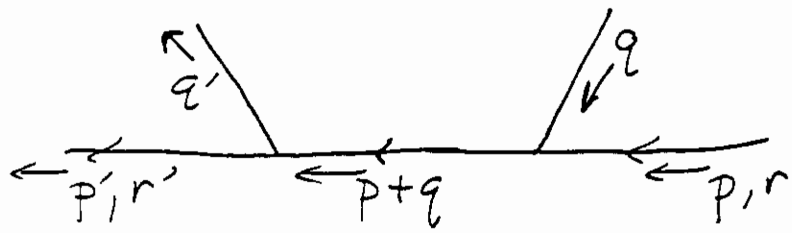
The exponentials go along with an interpretation. At x_1 an electron with momentum p' and a meson with momentum q' are created and a virtual electron with momentum l is absorbed (+ signs go with creation, − with absorption). At x_2 an electron with momentum p is absorbed and a meson with momentum q is absorbed, while a virtual electron with momentum l is created. This can happen at any space-time points x_1 and x_2 so they are integrated over, and in fact the integrals are trivial to do.

$$(-ig)^2 \int \frac{d^4 l}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p' - l + q') (2\pi)^4 (-p + l - q)$$
$$\bar{u}^{(r')}_{\vec{p}'} \Gamma \frac{i}{\not{l} - m + i\epsilon} \Gamma u^{(r)}_{\vec{p}}$$

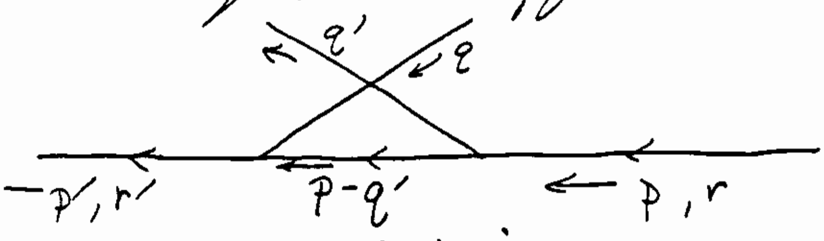
The l integration can be done because the energy momentum of the internal electron is fixed by the δ function. We get

$$(-ig)^2(2\pi)^4\delta^{(4)}(p'+q'-(p+q))\,\bar u^{(r')}_{\vec p'}\,\Gamma\,\frac{i}{\not p+\not q-m+i\epsilon}\,\Gamma\,u^{(r)}_{\vec p}$$

The Feynman diagram for this contribution to $N+\phi\to N+\phi$ is



What about the second term in $\langle q'|:\phi(x_1)\phi(x_2):|q\rangle$? It contributes to the same process, but the diagram is different:



The contribution is

$$(-ig)^2(2\pi)^4\delta^{(4)}(p'+q'-(p+q))\,\bar u^{(r')}_{\vec p'}\,\Gamma\,\frac{i}{\not p-\not q'-m+i\epsilon}\,\Gamma\,u^{(r)}_{\vec p}$$

Lets look at another process, $\bar N+\phi\to\bar N+\phi$ that the exact same operator in the Wick expansion can contribute to.

outgoing positron and meson　　　　incoming positron and meson

$$\langle p',r';q'|(-ig^2)\int d^4x_1\,d^4x_2 :\bar\psi\Gamma\psi\phi(x_1)\bar\psi\Gamma\psi\phi(x_2):|p,r;q\rangle$$

The field $\psi(x_2)$ has to create the outgoing positron. But, it has to be anticommuted past three Fermi fields to do it. The coefficient of $c^{(r')\dagger}_{\vec p'}$ in $\psi(x_2)$ is $e^{ip'\cdot x_2}v^{(r')}_{\vec p'}$, so we get

actually 1: two have been contracted

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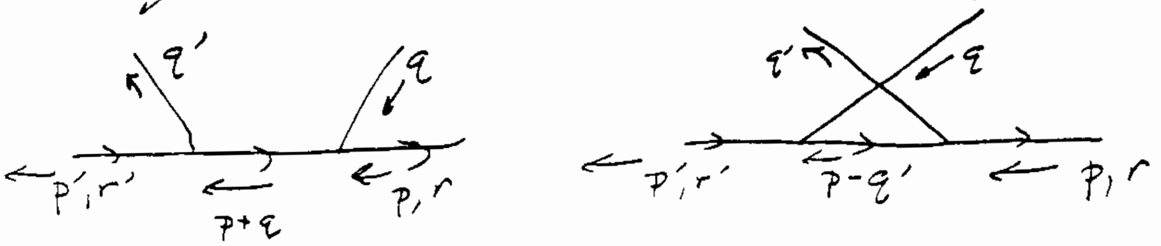
$$-\langle q'|(-ig)^2\int d^4x_1 d^4x_2 :\overline{\Psi}\Gamma\overbrace{\Psi\,\phi(x_1)\overline{\Psi}}\Gamma v_{\vec p'}^{(r')}e^{ip'\cdot x_2}\phi(x_2):|p,r;q\rangle$$

The $\overline{\Psi}$ field then only has to get by two (0) Fermi fields to annihilate the positron on the right. The coefficient of $c_{\vec p}^{(r)}$ in $\overline{\Psi}(x_1)$ is $e^{-ip\cdot x_1}\,\overline{v}_{\vec p}^{(r)}$ so I get

$$-(-ig)^2\int d^4x_1 d^4x_2\int\frac{d^4l}{(2\pi)^4}\,e^{-ip\cdot x_1}\,e^{ip'\cdot x_2}\,e^{-il\cdot(x_1-x_2)}$$

$$\overline{v}_{\vec p}^{(r)}\Gamma\frac{i}{\not l-m+i\epsilon}\Gamma v_{\vec p'}^{(r')}\langle q'|:\phi(x_1)\,\phi(x_2):|q\rangle$$

The main difference to notice for this process is the overall minus sign, and the change from u's to v's. There are still going to be two Feynman diagrams from the two terms in $\langle q'|:\phi(x_1)\phi(x_2):|q\rangle$. I'll just write down the result of doing the x_1, x_2 and l integrals.



$$-(-ig^2)(2\pi)^4\delta^{(4)}(p+q-(p'+q'))\;\overline{v}_{\vec p}^{(r)}\Gamma\left(\frac{i}{-\not p-\not q-m+i\epsilon}+\frac{i}{-\not p+\not q'-m+i\epsilon}\right)\Gamma v_{\vec p'}^{(r')}$$

The initial antinucleon state gave us a $\overline{v}_{\vec p}^{(r)}$. If the initial state had been specified simply by one spinor v, rather than a type r, one of our basis spinors), The amplitude is antilinear in v, that is linear in $\overline{v}$.

why do we expect this?

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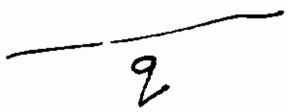
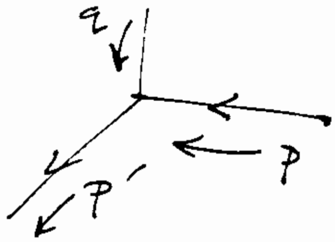
Feynman Rules - Factors



internal meson $\dfrac{i}{q^2-\mu^2+i\epsilon}$

internal nucleon $\dfrac{i}{\not{p}-m+i\epsilon}$

p oriented along arrow (along charge flow for positively charged particle)



$$-ig\,\Gamma(2\pi)^4\delta^{(4)}(p+q-p')$$

Integrate over internal momenta

For every incoming nucleon (annihilated by a ψ) get a u
" antinucleon (annihilated " by a $\bar{\psi}$) $\bar{v}$
outgoing nucleon (created by " a $\bar{\psi}$) $\bar{u}$
" antinucleon (created " by a ψ) v

Because Fermions always appear bilinearly. (we'll soon see quadrilinears are ruled out) in a Lagrangian, a Fermion line either goes all the way through a graph or in a loop. Since we haven't done any examples with a Fermion loop yet, we'll first do the matrix multiplication rules for a line going all the way through a graph first.

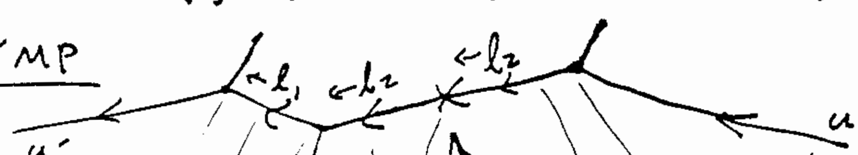
Feynman Rules – Matrix multiplication

Go to the head of any Fermion line that goes all the way through a graph. You will either be at an incoming antinucleon, in which case you write down a $\bar{v}$, or at an outgoing nucleon in which case you write down a $\bar{u}$. (I start at the head of the line because I habitually write from left to right, so I start with the row vector, then go through the matrices and finish with a column vector.)

Working against the arrows, the next thing you get is a vertex. Write down the matrix for the vertex. Then you'll get an internal line followed by another vertex some number of times. Write down the propogator matrix, then the interaction matrix each time.

When you get to the tail end of the line, you'll either be at an incoming nucleon, in which case write down a u, or at an outgoing antinucleon, in which case write down a v.

EXMP



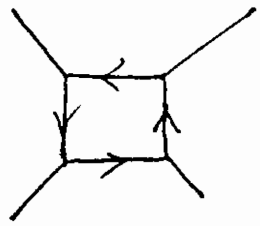
mass renormalization; comes from $-B\bar{\psi}\psi$ term in $\mathcal{L}_{CT}$

an identity matrix

$$\bar{u}'\,\Gamma\,\frac{i}{\not{k}_1 - m + i\epsilon}\,\Gamma\,\frac{i}{\not{k}_2 - m + i\epsilon}\,\frac{i}{\not{k}_2 - m + i\epsilon}\,\Gamma\,u$$

What you finally obtain is a number. You get another product like this for each Fermi line that goes through a graph.

What about Fermi lines that go in loops? Here is a graph:  . Another is



This is an order g^4 diagram for 2 meson → 2 meson scattering. The factor of interest in this contribution to the process is

$$\overline{\psi}\Gamma\psi(x_1)\overline{\psi}\Gamma\psi(x_2)\overline{\psi}\Gamma\psi(x_3)\overline{\psi}\Gamma\psi(x_4)$$

I'll rewrite this as

$$-\mathrm{Tr}\ \psi(x_4)\overline{\psi}\Gamma\psi(x_1)\overline{\psi}\Gamma\psi(x_2)\overline{\psi}\Gamma\psi(x_3)\overline{\psi}(x_4)\Gamma$$

Notice the minus sign. To put this in standard form for replacing $\psi\overline{\psi}$ by our integral over $\frac{i}{\not{p}-m}$, I not only had to move $\psi(x_4)$ all the way to the left and write the sum on α in $\Gamma_{\beta\alpha}\psi_\alpha(x_4)$ as a trace, I had to anticommute $\psi(x_4)$ by an odd number of Fermi fields to get it there. First $\overline{\psi}(x_4)$, then a bunch of bilinears. This is why there is a minus sign out front.

To conclude the matrix multiplication rules the rule is thus: If a Fermi line goes in a loop, start anywhere in the loop, and working against the arrows write down vertex and propagator matrices until you get back to where you started. Then take the trace to get a number. You get a factor like this for each Fermi loop. There are no u's or v's in the factor coming from a Fermi loop.

Feynman Rules + Fermi minus signs

One thing is for sure, each Fermi loop gives you a minus sign.

To get the rest of the minus signs, I'll do an example. Back on OCT. 28, we did 'nucleon'-'nucleon' scattering at $O(g^2)$. We'll do the analogous calculation for nucleon-nucleon scattering (no quotes). Halfway down page 2 is the expression whose spinor analog is going to give us some troublesome minus signs. We need to simplify:

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$$\frac{1}{2!}\langle p',r';q',s'|:\bar{\Psi}\Gamma\psi(x_1)\bar{\Psi}\Gamma\psi(x_2):|p,r;q,s\rangle$$

It is slightly ambiguous to write $|p,r;q,s\rangle$ and $\langle p',r';q',s'|$. The two possibilities for the ket are

$$|p,r;q,s\rangle = \underbrace{b^r(p)^\dagger}_{\text{relativistically normalized}} b^s(q)^\dagger|0\rangle \quad\text{and}\quad \{b^r(p), b^s(p)^\dagger\} = (2\pi)^3\delta^{rs}2E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{p}\,')$$

$$|p,r;q,s\rangle = b^s(q)^\dagger b^r(p)^\dagger|0\rangle = -b^r(p)^\dagger b^s(q)^\dagger|0\rangle$$

It doesn't matter which choice you take — I'll take the second — as long as you choose $\langle p',r';q',s'|$ to be the corresponding bra.

$$\langle p',r';q',s'| = \left(|p',r';q',s'\rangle\right)^\dagger = \langle 0|\,b^{r'}(p')\,b^{s'}(q')$$

That way the forward scattering amplitude when there is no interaction (or at zeroth order when there is) is positive, not negative. So what we have to simplify is

$$\frac{1}{2!}\langle 0|\,b^{r'}(p')\,b^{s'}(q'):\bar{\Psi}\Gamma\psi(x_1)\bar{\Psi}\Gamma\psi(x_2):b^s(q)^\dagger b^r(p)^\dagger|0\rangle$$

$\psi(x_1)$ and $\psi(x_2)$ both contain operators which could annihilate either of the incoming nucleons. Let's say $\psi(x_2)$ annihilates the ~~meson~~ nucleon with momentum q. If $\psi(x_1)$ annihilates the nucleon with momentum q, we can rewrite what follows, and the only difference will be $x_1\leftrightarrow x_2$. Since these are dummy variables in an otherwise symmetric integration, I'll ignore this second case and cancel the $\frac{1}{2!}$. The coefficient of $b^s(q)$ in $\psi(x_2)$ is $u^s_q\frac{e^{-iq\cdot x_2}}{(2\pi)^3 2E_{\vec{q}}}$, so what I have is

$$\langle 0|\,b^{r'}(p')\,b^{s'}(q'):\bar{\Psi}\Gamma\psi(x_1)\,\bar{\Psi}(x_2)\Gamma u^s_q:b^r(p)^\dagger|0\rangle\, e^{-iq\cdot x_2}$$

Move the bilinear past $\overline{\Psi}(x_2)$ and let $\Psi(x_1)$ annihilate the remaining incoming nucleon:

$$\langle 0|\, b^{r'}(p')\, b^{s'}(q') : \overline{\Psi}(x_2)\Gamma u^s{}_q\, \overline{\Psi}(x_1)\Gamma u^r_p : |0\rangle e^{-iq\cdot x_2} e^{-ip\cdot x_1}$$

Now either $\overline{\Psi}(x_2)$ or $\overline{\Psi}(x_1)$ can take care of the outgoing nucleon with momentum p', but this does not just give us another factor of 2. These two possibilities are different because a distinction between x_1 and x_2 has now been made by the way we annihilated the incoming nucleons.

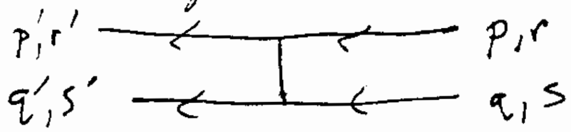
Suppose $\overline{\Psi}(x_2)$ takes care of the outgoing nucleon with momentum q'. The coefficient of $b^{s'}(q')^+$ in $\overline{\Psi}(x_2)$ is $\dfrac{\bar{u}^{s'}{}_{q'}\, e^{+iq'\cdot x_2}}{(2\pi)^3 2E_{\vec{q}'}}$, so what I have is

$$\langle 0|\, b^{r'}(p')\ \bar{u}^{s'}{}_{q'}\Gamma u^s{}_q\ \overline{\Psi}(x_1)\Gamma u^r_p\ |0\rangle e^{-iq\cdot x_2} e^{-ip\cdot x_1} e^{iq'\cdot x_2}$$

The final simplification gives

$$\bar{u}^{s'}{}_{q'}\Gamma u^s{}_q\, \bar{u}^{r'}{}_{p'}\Gamma u^r{}_p\ e^{-iq\cdot x_2} e^{-ip\cdot x_1} e^{iq'\cdot x_2} e^{ip'\cdot x_1}$$

There are other factors, but the graph these factors are from is



p is absorbed at the same spacetime point as p' is created; q is absorbed at the same spacetime point as q' is created.

What about the contribution where $\overline{\Psi}(x_1)$ takes care of the outgoing nucleon with momentum q'? First I'll anticommute the two $\overline{\Psi}$ fields to get:

$$-\langle 0|\, b^{r'}(p')\, b^{s'}(q') : \bar\Psi(x_1)\Gamma a^r_p\ \bar\Psi(x_2)\Gamma u^s_q|0\rangle e^{-iq\cdot x_2} e^{-ip\cdot x_1}$$

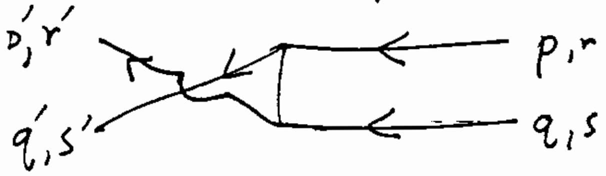
now the coefficient of $b^{s'}(q')^{\dagger}$ in $\bar\Psi(x_1)$ is $\dfrac{\bar u^{s'}_{q'}\, e^{iq'\cdot x_1}}{(2\pi)^3 2E_{\vec q'}}$
so I get

$$-\langle 0|\, b^{r'}(p')\quad \bar u^{s'}_{q'}\Gamma u^r_p\ \bar\Psi(x_2)\Gamma u^s_q\,|0\rangle e^{-iq\cdot x_2 - ip\cdot x_1 + iq'\cdot x_1}$$

The final simplification gives

$$-\bar u^{s'}_{q'}\Gamma u^r_p\ \bar u^{r'}_{p'}\Gamma u^s_q\ e^{-iq\cdot x_2 - ip\cdot x_1 + iq'\cdot x_1 + ip\cdot x_2}$$

The differences with the previous expression worth noting are the minus sign and the different spinor structure. The exponential factors are different in the expected way. The graph is



Having obtained the spinor factors and the minus sign you can continue on by doing the x integrations in the fashion leading up to the expression at the top of (OCT. 28) p.5 for the "nucleon"-nucleon" scattering.

It was how reordering Fermi operators gives minus signs. I don't have a tidy little rule to summarize the sign of the general case, but of course you won't have to be so detailed when you are just checking how many reorderings of Fermi fields it takes to get a given contribution to a matrix element.

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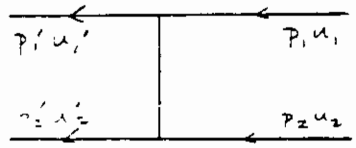
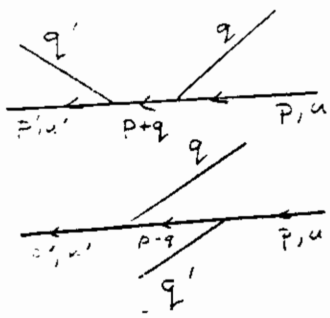
EXMP COMPLETE EXPRESSION FOR $N+N \to N+N$ $\quad \Gamma = i\gamma^5$



$$iA = (-ig)^2\left[\bar{u}_1' i\gamma^5 u_1 \bar{u}_2' i\gamma^5 u_2 \frac{i}{(p_1-p_1')^2-\mu^2}(+1) + \bar{u}_1' i\gamma^5 u_2 \bar{u}_2' i\gamma^5 u_1 \frac{i}{(p_1-p_2')^2-\mu^2}(-1)\right]$$

+ crossed graph

EXMP COMPLETE CALCULATION OF $N+\varphi \to N+\varphi$ $\quad \Gamma = i\gamma^5$



$$iA = (-ig)^2\left[\bar{u}' i\gamma^5 \frac{i}{\not p + \not q - m + i\epsilon} i\gamma^5 u + \bar{u}' i\gamma^5 \frac{i}{\not p' - \not q - m + i\epsilon} i\gamma^5 u\right]$$

$$A = g^2 \bar{u}' \gamma^5 \left[\frac{\not p + \not q + m}{(p+q)^2 - m^2} + \frac{\not p' - \not q + m}{(p'-q)^2 - m^2}\right]\gamma^5 u$$ using $\gamma^{5\,2} = 1$ and $\{\gamma^5, \not p\} = 0$

$$A = g^2 \bar{u}' \left[\frac{-\not p - \not q + m}{(p+q)^2 - m^2} + \frac{-\not p' + \not q + m}{(p'-q)^2 - m^2}\right] u$$ using $\not p u = m u$ and $\bar{u}' \not p' = \bar{u}' m$

$$= g^2 \bar{u}' \not q u \left[\frac{1}{(p'-q)^2 - m^2} - \frac{1}{(p+q)^2 - m^2}\right]$$

SPIN AVERAGING AND SPIN SUMMING

DO NOT SUM OR AVERAGE THE FEYNMAN AMPLITUDE. AVERAGE OR SUM PROBABILITY

INITIAL SPINS IN EXPT. UNKNOWN AVERAGE TRANS. PROB OVER THEM
FINAL " " " " SUM " " " "

$$\mathcal{M} = \bar{u}_1 \overset{\text{MATRIX}}{O^I} u_2 \left(\overset{\text{KINEMATIC FACTORS}}{\quad} \right)_{I \text{ K COLLECTIVE INDICES}}$$

i.e. $A = \bar{u}_1 \gamma^\mu u_2 \, q_\mu$

$$|\mathcal{M}|^2 = \bar{u}_{2\gamma}\left[\gamma^0 O^\dagger \gamma^0 u_1 \bar{u}_1 O\right]_{\gamma\delta} u_{2\delta} \left[\quad\right]$$

IN COLLECTIVE INDICES

USING $\sum_{\text{SPIN } 1} u_{1\alpha}\bar{u}_{1\beta} = (\not p_1 + m)_{\alpha\beta}$

$$= \left[\gamma^0 O^\dagger \gamma^0 \frac{\not p_1 + m}{2m} O\right]_{\delta\varepsilon} u_{2\varepsilon}\bar{u}_{2\delta}\left[\quad\right]$$

USING $\sum_{\text{SPIN } 2} u_{2\alpha}\bar{u}_{2\beta} = (\not p_2 + m)_{\alpha\beta}$

$$= \mathrm{Tr}\left[\gamma_0 O^\dagger \gamma^0 \frac{\not p_1 + m}{2m} O \frac{\not p_2 + m}{2m}\right]$$

In other calculations you will need

$$\sum_{\text{SPIN } 3} v_3 \bar{v}_3 = \not p_3 - m$$

$$\underbrace{\frac{1}{2}\sum_{r}}_{\text{initial av}} \underbrace{\sum_{s}}_{\text{final sum}} |a_{rs}|^2 = \frac{1}{2}|F|^2 \sum_{r,s}^{\mathrm{Tr}}\left[\bar{u}_{p'}^{(r)} \not q u_p^{(s)} \bar{u}_p^{(s)} \not q u_{p'}^{(r)}\right]$$

$$= \frac{1}{2}|F|^2 \; \mathrm{Tr}\, \not q (\not p + m) \not q (\not p' + m)$$

$$= \frac{1}{2}|F|^2 \left(4m^2\mu^2 + 8 p\cdot q \, p'\cdot q - 4 p\cdot p' \mu^2\right)$$

P, C, and PT for spinor fields

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We know what P does to a Dirac field

$$P:\ \psi(\vec{x},t) \longrightarrow \beta\,\psi(-\vec{x},t)$$

We expect (general QM theorem coming from uniqueness of canonical commutation relations) that in the quantum theory there is a unitary operator effecting this change.

$$P:\ \psi(\vec{x},t) \longrightarrow U_P^{\dagger}\,\psi(\vec{x},t)\,U_P = \beta\,\psi(-\vec{x},t)$$

U_P is a unitary operator in Hilbert space. It has no (spinor indices. (a Dirac field has a spinor index; each entry in the Dirac field is itself an operator in Hilbert space.)

From the effect of U_P on ψ and the expansion of ψ in terms of creation and annihilation operators, we get the effect of U_P on the creation and annihilation operators.

$$\psi(\vec{x},t) = \sum_r \int \frac{d^3p}{(2\pi)^{3/2}\,2E_{\vec{p}}}\Big[\,b_{\vec{p}}\,\underbrace{u^{(r)}_{\vec{p}}\,e^{-ip\cdot x}}_{\text{pos freq sol'n of Dirac eqn.}} + c^{(r)\dagger}_{\vec{p}}\,\underbrace{v^{(r)}_{\vec{p}}\,e^{ip\cdot x}}_{\text{negative frequency}}\Big]$$

We need an expression for $\beta\,\psi(-\vec{x},t)$. The first step is to evaluate

$\beta\, u^{(r)}_{\vec{p}}$ and the first step to evaluating that is to find $\beta\, u^{(r)}_{\vec{0}}$. But thats easy, $\beta\, u^{(r)}_{\vec{0}} = u^{(r)}_{\vec{0}}$ comes right from $(\not{p}-m) = 0$ for $p=(m,\vec{0})$.

To get the effect of β on $u^{(r)}_{\vec{p}}$ we now use $u^{(r)}_{\vec{p}} = e^{\alpha\cdot\vec{e}\,\phi/2}\,u^{(r)}_{\vec{0}}$

$$\beta u^{(r)}_{\vec p} = \beta e^{\vec\alpha\cdot\vec e\,\phi/2}\, u^{(r)}_{\vec 0} \qquad \vec e = \frac{\vec p}{|\vec p|} \quad \sinh\phi = \frac{|\vec p|}{m}$$

$$= e^{-\vec\alpha\cdot\vec e\,\phi/2}\,\beta u^{(r)}_{\vec 0} \qquad (\text{because } \{\beta,\alpha_i\}=0)$$

$$= e^{-\alpha\cdot\vec e\,\phi/2}\, u^{(r)}_{\vec 0}$$

$$= u^{(r)}_{-\vec p}$$

We have shown that parity does to positive frequency solutions of the Dirac equation what you would expect it to; it reverses the direction of motion but doesn't do anything to the spin

So the effect of U_P on $b^{(r)}_{\vec p}$ must be

$$U_P^{+}\, b^{(r)}_{\vec p}\, U_P = b^{(r)}_{-\vec p} \quad (\text{and the h.c. equation } U_P^{+}\, b^{(r)+}_{\vec p}\, U_P = b^{(r)+}_{-\vec p})$$

a very similar argument goes through for the $c^{(r)+}_{\vec p}$ which are multiplied by $v^{(r)}_{\vec p}$ except for one thing

$$\beta\, v^{(r)}_{\vec 0} = -v^{(r)}_{\vec 0}$$

Because of that minus sign the effect of U_P on $c^{(r)+}_{p}$ is

$$U_P^{+}\, c^{(r)+}_{\vec p}\, U_P = -c^{(r)+}_{-\vec p} \quad (\text{and the h.c. eqn}) \quad U_P^{+}\, c^{(r)}_{\vec p}\, U_P = -c^{(r)}_{-\vec p}$$

The b's have positive intrinsic parity, but the c's have negative intrinsic parity. U_P acting on a state with n electrons does nothing. U_P acting on a state with n positrons gives $(-1)^n$. Fermion and antifermion have opposite intrinsic parity. This is unlike the charged scalar, where both particle and antiparticle were scalar or pseudoscalar.

$$p + \bar p \to 2\pi \qquad \begin{matrix} l=0 & s=0 & j=0 \\ & s=1 & j=1 \end{matrix} \quad P=-1 \Big\} = P \to \quad 2\pi \begin{cases} j=0 & l=0 & P=1 \\ j=1 & l=1 & P=-1 \end{cases}$$

forbidden for $s=0$

i.e. could I redefine P to be P × some internal symmetry which would not have this relative minus sign.

Is this just an artifact of some unfortunate convention, or does this counterintuitive result have observable consequences.

Consider the process $N + \bar{N} \to 2\pi$. The nucleon and antinucleon are taken to be at rest, because this implies (no momentum → no angular momentum) that they are in an $l=0$ state. There are two possible $l=0$ states, either $S=0$ or $S=1$.

$$\begin{array}{llll} l=0 & S=0 & J=0 & P=-1 \\ l=0 & S=1 & J=1 & P=-1 \end{array}$$

The total angular momentum J is absolutely conserved, and the parity is -1 because of what we have just found.

The two pseudoscalar pions either have

$$\begin{array}{lll} l=0 & J=0 & P=+1 \\ l=1 & J=1 & P=-1 \end{array}$$

($l=0$ is an even function of relative momentum so it is positive parity, and $l=1$ is an odd function of relative momentum so it has negative parity; in general you get $(-1)^l$.) The fact that the pion is pseudoscalar doesn't affect the outcome because there are two (an even number) of them.

Looking at the possibilities, in particular at the J and P, you see that this process is forbidden. (Except by the P violating weak interactions, but the strong interactions which are P conserving and which would make this process occur very quickly compared to the weak interactions aren't allowed to do it.)

This is a convention independent consequence of the opposite intrinsic parity of N and $\bar{N}$

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C Charge conjugation

Recall how charge conjugation acted on a charged scalar:

$$C: \ \phi(x) \to \phi^*(x) \quad (\phi^*(x) \to \phi(x))$$

was a symmetry of the action. alternatively, you could see that $\phi^*(x)$ was a solution of the equations of motion whenever ϕ was. In the free case:

$$(\Box + m^2)\phi = 0 \iff (\Box + m^2)\phi^* = 0$$

This follows because $\Box + m^2$ is real.

(The free case is worth looking at because in general, when you add interactions you break symmetries, not create them.)

In the Dirac theory

$$(i\not\partial - m)\psi = 0 \not\Leftrightarrow (i\not\partial - m)\psi^* = 0$$

Unless we are in a representation in which all of the γ^μ are purely imaginary. [This doesn't mean that charge conjugation does not a symmetry worth calling exist in a general basis; It just means it takes a more complicated form.]

I'll show that a representation in which all of the γ^μ are purely imaginary exists by constructing one. Any real change of coordinates preserves this property, so there are lots of possibilities. You already know four matrices with square one that anticommute with each other:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}.$$

of these four one is pure imaginary $\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$, and the others are pure real.

So

$$\gamma'^0 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \quad \gamma'^1 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma'^2 = i\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} \quad \gamma'^3 = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}$$

satisfies $\{\gamma'^\mu, \gamma'^\nu\} = 2g^{\mu\nu}$ and

$$\gamma'^{\mu *} = -\gamma'^\mu, \text{ the "Majorana" condition.}$$

In this basis $C: \psi(x) \to \psi^*(x)$ is a symmetry of the free equation of motion and the free Dirac action (provided the classical field is thought of as anticommuting).

Thinking quantum mechanically, we expect a unitary operator, U_c, will exist such that

$$U_c^\dagger \psi(x) U_c = \psi^*(x)$$

Why write $\psi^*(x)$ and not $\psi^\dagger(x)$? Because we have been thinking of $\psi^\dagger$ as a row vector. Not only have we been using $\dagger$ to mean complex conjugation of numbers and hermitian conjugation of operators, but transpose in spinor indices. We don't always want to do a transpose when we hermitian conjugate a spinor, so we'll use $\psi^*(x)$ to mean hermitian conjugation without transposition in spinor indices. If you like, $\psi^*(x) = \psi^{\dagger T}(x)$

[There is always a similarity transformation between any two sets of γ^μ (that's Pauli's theorem), and so as not to completely jettison the approach that most books take to charge conjugation, I'll show what charge conjugation looks like in a general basis. Let

$$\psi_S = S\psi_M \qquad C: \psi_M \to \psi_M^*$$

Some other basis like the standard one (ψ_S); Dirac spinor in Majorana basis (ψ_M)

What does C do to ψ_S?

$$C: \psi_S \to S\psi_M^* = S S^{*-1} S^* \psi_M^* = S S^{*-1}(S\psi_M)^*$$
$$= S S^{*-1} \psi_S^*$$

The matrix SS^{*-1} is usually denoted $\mathcal{C}$. The representation dependent computations that we'll do are so much simpler, that this is all we'll have to say about $\mathcal{C}$.]

LORENTZ TRANSFORMATIONS IN A MAJORANA BASIS

$$M_i = \frac{i\alpha_i}{2} = \frac{i\gamma^0\gamma^i}{2} \quad \text{so} \quad M_i = -M_i^*$$

$$L_k = -i\epsilon_{ijk} M_i M_j \quad \text{so} \quad L_k = -L_k^*$$

$$D(A(\vec{e}\phi)) = e^{-i\vec{M}\cdot\vec{e}\phi} = D(A)^*$$

$$D(R(\vec{e}\theta)) = e^{-i\vec{L}\cdot\vec{e}\theta} = D(R)^*$$

so $\boxed{D(\Lambda) = D(\Lambda)^*}$ real

Charge conjugation commutes with Lorentz transformations

The utility of this simple little result will become clear on the next page.

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We want to find the effect of U_c on creation and annihilation operators. You get that from the expansion of ψ in terms of creation and annihilation operators and the assumed effect of U_c on ψ; $U_c^\dagger \psi(x) U_c = \psi^*(x)$. We need to get the expansion of $\psi^*(x)$ in terms of creation and annihilation operators explicitly. Thus we need to know what $u_{\vec p}^{(r)*}$ and $v_{\vec p}^{(r)*}$ are. From the eqn.

$(\not p - m) u_{\vec p}^{(r)} = 0$, complex conjugated in a Majorana basis,

we have $\quad (-\not p - m) u_{\vec p}^{(r)*} = 0$, i.e. $(\not p + m) u_{\vec p}^{(r)*} = 0$.

This says that the complex conjugate of a solution of the Dirac equation that has positive frequency is a solution of the Dirac equation with negative frequency.

We are free to choose the $u_{\vec p}^{(r)}$ and $v_{\vec p}^{(r)}$ any way we want. So to make the action of charge conjugation simple, choose

$$v_{\vec p}^{(r)} = u_{\vec p}^{(r)*} \qquad v_{\vec 0}^{(r)} = u_{\vec 0}^{(r)*}$$

This is consistent with

$$u_{\vec p}^{(r)} = D(A(\vec e \phi))\, u_{\vec 0}^{(r)}$$
$$v_{\vec p}^{(r)} = D(A(\vec e \phi))\, v_{\vec 0}^{(r)}$$

because charge conjugation commutes with Lorentz transformations (in a Majorana basis).

Let's see what this implies about the effect of charge conjugation on spin. From the complex conjugate of

$L_z u_{\vec 0}^{(1)} = \frac{1}{2} u_{\vec 0}^{(1)}$ we have $-L_z u_{\vec 0}^{(1)*} = \frac{1}{2} u_0^{(1)*}$ i.e.

$L_z v_{\vec 0}^{(1)} = -\frac{1}{2} v_{\vec 0}^{(1)}$ Since u multiplies an annihilation operator and v multiplies a creation operator, this is exactly what we expect for charge conjugation. It shouldn't have any effect on spin.

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$$\psi(x) = \sum_r \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_{\vec p}}} \left[b^{(r)}_{\vec p}\, u^{(r)}_{\vec p}\, e^{-ip\cdot x} + c^{(r)\dagger}_{\vec p}\, v^{(r)}_{\vec p}\, e^{ip\cdot x} \right]$$

So,

$$\psi^*(x) = \sum_r \int \text{———} \left[b^{(r)\dagger}_{\vec p}\, u^{(r)*}_{\vec p}\, e^{ip\cdot x} + c^{(r)}_{\vec p}\, v^{(r)*}_{\vec p}\, e^{-ip\cdot x} \right]$$

$$= \sum_r \int \text{———} \left[b^{(r)\dagger}_{\vec p}\, v^{(r)}_{\vec p}\, e^{ip\cdot x} + c^{(r)}_{\vec p}\, u^{(r)}_{\vec p}\, e^{-ip\cdot x} \right]$$

We equate this with

$$U_c^\dagger \psi(x) U_c = \sum_r \int \text{———} \left[U_c^\dagger b^{(r)}_{\vec p} U_c\, u^{(r)}_{\vec p}\, e^{-ip\cdot x} + U_c^\dagger c^{(r)\dagger}_{\vec p} U_c\, v^{(r)}_{\vec p}\, e^{ip\cdot x} \right]$$

Matching coefficients gives $U_c^\dagger \psi(x) U_c^\dagger = \psi^*(x)$

$$U_c^\dagger b^{(r)}_{\vec p} U_c = c^{(r)}_{\vec p} \quad \text{and} \quad U_c^\dagger c^{(r)\dagger}_{\vec p} U_c = b^{(r)\dagger}_{\vec p}$$

The h.c equations are

$$U_c^\dagger b^{(r)\dagger}_{\vec p} U_c = c^{(r)\dagger}_{\vec p} \quad \text{and} \quad U_c^\dagger c^{(r)}_{\vec p} U_c = b^{(r)}_{\vec p}$$

This couldn't be simpler. Thanks to the way we set up the correspondence, complex conjugation does not mix up the spin ups and spin downs. Spin ups transform into spin ups and spin downs transform into spin downs, exactly as if the spin up electron was a boson whose antiparticle is a spin up positron.

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Construction of nucleon-antinucleon state.

Scalar case ("nucleon" – "antinucleon" state for warmup and comparison)

$$c^{\dagger}_{\vec{p}'}\, b^{\dagger}_{\vec{p}} |0\rangle$$

adds an "anti-nucleon" creates "nucleon"

$$U_c\, c^{\dagger}_{\vec{p}'}\, b^{\dagger}_{\vec{p}} |0\rangle = b^{\dagger}_{\vec{p}'}\, c^{\dagger}_{\vec{p}} |0\rangle$$

$$|\psi\rangle \equiv \int d^3p\, d^3p'\, f(\vec{p}, \vec{p}')\, c^{\dagger}_{\vec{p}'}\, b^{\dagger}_{\vec{p}} |0\rangle$$

Then $U_c|\psi\rangle = \pm|\psi\rangle$ if $f(\vec{p}', \vec{p}) = \pm f(\vec{p}, \vec{p}')$

Fermionic Case

$$|\psi\rangle = \sum_{rs} \int d^3p\, d^3p'\, f_{rs}(\vec{p}, \vec{p}')\, c^{(s)\dagger}_{\vec{p}'}\, b^{(r)\dagger}_{\vec{p}} |0\rangle$$

$$U_c|\psi\rangle = \mp|\psi\rangle \quad \text{if} \quad f_{sr}(\vec{p}, \vec{p}') = \pm f_{rs}(\vec{p}', \vec{p})$$

An antisymmetric state of fermion and antifermion is charge conjugation even!

Came from (anti-)commutation relations.

Charge conjugation properties of fermion bilinears. JAN. 6 10

Consider $\bar{A}MB$. So we can anticommute $\bar{A}$ and B without worrying about the anticommutator, either consider the Fermi fields A and B to be classical Fermi fields, or consider the normal ordered product, $:\bar{A}MB:$. M is just some matrix in spinor space, like $i\gamma_5$.

Under charge conjugation

$$U_c^\dagger A U_c = A^* \qquad U_c^\dagger B U_c = B^*$$

From this

$$U_c^\dagger A^* U_c = A \qquad U_c^\dagger B^* U_c = B$$

(or $\quad U_c^\dagger A^\dagger U_c = A^T \qquad U_c^\dagger B^\dagger U_c = B^T$

same statement as a row vector)

Now $\quad \bar{A} = A^\dagger \gamma^0$ so

$$U_c^\dagger \bar{A} U_c = U_c^\dagger A^\dagger U_c \gamma^0 = A^T \gamma^0$$
$$= A^{\dagger *}(-\gamma^{0*}) = -(A^\dagger \gamma^0)^*$$
$$= -\bar{A}^*$$

also $\quad U_c^\dagger \bar{B} U_c = -\bar{B}^*$

<u>Exercise</u> In quantum mechanics you frequently use $(\theta_1\theta_2)^\dagger = \theta_2^\dagger\theta_1^\dagger$. Prove this from the def'n of the adjoint, $(\varphi, \theta^\dagger\psi) = (\psi, \theta\varphi)^*$. Is this formula changed if θ_1 and θ_2 are Fermi? No.

$\bar{B}\bar{M}\bar{A}$

$\bar{\psi}\gamma^\mu\psi \to -\bar{\psi}\gamma^\mu\psi \qquad \bar{\psi}i\gamma_5\psi \to \bar{\psi}i\gamma_5\psi$

$\bar{\psi}\sigma^{\mu\nu}\psi \to -\bar{\psi}\sigma^{\mu\nu}\psi \qquad \bar{\psi}\gamma_\mu\gamma_5\psi \to \bar{\psi}\gamma_\mu\gamma_5\psi$

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$$U_c^\dagger :\bar{A}MB: U_c = :U_c^\dagger \bar{A} M B U_c:$$

This step is allowed because U_c does not mix up creation and annihilation operators

$$= :U_c^\dagger \bar{A} U_c M U_c^\dagger B U_c:$$
$$= - :\bar{A}^* M B^*:$$

To do the next step, I am going to explicitly display the spinor matrix multiplications so I don't have to worry about keeping matrices and spinors in a given order. The idea of the next step is to write this as the complex conjugate of something. We have

$$-:\bar{A}^* M B^*: = - :\bar{A}^*_\alpha M_{\alpha\beta} B^*_\beta:$$

remember * means adjoint without transpose and adjoint reverses order

$$= - :B_\beta M^*_{\alpha\beta} \bar{A}_\alpha:^*$$

$$= + :\bar{A}_\alpha M^*_{\alpha\beta} B_\beta:^*$$

Fermi fields anticommute inside normal ordered product

This last anticommutation puts things back in the right order to use the conventions of spinor matrix multiplication. What I have shown is

$$U_c^\dagger : \bar{A} M B: U_c = :\bar{A} M^* B:^*$$
$$= :\bar{B}\, \overline{M^*} A:$$

look back when we introduced the bar of a matrix

So all you have to do to calculate the effect of C on our 16 bilinears is to calculate things like

$$\overline{1}^* = 1$$
$$\overline{\gamma^\mu}^* = -\gamma^\mu \quad \text{in a majorana basis}$$
$$\overline{i\gamma_5}^* = i\gamma_5 \quad \text{" " "}$$

(you can be sloppy and not distinguish between $\overline{M}^*$ and $\overline{M^*}$ in a majorana basis)

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$$\overline{\gamma_5\gamma^\mu}^{\,*} = (\gamma_5\gamma^\mu)^* = \gamma_5\gamma_\mu$$
$$\overline{\sigma^{\mu\nu}}^{\,*} = -\sigma^{\mu\nu}$$

So $\bar\Psi\psi$ is charge conjugation invariant. That's good; it would be bad to find out that our mass term breaks C.

$$g_1\phi\,\bar\Psi\psi + g_2\phi\,\bar\Psi\gamma_5\psi$$ is charge conjugation invariant

$$-e A_\mu \bar\Psi\gamma^\mu\psi$$ is charge conjugation invariant only if A_μ is charge conjugation odd, $U_c^{+} A_\mu U_c = -A_\mu$.

vector mesons

$$a\,W_\mu\bar\Psi\gamma^\mu\gamma_5\psi + v\,W_\mu\bar\Psi\gamma^\mu\psi$$

is parity violating and C violating, but it preserves CP.

$$:\bar\Psi\sigma^{\mu\nu}\psi: \;\longrightarrow\; -:\bar\Psi\sigma^{\mu\nu}\psi:$$

10 odd fermion bilinears from symmetric combs and 6 even ones from antisymmetric combs.

$$D^{(1/2,1/2)}\otimes D^{(1/2,1/2)} = D^{(1,1)}\oplus D^{(0,0)} + D^{(0,1)}\oplus D^{(1,0)}$$

Bosons $|\Psi\rangle = \int d^3\vec p\; d^3\vec p\,'\; f(\vec p,\vec p\,')\, b^{+}_{p}\, c^{+}_{p'}|0\rangle$

$U_c|\Psi\rangle = \pm|\Psi\rangle \quad \pm$ if $f(\vec p,\vec p\,') = \pm f(\vec p\,',\vec p)$

Fermions

$U_c|\Psi\rangle = \pm|\Psi\rangle \quad$ if $f(\vec p,\vec p\,') = \mp f(\vec p\,',\vec p)$

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Example of Charge Conjugation in QED

The QED Lagrangian contains $A_\mu J^\mu$ where

$$J^\mu = e\bar{\psi}\gamma^\mu\psi$$

and A_μ is the photon field. If J^μ is charge conjugation odd, and if the Lagrangian is to preserve charge conjugation then A_μ must be charge conjugation odd, and a state with N photons satisfies

$$U_c|N\gamma\rangle = (-1)^N|N\gamma\rangle$$

We'll use this to evaluate the relative decay rates of ortho and para positronium, the two lowest nearly degenerate hydrogen like bound states of an electron and a positron

$$l=0\left\{\begin{array}{lll} S=1 \text{ ortho} & J=1 & C=-1 \\ S=0 \text{ para} & J=0 & C=+1 \end{array}\right.$$

("para" means opposite to)

The charge conjugation properties are deduced: a state of one electron and one positron stands a chance of being a charge conjugation eigenstate. The orbital wave function when l is even is symmetric. When two spin 1/2 are put together in a symmetric combination, you get a spin 1 state. When they are put together antisymmetrically you get a spin 0 state. These facts, and the crucial Fermi minus sign from anticommuting particle and antiparticle

creation operators
gives the charge conjugation eigenvalue. Now what are the possible decay products? Electron and positron are the lightest charged particle antiparticle pair, so the decay must be into n photons. By kinematics alone, one photon is not allowed. Two or more photons are kinematically allowed, but each additional photon comes with a factor of e in the amplitude, or e^2 in the probability. assuming there are no numerical surprises! (without doing some calculations there is no way to rule out factors like $(2\pi)^4$ in relative amplitudes), the partial decay rate into $n+1$ photons should be down by a factor of $e^2 \approx \frac{1}{137}$ compared to the partial decay rate into n photons, assuming they are both allowed. The decay that go fastest will be the one that go into the lowest number of photons. The lowest possibilities are

2 photons $\quad C = +1$
3 photons $\quad C = -1$

para $\rightarrow 2\gamma$ allowed

ortho $\rightarrow 2\gamma$ not allowed

ortho $\rightarrow 3\gamma$ allowed.

For another explanation along these lines, see I+Z p. 154 where the experimental values are also given.

$$U_C U_P = U_P U_C (-1)^{N_f} = U_P U_C \, U(R(2\pi\hat{z}))$$

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Show $\quad U_C U_P = U_P U_C \, U(R(\vec{e}\, 2\pi))$

$U(R(\vec{e}\, 2\pi))$ is a fancy way of writing the operator $(-1)^{N_F}$ (N_F = # of fermions)
Write $(-1)^{N_F}$ this way because it reminds you that it is a symmetry of the theory.

One way to show this is to show it is true when acting on an arbitrary state, or at least a basis. Consider

$$b^\dagger_{\vec{p}_1} \cdots b^\dagger_{\vec{p}_n} \, d^\dagger_{\vec{p}_1} \cdots d^\dagger_{\vec{p}_n} |0\rangle$$

States of this form are a basis and on states of this form it is easy to convince yourself the identity is true.

PT

Just as in a scalar theory, Lorentz invariance makes it easier to consider PT than T.
Recall PT in the classical scalar theory.

$$(\Box + m^2)\phi(x) = 0 \Rightarrow (\Box + m^2)\phi(-\vec{x}, -t) = 0$$

so we expect there to be an antiunitary operator having the effect

$$\Omega_{PT}^{-1} \phi(x) \Omega_{PT} = \phi(-x)$$

In the Dirac theory

$$(i\partial\!\!\!/ - m)\psi(x) = 0 \not\Rightarrow (i\partial\!\!\!/ - m)\psi(-x) = 0$$

Now an operation worth calling PT can have a more general form.

$$PT: \psi(x) \longrightarrow M\psi(-x)$$

where M is some four-by-four matrix in spinor space. What we need is

$$(i\partial\!\!\!/ - m)\psi(x) \Longrightarrow (i\partial\!\!\!/ - m)\, M\,\psi(-x) = 0$$

i.e. M must anticommute with γ^μ. We'll take $M = i\gamma_5$. Up to a factor this choice is unique.

[Proof: Suppose there is a second M anticommuting with the γ^μ, call it M'. Then MM' commutes with every one of the 16 Γ matrices, MM' must be proportional to the identity, and M' must be proportional to $\gamma_5^{-1} = \gamma_5$.]

$$PT: \psi(x) \longrightarrow i\gamma_5\psi(-x) = \Omega_{PT}^{-1}\psi(x)\Omega_{PT}$$

Consider applying PT twice

$$\Omega_{PT}^{-2}\psi(x)\Omega_{PT}^{2} = \Omega_{PT}^{-1} i\gamma_5\psi(-x)\Omega_{PT}$$

$i\gamma_5$ is real so it goes through Ω_{PT}

$$= i\gamma_5\,\Omega_{PT}^{-1}\psi(-x)\,\Omega_{PT}$$

$$= i\gamma_5\, i\gamma_5\,\psi(x) = -\psi(x) \qquad \gamma_5^2 = 1$$

$$= U(R(\vec{e}\,2\pi))\,\psi(x)$$

PT is a rotation half way around the rotation group. This proof is unaffected by giving M an arbitrary phase.

$$\Omega_{PT}^{2} = U(R(\vec{e}\,2\pi))$$

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Using $(\Omega^{-1} A \Omega)^\dagger = \Omega^{-1} A^\dagger \Omega$

[Proof:

$$(b, (\Omega^{-1} A \Omega)^\dagger a) \xrightarrow{\text{def'n of adjoint of } \Omega^{-1}A\Omega} = (a, \Omega^{-1} A \Omega b)^* \quad \text{antiunitarity of } \Omega$$

$$= (\Omega a, \Omega \Omega^{-1} A \Omega b)$$

$$= (\Omega a, A \Omega b) \quad \text{def'n of adjoint of } A$$

$$= (\Omega b, A^\dagger \Omega a)^*$$

$$= (\Omega^{-1} \Omega b, \Omega^{-1} A^\dagger \Omega a) \quad \text{antiunitarity of } \Omega^{-1}$$

$$= (b, \Omega^{-1} A^\dagger \Omega a) \quad]$$

$$\Omega_{PT}^{-1} \psi^\dagger(x) \Omega_{PT} = (i\gamma_5 \psi(-x))^\dagger$$

$$= -\psi^\dagger(-x) i\gamma_5$$

This tells how $\bar\psi = \psi^\dagger \beta$ transforms

$$\Omega_{PT}^{-1} \bar\psi \Omega_{PT} = \Omega_{PT}^{-1} \psi^\dagger \Omega_{PT} \underbrace{\beta^*}_{-\beta}$$

$$= -\psi^\dagger(-x) i\gamma_5 (-\beta)$$

$$= -\bar\psi(-x) i\gamma_5$$

So

PT: $\bar\psi(x)\psi(x) \longrightarrow \bar\psi(-x)\psi(-x)$

$\Omega_{PT}^2 |a\rangle = |a\rangle$ $\quad \Omega_{PT}$

$|a_+\rangle = |a\rangle + \Omega_{PT}|a\rangle \Rightarrow \Omega_{PT}|a_+\rangle = |a_+\rangle$

$|a_-\rangle = i(|a\rangle - \Omega_{PT}|a\rangle) \Rightarrow \Omega_{PT}|a_-\rangle = |a_-\rangle$

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PT (CONT'D)

(PT commutes with Lorentz transformation)

Ω_{PT} on states.

We expect PT to do nothing to momenta

$$\vec{k} \xrightarrow[P]{} -\vec{k} \xrightarrow[T]{} \vec{k}$$

Reflection turns the momenta around, but running the movie backward does it again.

The spin of a particle is affected though!

$$\vec{S} \xrightarrow[P]{} \vec{S} \xrightarrow[T]{} -\vec{S}$$

Thus we can't expect to find a basis where single particle states are left unchanged. I.e.

$$\Omega_{PT}^{-1} \begin{Bmatrix} b_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)} \end{Bmatrix} \Omega_{PT} = \begin{Bmatrix} b_{\vec{p}}^{(r)\prime} \\ c_{\vec{p}}^{(r)\prime} \end{Bmatrix}$$

where $b_{\vec{p}}^{(1)\prime} = \text{some phase} \times b_{\vec{p}}^{(2)}$

$b_{\vec{p}}^{(2)\prime} = \text{some other phase} \times b_{\vec{p}}^{(1)}$

So that Ω_{PT} applied twice gives $-\begin{Bmatrix} b_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)} \end{Bmatrix}$

some other phase = − some phase*

$\Omega_{PT}^2 |0\rangle = -|0\rangle$

$\Omega_{PT} |0\rangle = \lambda |0\rangle \qquad \Omega_{PT}^2 |0\rangle = \Omega_{PT} \lambda |0\rangle = \lambda^* \lambda |0\rangle \neq -|0\rangle$

[illegible]

Tricky choice of spinor basis

If $b_{\vec{p}}^{(r)}$ is associated with the solution of the Dirac equation $u_{\vec{p}}^{(r)}$, let $b_{\vec{p}}^{(r)\prime}$ be associated with $u_{\vec{p}}^{(r)\prime} \equiv -i\gamma_5 u_{\vec{p}}^{(r)*}$

If $c_{\vec{p}}^{(r)}$ is associated with $v_{\vec{p}}^{(r)}$, let $c_{\vec{p}}^{(r)\prime}$ be associated with $v_{\vec{p}}^{(r)*}$

What's good about this choice? It makes the action of $\mathcal{P}_{PT}$ on creation and annihilation operators simple. Furthermore it agrees with the expectations of the previous page.

If $(\not{p}-m)u=0$, then (taking c.c.)

$$(-\not{p}-m)u^* = 0 \quad \text{and} \quad (\not{p}-m)(-i\gamma_5 u^*)=0$$

Given $L_z u_{\vec{0}}^{(1)} = +\frac{1}{2} u_{\vec{0}}^{(1)}$, then (taking c.c)

$$L_z u_{\vec{0}}^{(1)} = -\frac{1}{2} u_{\vec{0}}^{(1)*} \quad \text{and}$$

$$L_z\left(-i\gamma_5 u_{\vec{0}}^{(1)*}\right) = -\frac{1}{2}\left(-i\gamma_5 u_{\vec{0}}^{(1)*}\right)$$

Because Lorentz transformations are real in a Majorana basis, this generalizes to moving states.

The action of $\mathcal{P}_{PT}$ on creation and annihilation operators is derived from the action of $\mathcal{P}_{PT}$ on the field, and the expansion of the field

$$v_{\vec{p}}^{(1)\prime} \equiv -i\gamma_5 v_{\vec{p}}^{(1)*}$$

So now will see that the definitions of $b_{\vec{p}}^{(r)\prime}$ and $c_{\vec{p}}^{(r)\prime}$ and

$$\Omega_{PT}^{-1}\begin{Bmatrix} b_{\vec{p}}^{(r)} \\ c_{\vec{p}}^{(r)} \end{Bmatrix}\Omega_{PT} = \begin{Bmatrix} b_{\vec{p}}^{(r)\prime} \\ c_{\vec{p}}^{(r)\prime} \end{Bmatrix}$$

are consistent with

$$\Omega_{PT}^{-1}\psi(x)\Omega_{PT} = i\gamma_5\psi(-x)$$

which is equivalent to

$$-i\gamma_5\Omega_{PT}^{-1}\psi(-x)\Omega_{PT} = \psi(x)$$

We can write the expansion of $\psi(x)$ two ways

kinematic factors unimportant to the argument

$$\psi(x) = \sum\int(\cdots)\left[b_{\vec{p}}^{(r)}u_{\vec{p}}^{(r)}e^{-ip\cdot x} + c_{\vec{p}}^{(r)\dagger}v_{\vec{p}}^{(r)}e^{ip\cdot x}\right]$$

or $$\psi(x) = \sum\int(\cdots)\left[b_{\vec{p}}^{(r)\prime}u_{\vec{p}}^{(r)\prime}e^{-ip\cdot x} + c_{\vec{p}}^{(r)\prime\dagger}v_{\vec{p}}^{(r)\prime}e^{ip\cdot x}\right]$$

Use the first way in the LHS and the second way in the right. Writing out the LHS, we have

LHS =

$$-i\gamma_5\Omega_{PT}^{-1}\int(\cdots)\left[b_{\vec{p}}^{(r)}u_{\vec{p}}^{(r)}e^{+ip\cdot x} + c_{\vec{p}}^{(r)\dagger}v_{\vec{p}}^{(r)}e^{-ip\cdot x}\right]\Omega_{PT}$$

$$= -i\gamma_5\int(\cdots)\left[\Omega_{PT}^{-1}b_{\vec{p}}^{(r)}\Omega_{PT}\,u_{\vec{p}}^{(r)*}e^{-ip\cdot x} + \Omega_{PT}^{-1}c_{\vec{p}}^{(r)\dagger}\Omega_{PT}\,v_{\vec{p}}^{(r)*}e^{ip\cdot x}\right]$$

$$= \int(\cdots)\left[b_{\vec{p}}^{(r)\prime}u_{\vec{p}}^{(r)\prime}e^{-ip\cdot x} + c_{\vec{p}}^{(r)\prime\dagger}v_{\vec{p}}^{(r)\prime}e^{ip\cdot x}\right]$$

= RHS

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Proof of PCT within perturbation theory [4]

For scalars TCP invariance of the S matrix was equivalent to

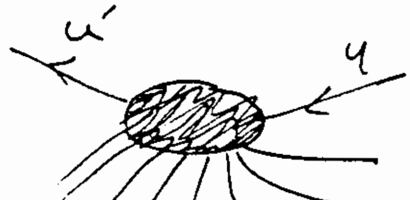
$$a(p_1, \dots, p_n) = a(-p_1, \dots, -p_n).$$

This says that the amplitude with all incoming particles turned into outgoing antiparticles with the same 3-momentum is the same

What is the corresponding statement when there are Dirac particles in the theory. We'll simplify by looking only at

1 fermion + any number of mesons

$\longrightarrow$ 1 fermion + any other number of mesons



a is of the form

$$a = \bar{u}' M(p_1, \dots, p_n) u$$

Instead of having an outgoing fermion characterized by $\bar{u}'$, the CPT reversed process has an incoming antifermion with the opposite spin characterized by $-i\bar{u}'\gamma_5$. Instead of an incoming fermion characterised by u, the CPT reversed process has an outgoing antifermion characterised by $-i\gamma_5 u$. If you want to understand this in two steps

$$u \xrightarrow[PT]{} -i\gamma_5 u^* \xrightarrow[C]{} -i\gamma_5 u$$

describes an incoming fermion with some spin — describes an outgoing fermion of the opposite spin — describes an outgoing antifermion of the opposite spin

This is a definite choice for the transformed spinor, a choice of another phase would screw up the CPT thm.

There is also an additional minus sign in the amplitude Because an operator has to have an odd number of reorderings of Fermi fields to contribute to this CPT reversed process.

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Equality of the amplitude for the process and the CPT transformed ~~process~~ is this switching the operators

$\bar{u}' M(p_1, \dots, p_n) u = -(-i)^2\, \bar{u}' \gamma_5 M(-p_1, \dots, -p_n) \gamma_5 u$

The proof that these two are equal only uses L.I. of the Feynman rules

Whatever the Feynman rules are, L.I. tells us that

$$\bar{u}' M(p_1, \dots, p_n) u = \bar{u}' \overline{D(\Lambda)} M(\Lambda p_1, \dots, \Lambda p_n) D(\Lambda) u$$

Consider the case when Λ is a boost in any direction by an angle ϕ. Is the RHS an analytic function of ϕ? $\overline{D(\Lambda)}$ contains complex conjugation, so we're off to a bad start. However, $\overline{D(\Lambda)} = D(\Lambda)^{-1}$. In this form, and using that

$$D(A(\vec{e}\phi)) = e^{\vec{\alpha} \cdot \vec{e}\phi/2} \qquad \Lambda = R(e_z\pi) A(\vec{e}_z\pi)$$

we see that both $D(\Lambda)$ and $\overline{D(\Lambda)}$ are analytic; they are just exponentials.

What about $M(\Lambda p_1, \dots, \Lambda p_n)$. M is of the form

$$M = \int d^4k_1 \cdots d^4k_m \; \frac{N(p_1, \dots, p_n; k_1, \dots, k_m)}{D(p_1, \dots, p_n; k_1, \dots, k_m)}$$

The denominator is Lorentz invariant. The numerator may be an unbelievably complex matrix, but at any finite order in perturbation theory, it is still a polynomial. So the whole RHS is an analytic function of ϕ, and we can use the equation for complex ϕ.

[If $LHS(\phi) = RHS(\phi)$ for real ϕ, and if both sides are analytic functions of ϕ in some domain (the complex plane containing a segment of the real line), then both sides are equal in that domain.]

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Consider the Lorentz transformation

$$\Lambda = R(\vec{e}_z\, \pi)\, A(\vec{e}_z\, i\pi)$$

$$A(\vec{e}_z\, \phi):\quad p^0 \to p^0 \cosh\phi + p^3 \sinh\phi$$
$$p^1 \to p^1$$
$$p^2 \to p^2$$
$$p^3 \to p^3 \cosh\phi + p^3 \sinh\phi$$

so $A(\vec{e}_z\, i\pi)$: $\quad p^0 \to -p^0 \quad p^1 \to p^1 \quad p^2 \to p^2$
$$p^3 \to -p^3$$

while $R(\vec{e}_z\, \pi)$: $\quad -p^0 \to -p^0 \quad p^1 \to -p^1 \quad p^2 \to -p^2$
$$-p^3 \to -p^3$$

so
$$\Lambda :\ p^\mu \to -p^\mu$$

a rotation by π in the z, it plane and a rotation by π in the x, y plane.

What is $D(\Lambda)$?

$$D(R(\vec{e}_z \pi)) = e^{-i L_z \pi}$$
$$= e^{\frac{\pi}{2}\gamma^1\gamma^2}$$
$$= \cos\frac{\pi}{2} + \gamma^1\gamma^2 \sin\frac{\pi}{2} = \gamma^1\gamma^2$$

$$\boxed{\begin{aligned} L_z &= \frac{i}{4}\epsilon_{3ij}\gamma^i\gamma^j \\ &= \frac{i}{4}(\gamma^1\gamma^2 - \gamma^2\gamma^1) \\ &= \frac{i}{2}\gamma^1\gamma^2 \end{aligned}}$$

$$D(A(\vec{e}_z i\pi)) = e^{\alpha_z i\pi/2} = e^{i\frac{\pi}{2}\gamma^0\gamma^3}$$
$$= \cos\frac{\pi}{2} + i\gamma^0\gamma^3 \sin\frac{\pi}{2} = i\gamma^0\gamma^3$$

So

$$D(\Lambda) = D(R(\vec{e}_z\pi))\, D(A(\vec{e}_z i\pi)) = \gamma^1\gamma^2\, i\gamma^0\gamma^3 = \gamma^5$$

PCT is the analytic continuation of the discrete transformation!

Since $\gamma_5^2 = 1$ $D(\Lambda)^{-1} = \gamma_5$ also.

Lorentz invariance says

$$\bar{u}' M(p_1, \dots, p_n) u = \bar{u}' D(\Lambda)^{-1} M(\Lambda p_1, \dots, \Lambda p_n) D(\Lambda) u$$
$$= \bar{u}' \gamma_5 M(-p_1, \dots, -p_n) \gamma_5 u,$$

and this is exactly the statement of equality between an amplitude and the CPT transformed amplitude.

Only analyticity of Feynman amplitudes was used in the proof of this theorem. This suggests that the theorem has very little to do with perturbation theory.

When we talked about parity invariance we had to hunt for the correct transformation of the field. Depending on the interactions that transformation may have to be chosen in various ways. A scalar meson may be forced to be scalar or pseudoscalar. For CPT invariance, you dont have to hunt for the right transformation. You just compute $D(\Lambda)$ for the funny Lorentz transformation with complex rapidity. It will be a symmetry of the Lagrangian, as long as the Lagrangian is Lorentz invariant and hermitian.

This proof easily generalizes to higher spin. You just compute $D(\Lambda)$ for the higher spin field.

(The restriction to one incoming and one outgoing fermion was totally inessential.)

NEXT renormalization of spinor fields

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8

Renormalization of spinor theories

To have a simple example in mind

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu_0^2\phi^2 + \bar\psi(i\not\partial - m_0)\psi$$
$$-\lambda_0\phi^4 - g_0\bar\psi i\gamma_5\psi\phi$$

The meson is pseudoscalar. This insures $\langle 0|\phi(x)|0\rangle = 0$. Not so if the meson nucleon interaction is $g_0\bar\psi\psi\phi$.

m_0 and μ_0 have no necessary connection with physical masses. g_0 and λ_0 have no necessary connection with the couplings measured in the standard scattering process. ϕ and ψ are not necessarily good fields from the standpoint of the LSZ reduction formula.

Define Z_3 by

$$\langle 0|\phi(0)|k\rangle \equiv Z_3^{1/2} \qquad \phi' \equiv Z_3^{-1/2}\phi$$

one meson $\quad \langle 0|\phi'(0)|k\rangle = 1$

ϕ' is a good field from the standpoint of the LSZ reduction formula (but of course it does not have conventionally normalized equal time commutation relations).

$\langle 0|\psi(x)|0\rangle = 0$ by Lorentz invariance. So it also only needs rescaling to get a good field for LSZ. However, the various components of this field may need different rescalings.

RELATIVISTICALLY NORMALIZED SO AS TO MAKE L.T. PROPERTIES SIMPLE.

Let $|r, p\rangle$ be a one fermion state with momentum p and spin labelled by r. We'll just study

$\langle 0|\psi(0)|r, p\rangle$ in the rest frame of p. Anything else can be obtained by a Lorentz transformation.

$\langle 0|\psi(x)|p, s\rangle$

physical vacuum $\qquad$ physical one nucleon

Matrix elements of $\bar\psi$ are related to matrix elements of ψ by CPT or just by C if the theory had that invariance. JAN. 8 9

For definiteness, label the $J_z = +\frac{1}{2}$ state by $r=1$ and $J_z = -\frac{1}{2}$ by $r=2$. Let

$$u_0 \equiv \langle 0 | \psi(0) | 1, p \rangle$$

we can obtain some restrictions on the form of u_0 by using L_z conservation

$$u_0 = \langle 0 | \psi(0) | 1, p\rangle = \underbrace{\langle 0 | e^{-iJ_z\theta}}_{\langle 0 |} \underbrace{e^{iJ_z\theta}\psi(0)e^{-iJ_z\theta}}_{e^{-iL_z\theta}\psi(0)} \underbrace{e^{iJ_z\theta}|1,p\rangle}_{e^{i\theta/2}}$$

$$= e^{-iL_z\theta} e^{i\theta/2} \langle 0 | \psi(0) | 1, p\rangle$$

$$= e^{-iL_z\theta} e^{i\theta/2} u_0$$

In the standard basis $L_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$

so in the standard basis this restricts u_0 to be of the form

$$u_0 = \begin{pmatrix} a \\ 0 \\ b \\ 0 \end{pmatrix}$$

A less formal way of getting what we have just shown is to say that of the four components of ψ, in the standard basis

these two $\begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$ lower J_z by $\frac{1}{2}$ and the other two raise J_z by $\frac{1}{2}$ so only the first two can have a nonzero $J_z = \frac{1}{2}$ to zero matrix element.

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9'

[ASIDE

Strong coupling scenarios could violate the assumption that the parity transformation property of the physical nucleon are the same as that of the bare nucleon. Starting with weak coupling, as you turn up the coupling a nucleon meson bound state may form. Turn up the coupling and it may become lighter than the nucleon. What you had called the nucleon is now unstable. If the meson is a pseudoscalar, the s wave bound state will not have the same parity: the perturbation theory nucleon did.]

To simplify life, let's also assume the theory has parity invariance

$$U_P^+ \psi(0) U_P = \beta \psi(0)$$

$U_P |1, p\rangle = |1, p\rangle$ remember we are in the rest frame of p.

$U_P |0\rangle = |0\rangle$

Now this is an assumption about the parity transformation properties of a physical nucleon, but in perturbation theory, the transformation properties of the physical nucleon should be the same as those of the bare nucleon for whatever symmetries are not broken by the interaction.

This assumption simplifies the possible form of u_0

$$u_0 = \langle 0 | \psi(0) | 1, p\rangle = \underbrace{\langle 0 | U_P}_{\langle 0|} \underbrace{U_P^+ \psi(0) U_P}_{\beta\psi(0)} \underbrace{U_P^+ | 1, p\rangle}_{|1,p\rangle}$$

$$= \beta \langle 0 | \psi(0) | 1, p\rangle$$

$$= \beta u_0$$

In the standard basis $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

so in the standard basis is now restricted to be $u_0 = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Define $Z_2^{1/2}$ by $a = Z_2^{1/2} \sqrt{2m}$

and ψ' by $\psi' = Z_2^{-1/2} \psi$, then

$$\langle 0 | \psi'(0) | 1, p\rangle = \begin{pmatrix} \sqrt{2m} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For general p, $r=1,2$, and any x we then have

$$\langle 0|\psi'(x)|r,p\rangle = e^{-ip\cdot x}\, u_{\vec p}^{(r)}$$

which has been arranged to be exactly like the free theory. The LSZ reduction formula goes through as before. The Lagrangian you proceed from to do renormalized perturbation theory is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi')^2 - \frac{\mu^2}{2}\phi'^2 + \bar\psi'(i\not\partial - m)\psi'$$
$$- \lambda\phi'^4 - g\bar\psi' i\gamma_5\psi'\phi'$$
$$+ \frac{1}{2}A(\partial_\mu\phi')^2 - \frac{1}{2}B\phi'^2 + C\bar\psi' i\not\partial\psi' - D\bar\psi'\psi'$$
$$- E\bar\psi' i\gamma_5\psi'\phi' - F\phi'^4$$

[Digression on spinor renormalization in parity non conserving theories.

γ_5 commutes with Lorentz transformations, so $\gamma_5\psi(x)$ transforms in the same way as $\psi(x)$ under Lorentz transformations. In the standard rep

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{so}$$

$$\langle 0|\gamma_5\psi(0)|1,\vec p\rangle \overset{\text{at rest}}{=} \begin{pmatrix} b \\ 0 \\ a \\ 0 \end{pmatrix}$$

ψ and $\gamma_5\psi$ have opposite parity transformation properties

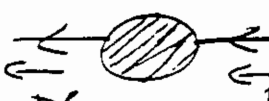
The field $\psi'(x) = \dfrac{a\psi(x) - b\gamma_5\psi(x)}{a^2 - b^2}$

is the one satisfying $\langle 0|\psi'(x)|r,p\rangle = e^{-ip\cdot x}\, u_{\vec p}^{(r)}$

As is usual in theories with fewer symmetries, more c.t. are needed and will have to be fixed in a theory without parity.

In parallel with the method for determining A and B order by order in perturbation theory done on NOV. 18 and 20, we'll show how C and D are determined order by order in perturbation theory.

Define



$$= \int d^4x\, d^4y\, e^{ip'\cdot x} e^{-ip\cdot y} \langle 0|T(\psi'(x)\bar{\psi}'(y))|0\rangle$$

$$\equiv (2\pi)^4 \delta^{(4)}(p'-p)\, S'(p)$$

where $S'(p)$ is some 4×4 matrix function of p (this form is dictated by translational invariance; $\langle 0|T(\psi'(x)\bar{\psi}'(y))|0\rangle$ is a function of $x-y$ alone)

[Let's check that the conventions are right by comparing with the free field theory result. On DEC. 18 pp. 4-5 we calculated

$$\overline{\psi(x)\bar{\psi}(y)} = \langle 0|T(\psi(x)\bar{\psi}(y))|0\rangle$$

$$= (i\partial\!\!\!/_x + m)\int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{i}{q^2-m^2+i\epsilon}$$

$$= \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{i(q\!\!\!/+m)}{q^2-m^2+i\epsilon} = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} \frac{i}{q\!\!\!/-m+i\epsilon}$$

in free field theory. Putting this in above we have

$$\int \frac{d^4q}{(2\pi)^4} \frac{i}{q\!\!\!/-m+i\epsilon} \int d^4x\, d^4y\, e^{-iq\cdot(x-y)} e^{ip'\cdot x} e^{-ip\cdot y}$$

$$= \int \frac{d^4q}{(2\pi)^4} \frac{i}{q\!\!\!/-m+i\epsilon} (2\pi)^4 \delta^{(4)}(p'-q)(2\pi)^4 \delta^{(4)}(p-q)$$

$$= (2\pi)^4 \delta^{(4)}(p'-p) \frac{i}{p\!\!\!/-m+i\epsilon}.$$ The conventions are right; this is what we write down upon seeing $\xleftarrow[p]{}$]

Lorentz invariance restricts the form of

$$\mathcal{S}'(p) = \int d^4x\, e^{ip\cdot x} \langle 0|T(\psi'(x)\bar{\psi}'(0))|0\rangle$$

$$= \int d^4x\, e^{ip\cdot x} \underbrace{\langle 0|U(\Lambda)}_{\langle 0|} U(\Lambda)^\dagger T(\psi'(x)\bar{\psi}(0))\, \underbrace{U(\Lambda)\, U(\Lambda)^\dagger |0\rangle}_{|0\rangle}$$

$$= \int d^4x\, e^{ip\cdot x} \langle 0|T\left(U(\Lambda)^\dagger \psi'(x)\, U(\Lambda) U(\Lambda)^\dagger \bar{\psi}(0)\, U(\Lambda)\right)|0\rangle$$

$$= \int d^4x\, e^{ip\cdot x}\, D(\Lambda)\, \langle 0|T(\psi'(\Lambda^{-1}x)\bar{\psi}'(0))|0\rangle\, \overline{D(\Lambda)}$$

$$= D(\Lambda) \int d^4x\, e^{ip\cdot\Lambda x} \langle 0|T(\psi'(x)\bar{\psi}'(0))|0\rangle\, \overline{D(\Lambda)}$$

$$= D(\Lambda) \int d^4x\, e^{i\Lambda^{-1}p\cdot x} \langle 0|T(\psi'(x)\bar{\psi}'(0))|0\rangle\, \overline{D(\Lambda)}$$

$$= D(\Lambda)\, \mathcal{S}'(\Lambda^{-1}p)\, \overline{D(\Lambda)}$$

You can use this and the Lorentz transformation properties of the 16 Γ matrices (which are a complete set of 4×4 matrices) to get

$$\mathcal{S}'(p) = a(p^2) + b(p^2)\gamma_5 + c(p^2)\gamma^\mu p_\mu + d(p^2)\gamma_5\gamma^\mu p_\mu + e(p^2)\sigma_{\mu\nu}p^\mu p^\nu$$

$e(p^2)\sigma_{\mu\nu}p^\mu p^\nu \to 0$ by antisymmetry

$b(p^2)\gamma_5$, $d(p^2)\gamma_5\gamma^\mu p_\mu$: Ruled out if we assume parity invariance.

$$= a(p^2) + c(p^2)\,\not{p}$$

Define a new function $S'(z) = a(z^2) + z\,c(z^2)$, a function of a single complex variable. Then because $\not{p}^2 = p^2$

$$\mathcal{S}'(p) = S'(\not{p})$$

The propagator is characterized by a single function of $\not{p}$, a function of one variable! (There is a one-to-one correspondence between functions of one variable and functions of 1 matrix. A function of two matrices is far more complicated than a function of two numbers, unless the two matrices commute)

$$\tilde{S}' = -\gamma_5 S \gamma_5 \qquad \tilde{\Psi} = \gamma_5 \Psi$$

$$S' = \frac{[illegible]}{\not{p} - m + i\epsilon} + \int \frac{\rho_+(\sigma^2)\, d\sigma^2}{\not{p} - \sigma + i\epsilon} + \int \frac{\rho_-(\sigma^2)\, d\sigma^2}{\not{p} + \sigma - i\epsilon} \qquad \rho_\pm \geq 0$$

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So far we have found


$$= (2\pi)^4 \delta^{(4)}(p'-p)\, S'(\not{p})$$

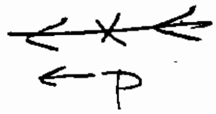
Now we define a one particle irreducible Green's function (defined to not include $(2\pi)^4\delta^{(4)}(p'-p)$ or external propogators)


$$= -i\Sigma'(\not{p})$$

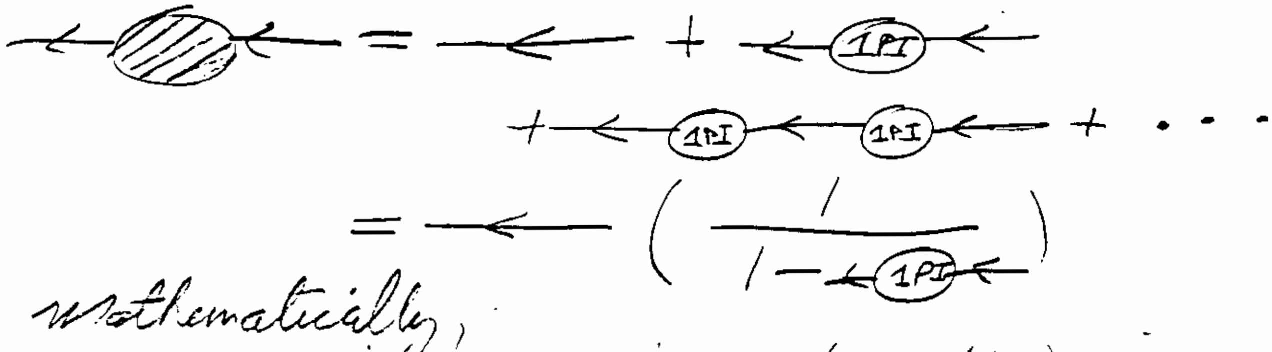
For example, if a term in the Lagrangian is

$$-\,\delta m\, \bar{\Psi}\Psi$$

There is a contribution to $-i\Sigma'(\not{p})$ of


$-i\delta m$ i.e to $\Sigma'(\not{p})$ of δm

The nice thing about the 1PI function is that it gives us an expression for the full Green's function (without the $(2\pi)^4\delta^{(4)}(p'-p)$, i.e it gives us $S'(\not{p})$).



mathematically,

$$S'(\not{p}) = \frac{i}{\not{p}-m+i\epsilon} + \frac{i}{\not{p}-m+i\epsilon}\left(-i\Sigma'(\not{p})\right)\frac{i}{\not{p}-m+i\epsilon}$$
$$+ \frac{i}{\not{p}-m+i\epsilon}\left(-i\Sigma'(\not{p})\right)\frac{i}{\not{p}-m+i\epsilon}\left(-i\Sigma'(\not{p})\right)$$
$$+ \cdots$$

which sums to

$$S'(\not p) = \frac{i}{\not p - m + i\epsilon}\left[1 + \frac{\Sigma'(\not p)}{\not p - m + i\epsilon} + \left(\frac{\Sigma'(\not p)}{\not p - m + i\epsilon}\right)^2 + \dots\right]$$

$$= \frac{i}{\not p - m + i\epsilon}\,\frac{1}{1 - \frac{\Sigma'(\not p)}{\not p - m + i\epsilon}} = \frac{i}{\not p - m - \Sigma'(\not p) + i\epsilon}$$

(Remember $\not p$ is the only matrix in the problem, so it commutes with every other matrix, and manipulations in which $\not p$ is treated like a number are correct. This simplification does not persist in the spin 3/2 problem.)

To get a spectral representation for $S'(\not p)$ we insert a complete set into $\langle 0|\psi'(x)\bar\psi'(y)|0\rangle$:

$$\sum \langle 0|\psi'(x)|n\rangle\langle n|\bar\psi'(y)|0\rangle$$

complete set $|n\rangle$ of momentum eigenstates

$$= \sum_{|n\rangle} e^{-iP_n\cdot(x-y)}\langle 0|\psi'(0)|n\rangle\langle n|\bar\psi'(0)|0\rangle$$

Now we break the $|n\rangle$ up into physical vacuum, physical one electron, one positron and all other states. One positron does not contribute because $\langle$one positron$|\bar\psi'|0\rangle = 0$ (fermion # conservation). As on NOV. 18 p. 8, we use the renormalization conditions to eliminate the physical vacuum contribution and to simplify the one electron contribution.

$$\langle 0|\psi'(x)\bar\psi'(y)|0\rangle = \sum_r \int \frac{d^3q}{(2\pi)^3 2\omega_{\vec q}} e^{-iq\cdot(x-y)}\langle 0|\psi'(0)|\underbrace{q,r\rangle\langle q,r}_{\text{one electron}}|\bar\psi'(0)|0\rangle$$

$$+ \sum_{\substack{\text{all other}\\ \text{states } |n\rangle}} e^{-iP_n\cdot(x-y)}\langle 0|\psi'(0)|n\rangle\langle n|\bar\psi'(0)|0\rangle$$

$$\langle 0|\psi(x)\,\overline{\psi}'(y)|0\rangle = \sum_r \int \frac{d^3q}{(2\pi)^3 2\omega_{\vec q}} e^{-iq\cdot(x-y)}\, u^{(r)}_{\vec q}\, \overline{u^{(r)}_{\vec q}}$$
$$+\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \sum_{\substack{\text{all other}\\ |n\rangle}} \delta^{(4)}(p-P_n)\langle 0|\psi'(0)|n\rangle\langle n|\overline{\psi}'(0)|0\rangle$$

The sum on $|n\rangle$ only contributes when the state $|n\rangle$ has spin $\frac{1}{2}$ in its rest frame. In a parity invariant theory. We can split these states into

$J^P = \frac{1}{2}^+$ like a nucleon and meson in a p wave

and $J^P = \frac{1}{2}^-$ like a nucleon and meson in an s wave.

The parity $+$ states only give nonzero contributions to the upper two components $\langle 0|\psi'(0)|n\rangle$ (in a standard basis, which is easiest for states at rest to work with). The parity $-$ state only give nonzero components to the lower two components of ψ. Furthermore the contribution of a $J_z = +\frac{1}{2}$ state to the top component is the same as a contribution to the second component of the same state but with the lowering operator $J_x - iJ_y$.

σ matrix which reduces to $2\sqrt{p^2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ← 2x2 blocks

(in the standard basis) when p is at rest and is covariant is $\not p + \sqrt{p^2}$, so

$$\sum_{\substack{\text{all other}\\ |n\rangle \\ \text{with } J^P = \frac{1}{2}^+}} \delta^{(4)}(p-P_n)\langle 0|\psi'(0)|n\rangle\langle n|\overline{\psi}'(0)|0\rangle$$

$$\frac{\theta(p^0)}{(2\pi)^3}\cdot \sigma_+(\sqrt{p^2})\,(\not p + \sqrt{p^2})$$

Similarly

Can shortcut some work by noticing that $\bar{\Psi}' = \gamma_5 \Psi'$ has the same matrix elements with a $\frac{1}{2}+$ state as Ψ' has with a $\frac{1}{2}-$ state. "As the $\frac{1}{2}+$ states are to Ψ, the $\frac{1}{2}-$ states are to $\gamma_5\Psi$."

$$\sum_{\substack{\text{all other}\\ |n\rangle \text{ with } J^P=\frac{1}{2}-}} \delta^{(4)}(p-p_n)\langle 0|\Psi'(0)|n\rangle\langle n|\bar{\Psi}'(0)|0\rangle$$

$$= \frac{\theta(p^0)}{(2\pi)^3}\,\sigma_-(\sqrt{p^2})\,(\not{p}-m)$$

σ_+ and σ_-, which are defined by these equations are both positive semidefinite by the positivity of the norm on Hilbert space. You might worry that $\not{p}-m$ is negative in the rest frame of p, but it should be because

$$\langle 0|\Psi'(0)|n\rangle\langle n|\bar{\Psi}'(0)|0\rangle$$

differs from $\langle 0|\Psi'(0)|n\rangle\langle n|\Psi'^{\dagger}(0)|0\rangle$ by the matrix γ^0 which is negative in its lower two components. In perturbation theory, $\sigma_+ = \sigma_- = 0$ when $p^2 < (m+\mu)^2$.

Putting this together, we have

$$\langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle = \int \frac{d^3q}{(2\pi)^3 2\omega_{\vec{q}}}\, e^{-iq\cdot(x-y)}\,(\not{q}+m)$$

$$+\int \frac{d^4p}{(2\pi)^3}\, e^{-ip\cdot(x-y)}\Big(\theta(p^0)\,\sigma_+(\sqrt{p^2})(\not{p}+\sqrt{p^2})$$
$$+\,\theta(p^0)\,\sigma_-(\sqrt{p^2})(\not{p}-\sqrt{p^2})\Big)$$

$$= (i\not{\partial}_x + m)\,\Delta(x-y)$$
$$+\int_0^\infty da\,\sigma_+(a)\int \frac{d^4p}{(2\pi)^3}\,\theta(p^0)\delta(p^2-a^2)\,(\not{p}+a)\,e^{-ip\cdot(x-y)}$$
$$+\int_0^\infty da\,\sigma_-(a)\int \frac{d^4p}{(2\pi)^3}\,\theta(p^0)\delta(p^2-a^2)\,(\not{p}-a)\,e^{-ip\cdot(x-y)}$$

$$\langle 0|\psi'(x)\bar\psi'(y)|0\rangle$$

$$= (i\not\partial_x + m)\Delta(x-y) + \int_0^\infty da\,\sigma_+(a)(i\not\partial_x + a)\Delta(x-y;a^2)$$

$$+ \int_0^\infty da\,\sigma_-(a)(i\not\partial_x - a)\Delta(x-y;a^2)$$

$$= \int_0^\infty da\left[\rho_+(a)(i\not\partial_x + a)\Delta(x-y) + \rho_-(a)(i\not\partial_x - a)\Delta(x-y)\right]$$

For compactness in the last step I have introduced

$$\rho_+(a) = \sigma_+(a) + \delta(a-m)$$

$$\rho_-(a) = \sigma_-(a)$$ and dropped the $;a^2$ in Δ.

Rather than redoing a lot of steps, I can get

$$\langle 0|\bar\psi'_\beta(y)\psi'_\alpha(x)|0\rangle$$

from what we've just calculated using Ω_{CPT} $(=\Omega)$ (In a theory with C invariance, it would be easier to just use U_c, but I'll be more general.) Then we'll be set to write down the time ordered product

$$\langle 0|T(\psi'_\alpha(x)\bar\psi'_\beta(y))|0\rangle = \theta(x^0-y^0)\langle 0|\psi'_\alpha(x)\bar\psi'_\beta(y)|0\rangle$$

$$-\theta(y^0-x^0)\langle 0|\bar\psi'_\beta(y)\psi'_\alpha(x)|0\rangle$$

I'll do the calculation in a Majorana basis.

$$\langle 0|\bar\psi'_\beta(y)\psi'_\alpha(x)|0\rangle = \langle 0|\Omega\Omega^{-1}\bar\psi'_\beta(y)\Omega\Omega^{-1}\psi'_\alpha(x)\Omega\Omega^{-1}|0\rangle$$

$$= \Big(\langle 0|\underbrace{\Omega^{-1}\bar\psi'_\beta(y)\Omega}_{-i(\gamma_5\gamma^0\psi'(-y))_\beta}\;\underbrace{\Omega^{-1}\psi'_\alpha(x)\Omega}_{-i(\bar\psi(-x)\gamma^0\gamma_5)_\alpha}|0\rangle\Big)^*$$

In this step when $\langle 0|\Omega$ is simplified to $\langle 0|$ the resulting matrix element must be complex conjugated because Ω is antiunitary.

$$= \left(-i\gamma_5\gamma^0\right)_{\beta\sigma}\langle 0|\psi'_\sigma(-y)\bar\psi'_\tau(-x)|0\rangle\left(\gamma^0\gamma_5\frac{1}{i}\right)^*$$

Notice the indices β, α come out in the wrong order, so to think of $\langle 0|T(\psi(x)\bar\psi(y))|0\rangle$ as a matrix, we actually need this thing transposed.

$$\langle 0|T(\psi'(x)\bar\psi'(y))|0\rangle = \theta(x^0-y^0)\int_0^\infty da\left[\rho_+(a)(i\not\partial_x+a)\Delta(x-y)+\rho_-(a)(i\not\partial_x-a)\Delta(x-y)\right]$$
$$+\theta(y^0-x^0)\int_0^\infty da\left[\rho_+(a)\gamma_5\gamma^0(i\not\partial_x+a)\Delta(x-y)\gamma^0\gamma_5+\rho_-(a)\gamma_5\gamma^0(i\not\partial_x-a)\Delta(x-y)\cdot\gamma^0\gamma_5\right]^{*T}$$

Now $(i\gamma_5\gamma^0\gamma^\mu\gamma^0\gamma_5)^{*T} = i\gamma^\mu$

and $(\gamma_5\gamma^0 1\gamma^0\gamma_5)^{*T} = 1$ so

(in this case we can pull the time derivative through the time ordered product)

$$\langle 0|T(\psi'(x)\bar\psi'(y))|0\rangle = \int_0^\infty da\left(\rho_+(a)(i\not\partial_x+a)+\rho_-(a)(i\not\partial_x-a)\right) \text{ acting on}$$
$$\left[\theta(x^0-y^0)\Delta(x-y)+\theta(y^0-x^0)\Delta(y-x)\right]$$

Using that $\rho_+(a)$ and $\rho_-(a)$ are real and that $\Delta(x-y)^* = \Delta(y-x)$

The object in brackets is

$$\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-a^2+i\epsilon}e^{-ip\cdot(x-y)}$$

$i\not\partial_x$ hitting this gives $\not p$. So the result is

$$\langle 0|T(\psi'(x)\bar\psi'(y))|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\int_0^\infty da\left(\rho_+(a)\frac{i(\not p+a)}{p^2-a^2+i\epsilon}+\rho_-(a)\frac{i(\not p-a)}{p^2-a^2+i\epsilon}\right)$$

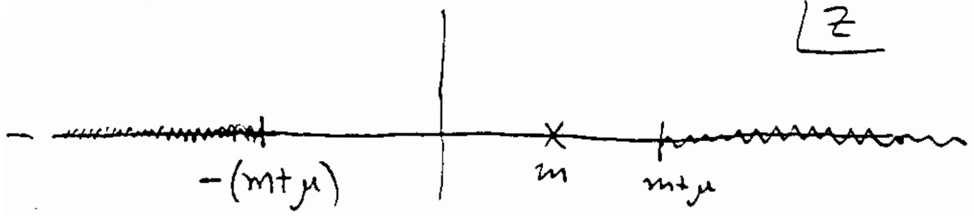
Or in a more suggestive form

$$\langle 0|T(\psi'(x)\bar{\psi}'(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)} \int_0^\infty da \left(\rho_+(a)\frac{i}{\not{p}-a+i\epsilon} + \rho_-(a)\frac{i}{\not{p}+a+i\epsilon}\right)$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, S'(\not{p})$$

where
$$S'(z) = \int_0^\infty da \left(\rho_+(a)\frac{i}{z-a+i\epsilon} + \rho_-(a)\frac{i}{z+a+i\epsilon}\right)$$

This result for $S'(z)$ has the renormalization conditions built in. They say S' has a pole at $z = m$ with residue i



Analytic structure of S' in pert. theory

Compare this with our other expression for S'.

$$S'(z) = \frac{i}{z - m - \Sigma'(z) + i\epsilon}$$

In terms of Σ', we see the renormalization conditions are

$\Sigma'(m) = 0$ pole is at m

$\left.\dfrac{d\Sigma'}{dz}\right|_{z=m} = 0$ residue is i

(often written $\left.\dfrac{d\Sigma'}{d\not{p}}\right|_{\not{p}=m} = 0$)

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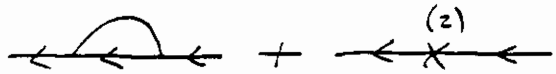
IN THE MODEL WITH   JAN. 13

C AND $\mathcal{L}' = -g\bar{\Psi}i\gamma_5\Psi\varphi + C\bar{\Psi}i\not{\partial}\Psi - D\bar{\Psi}\Psi$ 8

∞ POWER SERIES IN G

GET $\Sigma'(\not{p})$ TO ORDER g^2 $C^{(n)} \propto g^n$

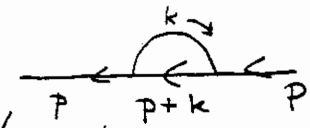


$$-i\Sigma'(\not{p}) = -i\Sigma^f(\not{p}) + iC^{(2)}\not{p} - iD^{(2)}$$

$$\Sigma'(\not{p}) = \Sigma^f(\not{p}) - \Sigma^f(m) - \frac{d\Sigma^f}{d\not{p}}\Big|_m(\not{p}-m)$$

ONLY KNOCKS OFF 1 POWER OF p

$\mathcal{L}' = -g\bar{\Psi}i\gamma_5\Psi\varphi$



THIS IS A PROPOGATOR A 4×4 MATRIX WITHOUT EXTERNAL PROPOGATORS INCLUDED

$$-i\Sigma^f = -(ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}\, i\gamma_5\, i\,\frac{\not{p}+\not{k}-m}{(p+k)^2-m^2+i\epsilon}\, i\gamma_5$$

$$= \frac{-ig^2}{(2\pi)^4}\int d^4k\,\frac{1}{k^2-\mu^2+i\epsilon}\,\frac{-\not{p}-\not{k}+m}{(p^2+2kp+k^2-m^2+i\epsilon)}$$

$$= \frac{-ig^2}{(2\pi)^4}\int d^4k\int_0^1 dx\,\frac{\not{p}-\not{k}-m}{[k^2+2kpx+p^2x-m^2x-\mu^2(1-x)+i\epsilon]^2}$$

$k+px$

$$= \frac{-ig^2}{(2\pi)^4}\int d^4k'\int_0^1 dx\,\frac{-\not{p}(1-x)+m-\not{k}'}{[k'^2+p^2x(1-x)-m^2x-\mu^2(1-x)+i\epsilon]^2}$$ — ODD

$$\Sigma' = \frac{-ig^2}{(2\pi)^4}\int d^4k'\int_0^1 dx\,\Bigg\{\frac{-\not{p}(1-x)+m}{[k'^2+p^2x(1-x)-m^2x-\mu^2(1-x)+i\epsilon]^2} - \frac{-m(1-x)+m}{[k'^2+m^2x(1-x)-m^2x-\mu^2(1-x)+i\epsilon]^2} - \frac{-m(1-x)+m}{[\quad]^2} + 2\,\frac{2m\,x(1-x)\quad m(1-x)+m}{[\quad]^3}\Bigg\}$$

DIVERGENT PART $\propto \int\frac{d^4k'}{k'^4}\left[-\not{p}(1-x)+m+m(1-x)-m-(\not{p}-m)(1-x)\right]$

$= 0$

A QUICKER WAY OF SEEING IF THE RESULT IS FINITE IS TO COMPUTE

$\frac{d^2\Sigma'}{d\not{p}^2}$. Σ' IS COMPLETELY DETERMINED BY THIS SECOND DERIVATIVE.

NOTE THAT ONE DERIVATIVE IS NOT ENOUGH TO GIVE A FINITE INTEGRAL. TWO DOES THE JOB ⇒ NEED TWO SUBTRACTIONS TO REMOVE [illegible], UNLIKE SCALAR CASE WHERE DIAGRAM WAS ONLY LOG DIVERGENT

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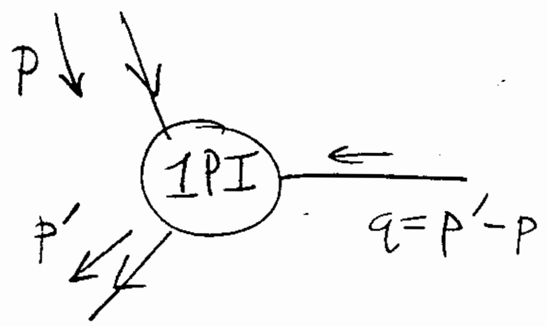
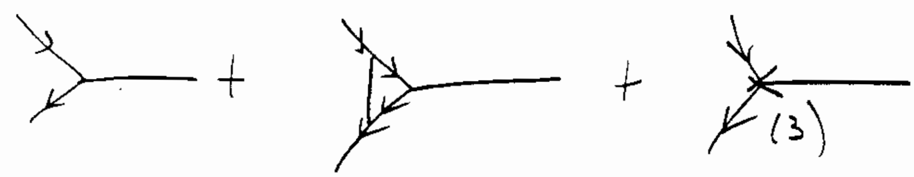
Coupling constant renormalization in spinor theory (parallels scalar case)



$$= -i\Gamma(p',p)$$

some awful 4x4 matrix

Contributions up to order g^3 are



You might give your renormalization condition as

$$\Gamma' = ig\gamma_5$$

at some astutely chosen value of p' and p. However, Γ may not be proportional to γ_5. It may have $\not{p}\gamma_5$ or $p_\mu p_\nu \sigma^{\mu\nu}\gamma_5$

(We can remedy this by cutting down the 4x4 matrix by sandwiching it between projectors.) Will show

$$(\not{p}'+m)\,\Gamma(p',p)\,(\not{p}+m)\Big|_{p^2=p'^2=m^2}$$

must be $\propto$ to $(\not{p}'+m)\gamma_5(\not{p}+m)$. $\frac{\not{p}+m}{2m}$ projects onto incoming nucleons & outgoing antinucl.

Consider this graph as contributing to

ϕ(off shell) $\longrightarrow N + \bar{N}$

and look at the process in the COM frame where

$$q = (q^0, \vec{0})$$

The initial state is $J^P = 0^-$

The two spin $\frac{1}{2}$'s in the final state can make $S=1$ or $S=0$. To get $J=0$ the only possible final states are

$l = 0 \quad S = 0$ which has $P = -1$

$l = 1 \quad S = 1$ which has $P = +1$

Only the first final state is allowed. There is only one amplitude (It may vary with q^0)

$$(\not{p}' + m)\, \gamma_5\, (\not{p} + m)$$

is nonzero when sandwiched between a $\bar{u}$ and a v (remember $p^0 < 0$ for this process with the momentum routing conditions

)

So

$$(\not{p}' + m)\, \Gamma(p, p'^2)\, (\not{p} + m)\Big|_{p^2 = p'^2 = m^2} = (\not{p}' + m)\, i\gamma_5\, (\not{p} + m)\, G(q^2)$$

We'll take

$$G(q^2 = \mu^2) \equiv g$$ as our renormalization condition.

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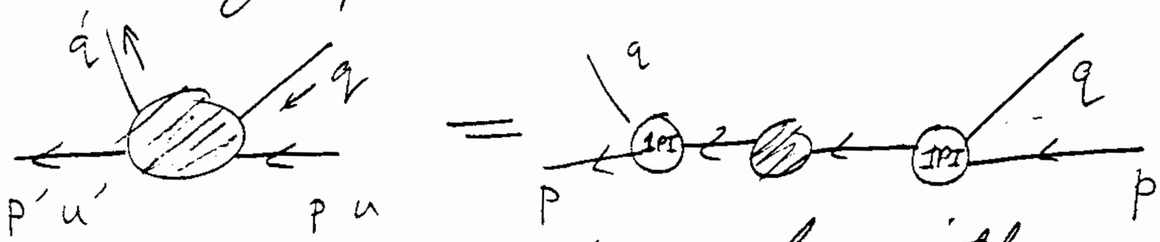
Utility of this choice



+ graphs with no pole at $(p+q)^2 = s = \mu^2$

Pole piece in $i\mathcal{A}$

$$-\bar{u}'\,\Gamma(p', p+q)\; S'(p+q)\,\Gamma(p+q, p)\,u$$

$$= -\bar{u}'\,\frac{\not{p}'+m}{2m}\,\Gamma(p', p+q)\; S'(p+q)\,\Gamma(p+q, p)\,\frac{\not{p}+m}{2m}\,u$$

insert, it is the identity on $\bar{u}'$

near $s=\mu^2$ this is
$$\frac{i(\not{p}+\not{q}+m)}{(p+q)^2-m^2+i\epsilon} + \text{analytic}$$

insert

So near $s=\mu^2$ we have Γ sandwiched between projection operators and we can use the renormalization condition to get that the pole piece in $i\mathcal{A}$ is

$$-\bar{u}'\, i\gamma_5 g\, \frac{i(\not{p}+\not{q}+m)}{(p+q)^2-m^2}\, i\gamma_5 g\; u$$

this simplification allows for unambiguous comparison with experiment to set g.

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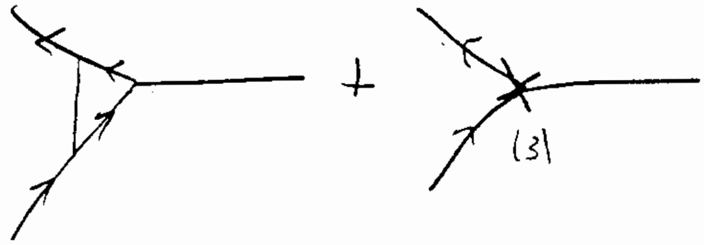
Is renormalization necessary and sufficient to get rid of ∞'s?

Let's look at the contributions to Γ at $O(g^3)$.



at high k, the integral for the first Feynman graph goes like

$$\int d^4k \; \frac{1}{k^2} \gamma_5 \frac{1}{\not k} \gamma_5 \frac{1}{\not k} \gamma_5 \sim \gamma_5 \int d^4k \frac{1}{k^4}$$

This is divergent, but only logarithmically divergent, and it multiplies γ_5. So the second graph cancels the divergent part.

So far our slovenliness has been good enough.

Regularization and renormalization

Throwing around ill-defined quantities, and discovering they always end up in convergent combinations isn't good enough.

The infinities come because the theory has an infinite # of degrees of freedom, both from the ∞ extent of spacetime (which gives IR ∞'s) and from the fact that in any given volume there is an ∞ # of degrees of freedom (which gives UV ∞'s). Next lecture we'll talk about ways to cut down the # of degrees of freedom in a given volume.

JAN. 15 1/22 (4F)

This lecture
I. Regularization
A. Regulator fields (Feynman)
B. Dimensional Regularization ('t Hooft-Veltman)
II. BPHZ renormalization

I. Regularization

No one knows of a quantum field theory that is nontrivial and finite.

In all theories worth studying, so as not to be making ad hoc cancellations of infinities with infinities you have to hack up the theory in some way to make it finite. For example, you could throw away all Fourier components in the Feynman integrals with momentum greater than some cutoff value Λ. Then you would renormalize as usual. Instead of making subtractions of ∞'s from ∞'s to satisfy the renormalization conditions, you will be subtracting finite (but big; proportional to Λ, Λ^2 or $\ln \Lambda$) things from other finite things to satisfy the renormalization conditions. Then you try to undo your hacket job by sending Λ to ∞. The big job is to prove that the properties you expect of the theory (Lorentz invariance, gauge invariance, positivity of the Hilbert space inner product) are recovered as $\Lambda \to \infty$, and that nothing depends on Λ in this limit. a scattering amplitude should not depend on the method some theorist used to make an infinity large but finite.

Regulator fields or Propogator Modification

a good regularization method should
(1) be analytically tractable
(2) ruin as few properties of the theory as possible. (The less you ruin, the less you have to laboriously prove you recover in the $\Lambda \to \infty$ limit.)

Regulator fields (or at least a variant we'll discuss March 3 called Pauli-Villars) only wreck positivity of the Hilbert space metric in QED with massive charged particles. and the only kinds of integrals that have to be evaluated are of the same type we have already studied. The idea is to replace propogators in the Feynman integrals by propogators that fall off faster at high momentum so that loop integrals will be finite. To do this we'll let

$\frac{i}{k^2-m^2}$ become a combination of propogators. For example

$$\frac{i}{k^2-m^2} \longrightarrow \frac{i}{k^2-m^2} - \frac{i}{k^2-M^2}$$

M plays the role of the cutoff. For $k^2 \gg M^2$ this combination falls off like

$\frac{1}{k^4}$ instead of $\frac{1}{k^2}$

Similarly $\frac{i}{\not{p}-m} \longrightarrow \frac{i}{\not{p}-m} - \frac{i}{\not{p}-M} \sim \frac{1}{p^2}$ at high p.

after modifying the propogators enough to make the diagrams convergent, you adjust the counterterms to satisfy the renormalization conditions, and then send $M \to \infty$.

making a propogator go like $\frac{1}{k^4}$ may not be enough to make the diagrams convergent. Here's how to make them go like $\frac{1}{k^{2n}}$ for n as big as you need.

Let

$$\frac{i}{k^2-m^2} \longrightarrow \frac{i}{k^2-m^2} + \sum_{r=1}^{n} \frac{i C_r^2}{k^2-M_r^2}$$

(I write the coefficient as c_r^2, but don't let me mislead you into thinking $c_r^2 > 0$.)

We can look at the behavior of this for high k^2 by expanding

$$\frac{1}{k^2-m^2} = \frac{1}{k^2}\left(\frac{1}{1-\frac{m^2}{k^2}}\right)$$

$$= \frac{1}{k^2}\left(1+\frac{m^2}{k^2}+\left(\frac{m^2}{k^2}\right)^2+\dots\right)$$

By choosing

$1+\sum_{r=1}^{n} C_r^2 = 0$ makes propogator $\sim \frac{1}{k^4}$

and $m^2+\sum_{r=1}^{n} M_r^2 C_r^2 = 0$ " " $\sim \frac{1}{k^6}$

and $m^4+\sum_{r=1}^{n} M_r^4 C_r^2 = 0$ " " $\sim \frac{1}{k^8}$

and $m^6+\sum_{r=1}^{n} M_r^6 C_r^2 = 0$ " " $\sim \frac{1}{k^{10}}$

etc.

$n = 1,\ \bar{c}_1 = i\ \ M_1 = M$ creates the simplest example on page 2. JAN. 15 4

By making n large enough, you can clearly make the propogator fall off as fast as you like, and still have freedom to send all the $M_r \to \infty$. (The c_r must remain finite.) Of course some of the c_r^2 are going to have to be less than zero, or you are just going to have i times a sum of things with the same sign at large k^2. There is no way this can happen in any realistic theory of the world. $c_r^2 > 0$ is a consequence of the Lehman-Kallen spectral representation.

We can construct an operator theory that is unrealistic that has these sicko propogators though.

Suppose the original theory had

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 + \mathcal{L}'(\phi)$$

The unrealistic theory that has these propogators is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2$$
$$+ \sum_{r=1}^{n} \frac{1}{2}(\partial_\mu \phi_r)^2 - \frac{M_r^2}{2}\phi_r^2$$
$$+ \mathcal{L}'(\Phi) \quad \text{where} \quad \Phi = \sum_{r=1}^{n} c_r \phi_r$$

I talk about why this gives the right propogator combination on March 3.

About this point you maybe wondering why we are trying to construct a Lagrangian that reproduces our hatchet job. Answer: It helps you ascertain what properties of the theory you have or haven't ruined.

Because some of the c_r^2 are less than zero, some of the c_r are imaginary, and the Hamiltonian is not Hermitian.

We can gain some insight into what is going on by defining a new inner product.

In the theory that embodies the simplest propogator modification,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{M^2}{2}\phi^2 + \tfrac{1}{2}(\partial_\mu\phi_1)^2 - \tfrac{M^2}{2}\phi_1^2$$
$$+ \mathcal{L}'(\Phi) \qquad \Phi = \phi + i\phi_1 \quad ,$$

define a new inner product

$$\langle a|b\rangle_{new} = \langle a|\,(-1)^{N_1}\,|b\rangle$$

N_1 counts the number of mesons of the sicko type. This metric is not positive definite.

$$\langle a|a\rangle_{new} < 0 \quad \text{if } |a\rangle \text{ has an odd number of } \phi_1 \text{ mesons in it.}$$

The great thing about this metric is that in it Φ is hermitian.

$$(\phi_1)^\dagger_{new} = -\phi_1$$

because ϕ_1 anticommutes with $(-1)^{N_1}$!

So $(\Phi)^\dagger_{new} = (\phi + i\phi_1)^\dagger_{new} = \Phi$

To summarize. In the old metric, which was positive definite, the Hamiltonian wasn't hermitian and thus didn't conserve probability. In the new metric, we have a new definition of probability, and although it is not always greater than zero, the Hamiltonian is hermitian, and the new probability is conserved.

Here is why you might hope that a sensible theory will be recovered ~~will~~ when the $M \to \infty$ limit is taken. We won't be interested in amplitudes that have those phony particles in the initial states, and when $M \to \infty$, it will be impossible to produce them in the final state, just for lack of energy.

The only initial and final states possible will thus be the ones with sensible particles in them, and for them, the inner product is normal.

The good things about regulator fields are that they preserve Lorentz invariance, internal symmetries in theories with massive particles (they spoil symmetries that depend on masslessness), conserve probability at energies low compared to the cutoff, with some modification, will be seen to preserve gauge invariance in QED, and they are computationally easy to introduce.

$\boxed{\text{A note on computation}}$ In practice you don't try to combine the various propogators to make the integrals manifestly convergent. You just work along with each propogator separately, and use the integral tables that are valid when you work with a convergent combination.

Dimensional Regularization

Begin with an example. Let's evaluate

$$I = \int \frac{d^d k}{(k^2+a^2)^n}$$ which is convergent if $n > \frac{d}{2}$

in an arbitrary number of Euclidean space dimensions d.

Here is a trick to turn a denominator into an exponential. Start with

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt$$

(which is sometimes taken as the defn of the Γ fn). Change variables in the integrand to λ given by $\alpha\lambda = t$; α real, >0.

$$\Gamma(n) = \int_0^\infty (\alpha\lambda)^{n-1} e^{-\alpha\lambda}\, d(\alpha\lambda)$$

or $$\frac{1}{\alpha^n} = \frac{1}{\Gamma(n)} \int_0^\infty \lambda^{n-1} e^{-\alpha\lambda}\, d\lambda$$

Our (Euclidean space) integral becomes

$$I = \frac{1}{\Gamma(n)} \int_0^\infty \lambda^{n-1}\, d\lambda \underbrace{\int d^d k\, e^{-\lambda(k^2+a^2)}}_{e^{-\lambda a^2}\left(\frac{\pi}{\lambda}\right)^{d/2}}$$

$$= \frac{\pi^{d/2}}{\Gamma(n)} \int_0^\infty \lambda^{n-\frac{d}{2}-1} e^{-\lambda a^2}\, d\lambda = \frac{\pi^{d/2}}{\Gamma(n)} \frac{\Gamma(n-\frac{d}{2})}{a^{2n-d}}$$

Here is 't Hooft and Veltman's whammy: adopt this formula for arbitrary complex d. If you stay away from even integers $d \geq 2n$, this expression is well defined. As you head towards $d=4$ ~~the~~ you approach poles in the Γ function. You do your renormalization in arbitrary d, and only after you have your expressions for the graphs plus counterterms in convergent combinations (that is with poles in $d-4$ cancelling) do you send $d \to 4$.

You have to be careful formulating a theory in an arbitrary ~~#~~ of dimensions.

You can't just maintain $\frac{e^2}{4\pi} = \frac{1}{137}$ in an arbitrary number of dimensions because only in four dimensions is e dimensionless.

There are simpler examples than QED to demonstrate the effect of this. Take

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi')^2 - \frac{m^2}{2}\phi'^2 - \lambda\phi'^4 + \mathcal{L}_{c.t.}$$

$[\phi'] = \frac{d-2}{2}$, so m is a mass as it appears.

But to keep the Lagrangian having dimension d, we must have

$$d = [\lambda] + 4[\phi'] \qquad [\lambda] = d - 2(d-2) = -d+4$$

Only in four dimensions is λ dimensionless. It cannot remain constant as we change d. It has to acquire dimension. So we rewrite the interaction:

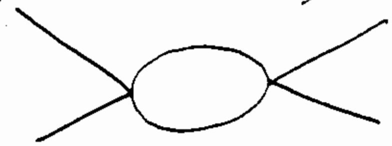
Now dimensionless for any d

$$\lambda \mu^{4-d} \frac{\phi^4}{4!}$$

where μ is a parameter that has appeared uninvited into the theory.

You might think that after renormalizing, when we set $d=4$, all μ dependence will go away.

We'll look at a contribution to the four point function. An $O(\lambda^2)$ diagram is



It leads to an integral like

$$(\lambda \mu^{4-d})^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+a^2)^2}$$

where a contains masses, external momenta and Feynman parameters, and I have supressed the Feynman parameter integral and a lot of factors.

Our result for this integral is

$$(\lambda \mu^{4-d})^2 \frac{\pi^{d/2}}{\Gamma(2)} \Gamma\left(2-\frac{d}{2}\right) a^{d-4}$$

Γ has a pole piece near $d=4$. For integer n,

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \overbrace{\psi(n+1)}^{\text{Some number like } \pi \text{ except more complicated}} + O(\epsilon)\right]$$

See Ramond p. 152

you can't just set $d=4$ in the rest of the expression. That would give you the right coefficient of the pole, but the wrong finite part.

Let's see what the finite part is. In our example $n=0$ and $\epsilon = 2-\frac{d}{2}$

We'll pull out the a factor of μ^{4-d} since that is the dimension of the Green's function (the lowest order contribution is proportional to $\lambda\mu^{4-d}$). We expand the dimensionless thing that is left.

$$\lambda^2 \mu^{4-d} \frac{\pi^{d/2}}{1!} \Gamma(2-\tfrac{d}{2}) a^{d-4}$$

$$= \lambda^2 \pi^2 \Gamma(2-\tfrac{d}{2}) \left(\frac{\mu}{a}\right)^{4-d} \pi^{\frac{d}{2}-2}$$

$$= \lambda^2 \pi^2 \Gamma(2-\tfrac{d}{2}) \left(\frac{\mu^2}{\pi a^2}\right)^{\frac{4-d}{2}}$$

$$= \lambda^2 \pi^2 \left[\frac{(-1)^0}{0!}\left(\frac{1}{2-\frac{d}{2}} + \psi(1) + O(d-4)\right)\right] \underbrace{e^{\frac{4-d}{2}\ln\frac{\mu^2}{\pi a^2}}}_{1+\frac{4-d}{2}\ln\frac{\mu^2}{\pi a^2}}$$

$$= \lambda^2 \pi^2 \left[\frac{1}{2-\frac{d}{2}} + \psi(1) + \ln\frac{\mu^2}{\pi a^2} + O(d-4)\right]$$

you would have lost the $\ln\frac{\mu^2}{\pi a^2}$ piece if you prematurely set $d=4$.

Now you can renormalize as usual, although you need an extension of the renormalization conditions for arbitrary dimension.

MINIMAL SUBTRACTION (or MS)

MS is another renormalization prescription, that is, a way of determining counterterms. It makes no reference to the physical mass and coupling so it is not good for comparison with experiment. It is a companion to dimensional regularization. Theorists like it because they no longer make comparison with experiment and the minimal subtraction renormalization prescription is easy. It amounts to just chucking the pole terms in the dimensionally regularized integrals. I'll do it in our example.

Again suppressing the Feynman parameter integral and whatever else, we have found

$$>\!\!\bigcirc\!\!< \; = \mu^{4-d} \lambda^2 \pi^2 \left[\frac{1}{2-\frac{d}{2}} + \text{finite as } d \to 4 \right]$$

The coefficient of the pole is unambiguous. Minimal subtraction says introduce a counterterm to exactly cancel it.

In this example we need a term in $\mathcal{L}_{c.t.}$ of

$$\mu^{4-d} \lambda^2 \pi^2 \frac{1}{2-\frac{d}{2}} \frac{\phi^4}{4!} \quad \text{(up to } i\text{'s and minus signs)}$$

In what follows, another renormalization prescription is heavily used. It also makes no reference to physical masses and couplings either. It's called BPH. It is useful for proving that renormalization removes the ∞'s.

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Renormalization and symmetry: a review for non-specialists
(1971)

1 Introduction

This lecture does not talk too much about regularization which is the first step in the regularization and renormalization process. This is because BPHZ tells you how to make sense of integrals and what c.t.'s to add in one set of instructions for defining integrands.

I suppose that as good a way as any of explaining the contents of this lecture is to explain the title. By 'renormalization' I mean the removal of infinities for Feynman amplitudes, in perturbation theory, for Lagrangian field theories with polynomial interactions. In particular non-perturbative renormalization (the work of Jaffe, Glimm, etc.) is outside the scope of this lecture, as are the properties of non-polynomial interactions (the work of Efimov, Salam, Lehmann, etc.). By 'renormalization and symmetry' I mean that we will be concerned not only with the renormalization of scattering amplitudes, but also with the renormalization of the matrix elements of conserved and partially conserved currents. In particular, we will discuss some fairly recent results of Symanzik, Benjamin Lee, Preparata, Weisberger, and others. By 'a review for non-specialists' I mean that I hope that this talk will be intelligible to people who can do nothing more complicated than remove the divergences from the self-energy of the electron.

Since renormalization theory has a well-deserved reputation for complexity, it is obvious that I will be able to do all this in a single lecture only by cheating. To be precise, I will explain a very powerful theorem due to Klaus Hepp, but not prove it (this is the cheat); then I will show how a wide variety of results can be obtained from this master theorem by elementary methods.[1]

2 Bogoliubov's method and Hepp's theorem

For simplicity, we will restrict ourselves to field theories involving spin-zero and spin-one-half fields only, which we will call Bose and Fermi fields, respectively. We will write the Lagrangian for such a theory in the form

$$\mathscr{L} = \mathscr{L}_0 + \sum_i \mathscr{L}_i, \qquad (1)$$

where $\mathscr{L}_0$ is a sum of free Lagrangians of standard form, one for each field, and each $\mathscr{L}_i$ is a monomial in the fields and their derivatives. For future use, it will be convenient to establish some notation, and denote by f_i the number of Fermi fields in $\mathscr{L}_i$, by b_i the number of Bose fields, and by d_i the number of derivatives. Thus, for example, the ps-ps meson–nucleon interaction

$$g\bar{\psi}\gamma_5\psi\phi,$$

has $f=2$, $b=1$, and $d=0$, while the ps-pv interaction

$$f\bar{\psi}\gamma_\mu\gamma_5\psi\partial^\mu\phi,$$

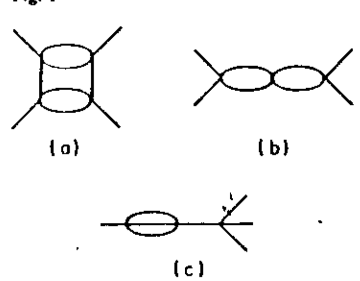
has $f=2$, $b=1$, and $d=1$.

If we attempt to calculate scattering amplitudes with such a Lagrangian, following the conventional Feynman rules, we soon encounter divergent diagrams, that is to say, infinite Feynman integrals. I will assume that we have cut off the theory in some way (say, by modifying the propagators) so that instead of divergent amplitudes we have cutoff-dependent ones. The renormalization procedure of Bogoliubov[2] consists of adding to the Lagrangian extra terms, the so-called renormalization counter-terms, whose function is to cancel the cutoff-dependence of the amplitude. First I will explain how these extra terms are constructed; later I will explain their physical meaning.

To explain the construction, three definitions are needed:

(1) ***One-particle-irreducible diagrams.*** A Feynman diagram is said to be one-particle-irreducible (abbreviated 1PI) if it is connected and cannot be disconnected by cutting any one internal line. Fig. 1 shows three Feynman diagrams in ϕ^4 theory. The first two are 1PI; the third is not. (If the horizontal line is cut, the diagram falls into two pieces.)

Fig. 1



General amplitude at point zero

$$\int \frac{d^4k_1 \cdots d^4k_L}{(q_1^2-m_1^2+i\varepsilon)(q_2^2-m_2^2+i\varepsilon)\cdots} N(q_1 \cdots q_I)$$

# of loop

# of int line

$$q_E = (iq_0, \vec{q})$$

$$\sim \int \frac{d^4k_{1E} \cdots d^4k_{LE}}{(q_{1E}^2+m_1^2)\cdots(q_{IE}^2+m_I^2)} \cdots$$

(2) ***Taylor expansions about the point zero.*** A Feynman amplitude with n external lines is a function of $n-1$ independent four-momenta. Furthermore, if there are no massless particles in the theory (as we shall assume from now on) it is an analytic function of these momenta in some neighbourhood of the point zero, the point where *all* external momenta vanish. Thus, it may be expanded in a Taylor series in these variables. For example, the third-order vertex diagram of ps-ps meson–nucleon theory, shown in Fig. 2, has an expansion of the form Photon self energy

$$a\gamma_5$$
$$+ b\gamma_5\gamma_\mu p^\mu + c\gamma_5\gamma_\mu p'^\mu$$
$$+ d\gamma_5 p^2 + e\gamma_5 p'^2 + f\gamma_5 p\cdot p'$$
$$+ \cdots,$$

$a g_{\mu\nu}$ $\quad D=0$

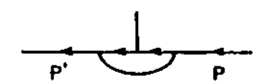
$c g_{\mu\nu} p^2 + d p_\mu p_\nu \quad D=2$

$e g_{\mu\nu} p^4 + f p^2 p_\mu p_\nu \quad D=4$

where a, b, c, etc. are constants. The term on the first line is called a term of zeroth order, those on the second line terms of first order, those on the third line terms of second order, etc.

order 0

Fig. 2



(3) ***Superficial degree of divergence.*** A Feynman amplitude is, in general, a multiple integral. The superficial degree of divergence of such an integral is the difference between the number of momenta in the numerator of the integral (arising from loop integration variables and from explicit momenta at vertices due to derivative interactions) and the number of momenta in the denominator (arising from propagators). Fig. 3 shows three Feynman diagrams from ϕ^4 theory, with their superficial degrees of divergences (denoted by D). The contribution from numerator and denominator are separately displayed. If $D=0$, we say the diagram is superficially logarithmically divergent, if $D=1$, that it is superficially linearly divergent, etc. If D is less than zero, we say it is superficially convergent.

Fig. 3(c) demonstrates the reason for the pejorative adjective 'superficial'. Although the diagram is superficially convergent, it is in fact divergent; the integration along the lower loop is logarithmically divergent no matter what happens in the rest of the diagram.

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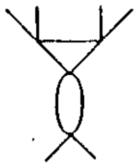
Fig. 3

(a)  $D = 4 - 4 = 0$ Superficially logarithmic divergent

(b)  $D = 8 - 6 = 2$ superficially quadratically divergent

(c)  $D = 8 - 10 = -2$ superficially convergent

It will be convenient later to have a general expression for the superficial degree of divergence of a connected Feynman diagram. For such a diagram let

- B be the number of external boson lines,
- IB be the number of internal boson lines,
- F be the number of external fermion lines,
- IF be the number of internal fermion lines, and
- n_i be the number of vertices of the ith type, i.e. those that come from the ith term in the Lagrangian (1).

There is an elementary relation between these numbers. Since a vertex of the ith type has b_i boson line ends attached to it, and since every internal boson line has two ends attached to vertices and every external boson line has one, we can readily deduce that

$$B + 2(IB) = \sum n_i b_i,$$

'the law of conservation of boson ends'. By the same reasoning, we can deduce 'the law of conservation of fermion ends',

$$F + 2(IF) = \sum n_i f_i.$$

It is also elementary to compute the superficial degree of divergence:

$$D = \sum n_i d_i + 2(IB) + 3(IF) - 4 \sum n_i + 4.$$

The five terms in this formula have the following origins. (1) Every derivative in an interaction puts a momentum in the numerator of the Feynman

Because there are pieces in the meson propagator you cannot ignore if the current is not conserved that do not fall like $\frac{1}{k^2}$ for large k^2. In particular the piece is $\frac{k_\mu k_\nu}{\mu^2} \cdot \frac{1}{k^2 - \mu^2}$ ∞

integral. (2) Every internal boson line puts four integration momenta in the numerator and two propagator momenta in the denominator. (3) Every internal fermion line puts four integration momenta in the numerator and one in the denominator. (4) Every vertex has a four-dimensional delta-function attached to it, which, upon integration, cancels four integration momenta, (5) except for one delta-function that is left over to give overall four-momentum conservation.

Putting all of this together, we find that

$$D = -B - \tfrac{3}{2}F + 4 + \sum n_i \delta_i, \tag{2}$$

where δ_i, 'the index of divergence of $\mathscr{L}_i$', is given by

$$\delta_i = b_i + \tfrac{3}{2} f_i + d_i - 4. \tag{3}$$

It is worth remarking that, for the cases we are considering,

$$\delta_i = \dim \mathscr{L}_i - 4, \tag{4}$$

where dim $\mathscr{L}_i$ is the dimension of $\mathscr{L}_i$, in the usual sense of dimensional analysis, in units of mass. (This is, however, special to the theories we are considering; eq. (4) is not true, for example, for the interactions of a massive vector meson coupled to a non-conserved current.)

This completes our three definitions (plus one long digression). We are now in a position to state the renormalization prescription of Bogoliubov.[2] As advertised, this is an iterative procedure; as we calculate in perturbation theory, to each order we change the Lagrangian, adding to it extra terms. The procedure is as follows:

(1) Calculate in perturbation theory until you encounter an 1PI diagram whose superficial degree of divergence, D, is greater than or equal to zero.

(2) Add to the Lagrangian extra terms (the counterterms) chosen to precisely cancel, to this order, all terms in the Taylor expansion of this diagram of order D or less.* order of a term O, defined on p. 101 $O = 0, \ldots, D$ $\delta_O = D - D$

As an example of this procedure, let us consider $\lambda\phi^4$ theory, for which the Lagrangian (1) is

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{5}$$

In order λ^2, we encounter the divergent diagrams 3(a) and 3(b). For the first of these, $D = 0$, for the second $D = 2$. Thus we change the Lagrangian

* Please note that it follows from this and Eqs. (2) and (3) that the counterterms induced have index of divergence, δ, less than or equal to the sum of the indices of divergence of the interactions occurring in the diagram. This observation has been stuck in a footnote because it is not important now, but it will be useful later.

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PROOF OF FOOTNOTE (W.O. FERMIONS)

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B
IB
$n_i \quad b_i \quad d_i$

$$D = \sum n_i d_i + 2IB - 4\sum n_i + 4$$

$$B + 2IB = \sum_i n_i b_i$$

$$D = \sum n_i (d_i + b_i - 4) + 4 - B$$

$$\delta_i = d_i + b_i - 4$$

Given a diagram with $D>0$ we are instructed to form a Taylor series expansion, and add terms to the Lagrangian to identically cancel the terms of the power series up to order D. The counterterm of order $\mathcal{O}$ has the same number of external boson lines and it has $d_{\mathcal{O}} = \mathcal{O}$.

Thus

$$\delta_{\mathcal{O}} = B + \mathcal{O} - 4 \leq B + D - 4$$

$$= \sum_i n_i \delta_i$$

$\phi\ \phi^2\ \phi^3\ \phi^4$

$\partial_\mu\phi\,\partial^\mu\phi\ ,\ \bar\psi\psi$

$\bar\psi\gamma_5\psi\ ,\ \bar\psi\gamma^\mu\partial_\mu\psi$

$\bar\psi\gamma_5\gamma_\nu\partial^\mu\psi$

$\bar\psi\psi\phi\ ,\ \bar\psi\gamma_5\psi\phi$

(5) by adding to it extra terms

$$\mathcal{L}\to\mathcal{L}-\frac{A_2}{4!}\phi^4+\frac{1}{2}B_2\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}C_2\phi^2. \qquad (6)$$

(The subscript 2 is to remind you that these terms are of second order in λ; they are also cutoff-dependent, but that is not important at the moment). The A_2 term is chosen to cancel the zeroth-order term in the Taylor expansion of 3(a); the B_2 and C_2 terms to cancel the zeroth and second order terms in the Taylor expansion of 3(b). (There is no need for a first-order counterterm because Lorentz-invariance forbids a first-order term in the Taylor expansion.)

(3) Continue computing, now using the corrected Lagrangian.

Theorem (Hepp).[3] This procedure eliminates all divergences. That is to say, the resultant perturbation expansion is independent of the cutoff in the limit of infinite cutoff.[4]

For the moment, I would like you to think of this purely as a mathematical theorem about Feynman expansions; we will try to understand its physical significance shortly. However, there is one point I would like to make now – we can already begin to see why it is the superficial degree of divergence, rather than the true degree of divergence, that is important. Remember the order λ^4 diagram 3(c), which has $D=-2$. By our prescription, even though this diagram is in fact divergent, it does not induce a counterterm. We can now see the reason for this: there is another diagram of order λ^4, shown in Fig. 4, where the heavy dot is the A_2 term in eq. (6), the counterterm that was added to the Lagrangian in order λ^2. This diagram *automatically* cancels the divergence of Fig. 3(c). Speaking very roughly, we only need new counterterms at a given order of perturbation theory to take care of new divergences; old divergences, divergences caused by lower-order diagrams hiding inside higher-order ones, as 3(a) is hiding inside 3(c), are taken care of by old counterterms.

Fig. 4

This renormalization removes all cut off dependence as the cutoff $\to\infty$, from all green's functions to all orders in pt.

3 Renormalizable and non-renormalizable interactions

Let us look a little more closely at the theory defined by eq. (5). We have already classified the counterterms that arise in order λ^2; what

happens in an arbitrary order of perturbation theory? Equations (2) and (3) give us the answer; for the special case of a ϕ^4 interaction, eq. (3) becomes

$$\delta=4+0+0-4=0,$$

and eq. (4) becomes

$$D=4-B.$$

Thus the only superficially divergent diagrams are those with B equal to two or four (diagrams with odd numbers of external lines vanish because of the symmetry of the Lagrangian under $\phi\to-\phi$), and their superficial degrees of divergence are the same in a general order as in second order. The only effect of renormalization, to any order, is to change (5) into

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)^2-\frac{1}{2}\mu^2\phi^2-\frac{\lambda}{4!}\phi^4-\frac{A}{4!}\phi^4+\frac{B}{2}(\partial_\mu\phi)^2-\frac{1}{2}C\phi^2, \qquad (7)$$

where the constants A, B, and C are power series in λ, with coefficients that are, in general, cutoff-dependent. We can now see the physical meaning of the renormalization procedure; for if we define

$$\phi_u=(1+B)^{\frac{1}{2}}\phi, \qquad (8a)$$

$$\mu_0^2=(\mu^2+C)(1+B)^{-1}, \qquad (8b)$$

$$\lambda_0=(\lambda+A)(1+B)^{-2}, \qquad (8c)$$

then we may rewrite the Lagrangian (7) as

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi_u)(\partial^\mu\phi_u)-\frac{1}{2}\mu_0^2\phi_u^2-\frac{\lambda_0}{4!}\phi_u^4.$$

This is of the same form as our starting Lagrangian (5), except that the coefficients have been changed. The field ϕ_u is called the unrenormalized field; it obeys canonical commutation relations, but has cutoff-dependent matrix elements (because of the cutoff-dependent quantity B in eq. (8a)). The quantities μ_0 and λ_0 are called the bare mass and bare coupling constants. Thus, for this theory the content of Hepp's theorem is that if we choose the bare mass and coupling constants in an appropriate cutoff-dependent fashion, and rescale the fields in an appropriate cutoff-dependent way, all the divergences disappear, order by order, in perturbation theory. A Lagrangian that has this property is said to be renormalizable.

The Green's functions of ϕ have been arranged by the procedure to be finite in the limite of large Λ, that is, independent of Λ.

The field ϕ and the quantities μ and λ are not the renormalized field, mass, and coupling constants as usually defined; this is because they are defined in terms of Green's functions at the point zero, rather than at some astutely chosen mass-shell point. However, they are cutoff-independent parameters that characterize the theory; the usual parameters can be computed in terms of them to any order of perturbation theory, and, if

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one wishes, these expressions can be inverted in the standard way to obtain a perturbation theory in terms of the usual parameters. For our purposes they are more convenient than the usual parameters because it is easier to do power series expansions about the point zero than about mass-shell points, where one has to worry about possible singularities of Feynman integrals. To distinguish them from the usual parameters we will refer to μ and λ as the intermediate mass and coupling constants; likewise, we will refer to the Green's functions associated with the field ϕ as intermediate Green's functions.

Not all Lagrangians are renormalizable. For example, if we had added a ϕ^5 interaction to our starting Lagrangian, this would have δ equal to one, and the renormalization procedure would inexorably add to the Lagrangian, as we computed higher and higher orders of perturbation theory, higher and higher order monomials in the field and its derivatives. Once such expressions appear in the counterterms, there is no physical reason to exclude them from the starting Lagrangian. (After all, it was only mathematical convenience that made us choose the point zero for our renormalization prescription; if we had chosen a different point (or even a separate point for every Green's function, or different points in different orders of perturbation theory) we would have obtained different values for the coefficients of the counterterms.) Thus we would be led to a theory with an infinite number of parameters. Such theories are called non-renormalizable.

4 Symmetry and symmetry-breaking: Symanzik's rule.

We are now ready to begin pulling interesting results out of Hepp's theorem. I would like to begin by taking the observation made in a footnote to Sect. 2 and raising it to the dignified status of a

footnote on p. 103

Lemma. The counterterms induced by a given Feynman diagram have index of divergence, δ, less than or equal to the sum of the indices of divergence of all the interactions in the diagram.

I would also like to adopt a somewhat more stringent definition of renormalizability than usual: I will call a Lagrangian renormalizable only if all the counterterms induced by the renormalization procedure can be absorbed into a redefinition of the parameters in the Lagrangian. Thus, by this strict test, the theory of a single nucleon field interacting with a single pseudoscalar meson field through Yukawa coupling, $\bar\psi\gamma_5\psi\phi$, is *not* renormalizable, because renormalization induces a ϕ^4 counterterm, not present in the original Lagrangian. However, the same theory, *with* a ϕ^4 interaction in the original Lagrangian, *is* renormalizable, because now all the counterterms are of the same form as terms originally present.

With this definition, we can now state our:

Exercise, list all interactions of renormalizable type four dimensions with scalar spinor and vector fields.

First result. Given a set of spin-zero and spin-one-half fields, the most general Lagrangian constructed from this set containing all terms with δ less than or equal to zero (equivalently, with dimension less than or equal to four) is renormalizable.

This is a trivial consequence of the Lemma.

Second result. If we restrict the Lagrangians defined in the first result to only contain parity-conserving terms, they are still renormalizable. Likewise, if we restrict them to preserve some internal symmetry, such as isotopic spin, they are still renormalizable.

This is also trivial. Unless we have been so stupid as to introduce parity violation into our cutoff procedure, Feynman diagrams computed from a parity-conserving Lagrangian will be parity-conserving. Thus, they will have no parity-violating terms in their Taylor expansions about the point zero, and hence no parity-violating counterterms will be induced by the renormalization procedure. Ditto for internal symmetries. (In fact, ditto for chiral symmetries, such as those of the σ-model, although here one must be more clever than usual to construct a cutoff procedure that does not break the symmetry.)[5]

?

Third result. (Symanzik's rule for symmetry-breaking):[6] If we generalize the preceding set of Lagrangians to include symmetry-breaking terms, but only with dimensions less than or equal to n, where n is either 3, 2, or 1, they are still renormalizable.

Although this is our first 'new' (1970) result, it is also trivial.[7] The symmetric terms in the Lagrangian have $\delta \leq 0$; the symmetry-breaking terms have $\delta \leq n-4<0$. A symmetry-breaking counterterm can arise only from a diagram that involves at least one symmetry-breaking interaction. By the Lemma, this must also have $\delta \leq n-4$.

Thus, for example, if, in the standard isospin-symmetric theory of pions and nucleons, we choose to break isospin only by giving the charged and neutral pions different masses, then renormalization will not force us to change our intention and also introduce symmetry-breaking Yukawa couplings. Remember, though, that we are speaking here of the intermediate coupling constants. The physical renormalized coupling constants *do* display the effects of symmetry-breaking; the new terms we have added to the Lagrangian *do* affect the three-particle Green's functions. It is just that these effects are not divergent, and hence do not require counterterms. If we look at the equations that define the bare masses and coupling constants, discussed in the preceding section, we see that another way of

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stating this result is to say that the constraint that the internal symmetry be broken only by the bare masses, while the bare coupling constants remain symmetric, does not introduce any divergences. (Unfortunately it is the opposite case – equal bare masses but asymmetric coupling (to electromagnetism) – that is of greatest physical interest, for this is the problem of the electromagnetic mass differences within isotopic multiplets. Alas, we have to go, one way or another, beyond conventional renormalized field theory, to solve this problem.)

If you are trying to explain mass differences by interaction differences

The most important special case of Symanzik's rule is the renormalization of the outstanding example of a Lagrangian field theory obeying PCAC, the σ model. This can be characterized as the theory of the interactions of pions, sigmas (scalar isoscalar mesons) and nucleons, such that the chiral symmetry group $SU_2 \times SU_2$ is broken only by terms of dimension one (i.e., linear in the σ field). Symanzik's rule then immediately says that this model is renormalizable.[5]

5 Symmetry and symmetry-breaking: currents

Field theories with internal symmetries have the famous feature of possessing conserved currents, and frequently the matrix elements of these currents are objects of great physical interest (e.g. electromagnetic form factors). These currents are typically bilinear forms in *unrenormalized* fields and their derivatives. Thus, one would naively expect them to be doubly divergent – divergent because the unrenormalized fields are themselves divergent, and divergent also because we are bringing two fields together at the same space-time point. Thus the following result is as surprising as it is beautiful:

Fourth result. In a renormalizable field theory with internal symmetry, the matrix elements of the conserved currents associated with the symmetry are cutoff-independent in every order of perturbation theory.[8]

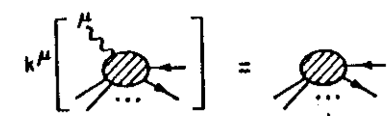
To prove this result we shall need two pieces of information. Firstly, we need to know how to compute the Green's functions for one current and a string of fields in a Lagrangian way, so we can apply Hepp's theorem, which is about Lagrangians. Fortunately, there is a standard trick for doing this: let j_μ be the current, and let $\mathcal{J}_\mu(x)$ be an arbitrary c-number function of space and time. Change the Lagrangian of the theory by adding to it an extra term:

$$\mathcal{L} \to \mathcal{L} + \mathcal{J}_\mu(x) j^\mu. \tag{9}$$

Compute all Green's functions to first order in the added term, and then functionally differentiate with respect to $\mathcal{J}_\mu$. The result is i times the Green's function with a current inserted.

Secondly, we need the Ward identities for conserved currents. I have discussed these in some detail in my other lectures at this school. Here we need the Ward identities only in the somewhat sketchy form depicted in Fig. 5. The blob on the left is a Green's function for one current and a string of Bose and Fermi fields, represented by the solid lines, without and with arrows. The current is represented by a wiggly line; it carries momentum k, and vector index μ. The right-hand side of the Ward identity is some linear combination of Green's functions without a current, represented by the blob on the right.

Fig. 5



A crucial property of this equation is that it involves the same number of fields on the right as on the left, and is therefore true for any normalization of the fields that respects the internal symmetry. In particular, it is true for the intermediate fields we have been using, and therefore, for this choice of fields, the right-hand side of the Ward identity is independent of the cutoff in the limit of large cutoff. This is the essential fact we will use in the sequel.

Now let us begin counting divergences. Let us, in the manner of eq. (9), add to the Lagrangian an extra term

$$\mathcal{L} \to \mathcal{L} + \mathcal{J}_\mu^{(x)}\Big(\sum_{i,j} \alpha_{ij}\bar\psi_i\gamma^\mu\psi_j + \sum_{i,j}\beta_{ij}\bar\psi_i\gamma^\mu\gamma^5\psi_j + \sum_{i,j}\gamma_{ij}\phi_i\partial^\mu\phi_j + \sum \varepsilon_i\partial^\mu\phi_i\Big), \tag{10}$$

where the αs, βs, γs, and εs are numerical coefficients, and the sums run over all the Fermi (or Bose) fields in the theory.[9] The interactions that give us the Green's functions for the conserved currents are certainly of this form, with special choices for the numerical coefficients. However, for the moment, let us consider a general interaction of the form (10), without asking whether or not it is associated with a conserved current. Now let us follow the renormalization procedure for this new interaction (but only going to first order in $\mathcal{J}_\mu$). Since (10) is the most general Lorentz-covariant interaction linear in $\mathcal{J}_\mu$ and of dimension three or less, the counterterms induced will also be of the form (10). That is to say, starting with any interaction of the form (10), we can obtain a cutoff-independent interaction, (i.e. one that leads to cutoff-independent Green's functions; the actual

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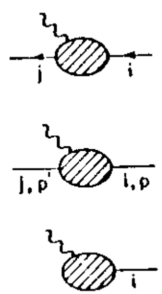
numerical coefficients in the interaction will be, of course, cutoff-dependent) order by order in renormalized perturbation theory, by appropriately adjusting the numerical coefficients.

We can choose a certain subset of these interactions – say, those that are generated by starting with interactions (10) for which all but one of the numerical coefficients vanishes – as a linearly independent set. Then any interaction of the form (10) is a linear combination of these with some coefficients. The value of these coefficients is completely determined by certain terms in the Taylor expansions of certain Green's functions about the point zero. The relevant Green's functions, and their expansions, are shown in Fig. 6, where the latin letters label the fields. The one-to-one correspondence between these coefficients and the terms in (10) is evident.

Fig. 6



$$= a_{ij}\gamma_\mu + b_{ij}\gamma_\mu\gamma_5 + \cdots$$

$$= c_{ij}p_\mu - c_{ji}p'_\mu + \cdots \quad (k = p' - p)$$

$$= e_i k_\mu + \cdots$$

If, for the particular case of the Green's functions of a conserved current, we can show that these expansion coefficients are cutoff-independent, we will have shown that these Green's functions are linear combinations of cutoff-independent Green's functions with cutoff-independent coefficients, and we will have the desired result. But this is just where the Ward identities come in, for they tell us that k^μ dotted into the expansions shown in Fig. 6 must be cutoff-independent, and it is trivial to check that this is enough to tell us that the coefficients themselves are cutoff-independent.

(Please note that if we had had to go to higher orders in the Taylor expansion, the Ward identities would not have been sufficient. For example, they tell us nothing about the coefficient of the following term which can occur in the expansion of the second line of Fig. 6:

$$p_\mu k^2 - k_\mu p\cdot k,$$

because k^μ dotted into this expression vanishes. We need the divergence-

counting of the renormalization procedure to tell us that all possible divergences are controlled if we can control only a few terms in the Taylor expansion. Only then can we use the Ward identities to control those terms.)

This completes the proof of our fourth result.

It is now fairly trivial to get a generalization.

Fifth result. The matrix elements of internal symmetry currents are cutoff-independent even if the symmetry is broken, provided: (1) it is broken in the manner described in the third result, that is to say, by terms of dimension three or less; and (2) the theory possesses no Bose fields with the same internal-symmetry transformation properties as the symmetry-breaking terms of dimension three. (This is a slight generalization of a result of Preparata and Weisberger.[10]

Here we proceed just as we did when establishing the third result. We treat the symmetry-breaking as a perturbation, and ask if it can introduce new divergences into current Green's functions – that is to say, whether it can induce new counterterms in the interaction (10). Since the symmetry-breaking has $\delta \leq -1$, and since (10) has dimension three or less, these new counterterms, if they exist, must be of dimension two or less. Thus, they must be proportional to the gradient of a Bose field. But such terms are excluded by hypothesis (2) above; they have the wrong internal-symmetry transformation properties.

Notes and references

1. Sometimes I will make further cheats. I will warn you about them in notes like this.
2. N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (Interscience, 1959), especially Chapter IV and references contained therein.
3. K. Hepp: *Comm. Math. Phys.* 1, 95 (1965).
4. A cheat: we will treat this theorem as if it had been proved for general cutoff procedures; in fact it has been proved only for a restricted class of cutoffs.
5. Cheating again! Here I am blatantly ignoring the fact that the σ-model displays the Goldstone phenomenon, and that we are, therefore, not perturbing about the solution with manifest symmetry, but the one with a Goldstone boson. This cheat is not so bad, though. What we are really interested in is whether the counterterms spoil the Ward identities of chiral symmetry; these are independent of whether we are in the manifest-symmetry mode or in the Goldstone mode. See B. W. Lee, *Nucl. Phys.* B9, 649 (1969).

Only infinite parts of counterterms are unambiguous; finite parts of the 1 c.t. can be chosen so as not to break chiral symmetry.

6. K. Symanzik in *Fundamental Interactions at High-Energies*, ed. by A. Perlmutter *et al.* (Gordon and Breach, 1970)
7. After it was done first by Symanzik.
8. Remember, we are discussing theories without vector mesons. The result is not true if the theory contains vector mesons with the same quantum numbers as the conserved currents, as does quantum electrodynamics.
9. As the γ_5 may indicate, these arguments work for chiral symmetries as well as for internal symmetries in the more usual sense. This may disturb those of you who know that the Ward identities for chiral theories sometimes contain anomalies, but don't worry – those with only one current have no anomalies, and those are the only ones we are using.
10. G. Preparata and W. Weisberger: *Phys. Rev.* 175, 1973 (1968).

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DISCUSSION

Chairman: Prof. S. COLEMAN

Scientific Secretary: B. W. KECK

DISCUSSION

— DE WITT:

This morning you gave the proof that matrix elements of conserved and partially conserved currents are finite. Do you know if the matrix elements of Green's functions involving the product of several currents are finite?

— COLEMAN:

You can prove for example for a meson-nucleon system and the current $\bar{\psi}\gamma^\mu\psi$ that such Green's functions, except for that containing just two currents, are finite, by using the same technique as in the lecture.

— BERLAD:

I have two questions, one technical and one general. *a*) Is it possible to generalize δi the quantity that tests the renormalizability of the interaction, to higher spin fields? *b*) Is the renormalization procedure, carried out separately in each order of perturbation theory, connected in any way with the convergence or divergence of the renormalized perturbation series itself?

— COLEMAN:

a) Yes, it is. However, in general, δ will no longer be simply related to the dimension of the interaction in the sense of dimensional analysis. This is because, for fields of spin one or higher, powers of momentum, divided by masses, appear in the numerator of the propagator. *b*) There is no known connection between renormalization and the convergence of the perturbation series.

— GASIOROWICZ:

Where, if at all, does the old notion of overlapping divergences rear its ugly head in the BPH programme?

— COLEMAN:

It rears its head in Hepp's proof. He has to do some complicated combinatorics. The result is that you never need to worry if you take his theorem on faith. An example is the diagram

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— KERNER:

In a quite recent series of papers by Veltman the perturbation theory of the Yang-Mills fields has been investigated. By using special Feynman rules the author suggested that in the mass zero limit the theory is renormalizable, but did not prove it conclusively. Can it be done in the way you have discussed?

— COLEMAN:

No. It has been known for many years that ordinary Feynman rules do not apply to zero mass nonabelian gauge fields. Rules were given by Feynman and developed by De Witt involving « ghost loops », too complicated to explain here. With these, one can use our analysis to prove renormalizability by repeatedly applying Ward identities. But unfortunately the internal consistency of the method depends (though Veltman may disagree) on the possibility of constructing a cutoff procedure that preserves the Ward identities. Neither Veltman nor I have been able to do this, except (by different methods) for diagrams with a single closed loop. (Note added in proof: Two gauge-invariant cutoff procedures have since been found, one by B. Lee and Zinn-Justin, the other by 'tHooft and Veltman).

— KRCK:

In Bogoliubov's presentation unitarity is acquired formally by making the S-matrix a time ordered exponential. It is not clear to me that it survives renormalization. Does Hepp show that it does?

— COLEMAN:

Hepp does not concern himself with unitarity. I can give you a handwaving argument for it, namely that one can think of the Pauli-Villars-Villars regularization as provided by the presence of particles of negative norm, which at a given energy do not contribute to the unitarity sum for sufficiently large cutoff. This is only handwaving. I think that there has been a proof combining Hepp and Cutkosky.

— MARTIN:

The problem of unitarity in perturbation theory has been recently settled by Glaser and Epstein, who use an alternative but equivalent renormalization procedure to that described by Prof. Coleman. Now can I come back to the question of Berlad. First of all we have a strong suspicion that $\lambda\varphi^4$ perturbation theory is divergent. However, we hope that all the information about the true theory is contained in the terms of the perturbation expansion. This is supported by i) the anharmonic oscillator model, for which Padé approximants converge (however, this is potential theory), ii) by work of B. Simon on a cutoff two dimensional field theory, for which he proves that one can calculate certain quantities by the Borel summation procedure, which is more general than Padé's. Here of course the result holds for unrenormalized quantities, since you have a cutoff.

$\hbar = c = 1$

$[ML] = 1$

$[S] = [\hbar] = 1$

$[\mathcal{L}] = M^4$

$[\Phi] = M \quad [\Psi] = M^{3/2} \quad [\partial_\mu] = M$

$\dim L_i$ (in mass units) $= b_i + \frac{3}{2} f_i + d_i = \delta_i + 4$

Given a set of bosons + fermions the most general interaction of renormalizable type defines a renormalizable theory

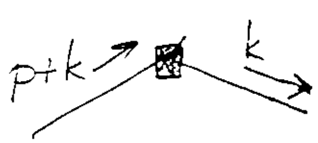
The same is true if we restrict the theory to be invariant under parity and internal symmetry

The same is true if we allow symmetry breaking interaction if we allow all sym-breaking interaction with dim $\leq n$ ($n = 3, 2, 1$)

As an example of a matrix element of a composite operator, lets calculate the matrix element of $\frac{1}{2}\phi^2$ between single nucleon states in our meson nucleon theory. As equation (9) (page. 18) suggests we add

$$\frac{1}{2}\mathcal{J}(x)\,\phi^2(x)$$

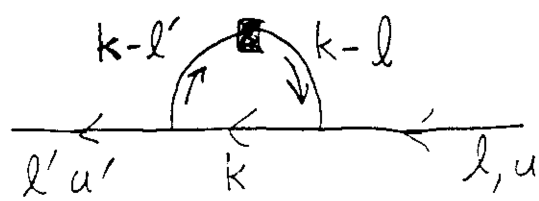
to $\mathcal{L}$. This gives us a new Feynman rule



$$\Longleftrightarrow \quad i\tilde{\mathcal{J}}(p)$$

To $O(g^2)$ $\quad \langle l', u' \,|\, \frac{1}{2}\phi^2(x) \,|\, l, u\rangle$

is calculated by evaluating



$$= i\tilde{\mathcal{J}}(l-l')(-ig)^2 \int \frac{d^4k}{(2\pi)^4}\, \bar{u}' \frac{i}{\not{k}-m} u \;\frac{i}{(k-l')^2-\mu^2}\;\frac{i}{(k-l)^2-\mu^2}$$

$$= -\tilde{\mathcal{J}}(l-l')\, g^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{\bar{u}'(\not{k}+m)u}{k^2-m^2}\;\frac{1}{(k-l')^2-\mu^2}\;\frac{1}{(k-l)^2-\mu^2}$$

$$= -2g^2\tilde{\delta}(\ell-\ell')\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}'(\not{k}+m)u}{\left\{(1-x-y)(k^2-m^2)+x\left[(k-\ell')^2-\mu^2\right]+y\left[(k-\ell)^2-\mu^2\right]\right\}^3}$$

$$= -2g^2\tilde{\delta}(\ell-\ell')\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}'(\not{k}+m)u}{\left[k^2-(1-x-y)m^2-2x\,k\cdot\ell'-2y\,k\cdot\ell-(x+y)\mu^2+x\ell'^2+y\ell'^2\right]^3}$$

$$k' = k - x\ell' - y\ell$$

$$= -2g^2\tilde{\delta}(\ell-\ell')\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4k'}{(2\pi)^4}\frac{\bar{u}'(\not{k}'+x\not{\ell}'+y\not{\ell}-m)u}{\left[k'^2+x(1-x)\ell'^2+y(1-y)\ell^2-2xy\,\ell\cdot\ell'-(1-x-y)m^2-(x+y)\mu^2\right]^3}$$

The $\not{k}'$ term in the numerator is seen to be odd. Also $\ell^2=\ell'^2=m^2$, $\not{\ell}u=mu$ and $\bar{u}'\not{\ell}'=m\bar{u}'$ are simplifications. We have (dropping prime on k)

$$-2g^2\tilde{\delta}(\ell-\ell')\int_0^1 dx\int_0^{1-x} dy\int\frac{d^4k}{(2\pi)^4}\frac{(x+y-1)m\,\bar{u}'u}{\left[k^2-M^2(x,y)\right]^3}$$

where

$$M^2(x,y)=\left[-x(1-x)-y(1-y)+1-x-y\right]m^2+(x+y)\mu^2+2xy\,\ell\cdot\ell'$$

The k integration is in our tables.

$$2ig^2\tilde{\delta}(\ell-\ell')\,m\bar{u}'u\;\frac{1}{32\pi^2}\int_0^1 dx\int_0^{1-x} dy\;\frac{1}{M^2(x,y)}$$

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Let's just call the result of the Feynman parameter integrations

$$F(m^2, \mu^2, (l-l')^2)$$

so what we have is

$$\frac{g^2}{16\pi^2}\, i\tilde{J}(l-l')\, m\bar{u}'u\, F(m^2, \mu^2, (\vec{l}-\vec{l}')^2)$$

To get $\langle l', u' | \frac{1}{2}\phi^2(x) | l, u \rangle$ from this we have to write $\tilde{J}(l-l')$ in terms of $J(x)$ divide by i and then take $\frac{\delta}{\delta J(x)}$

$$\langle l', u' | \tfrac{1}{2}\phi^2(x) | l, u \rangle = \frac{\delta}{\delta J(x)}\left[\frac{g^2}{16\pi^2}\int d^4x\, e^{-i(l-l')\cdot x} J(x)\, m\bar{u}'uF\right]$$

$$= \frac{g^2}{16\pi^2} e^{-i(l-l')\cdot x}\, m\bar{u}'u\, F(m^2, \mu^2, (\vec{l}-\vec{l}')^2)$$

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Renormalization of composite operators

Unfortunately even in a theory that was finite to some order in perturbation theory, the matrix elements of composite operators will not necessarily be finite to that order.

Redefinitions of the composite operator are necessary and additional renormalization conditions to make these redefinitions definite are needed.

In the method of getting Feynman rules for composite operators by adding a source coupled to the operator to $\mathcal{L}$ the redefinitions come as further additions multiplied by the same source. For example we will see that at order λ in a theory with a ϕ^4 interaction it is necessary to add to $\mathcal{L}$ in addition to

$$J(x)\,\tfrac{1}{2}\phi^2(x)$$

further terms

$$J(x)\left(\tfrac{A}{2}\,\phi^2(x) + B\right)$$

The total coefficient of $J(x)$

$$\tfrac{1}{2}\phi^2(x)\left(1+A\right) + B \equiv \tfrac{1}{2}\phi_R^{\,2}$$

will have cutoff independent matrix elements in the limit of large cutoff. The finite parts of A and B will be determined by renormalization conditions.

$$\mathcal{L} \to \mathcal{L} + J(x)\,\big(O(x) + \sum O\big)$$

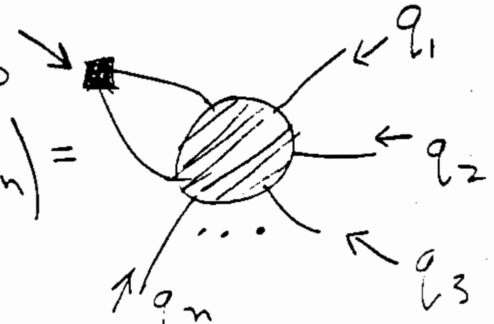
all ops.
with right
sym prop
of lower dim

Rather than calculate a matrix element of $\frac{1}{2}\phi_R^2$, let's calculate

$$\langle 0|T\left(\phi_R^2(x)\,\phi'(y_1)\,\phi'(y_2)\cdots\phi'(y_n)\right)|0\rangle$$

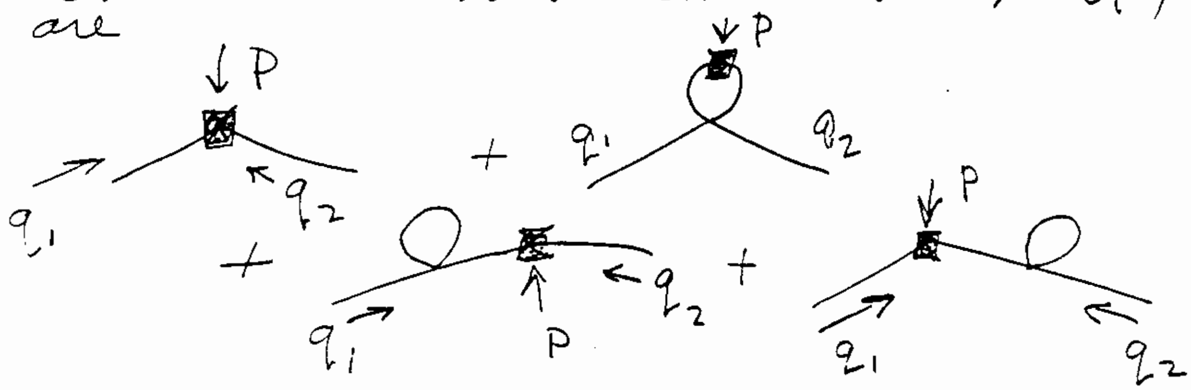
at least for $n=0$ and $n=2$, to order λ. We'll just calculate the Fourier transform

$$\tilde{G}(p;q_1,\dots,q_n) =$$



$$= \int d^4x\, d^4y_1\cdots d^4y_n\, e^{-ip\cdot x}\, e^{-i(q_1\cdot y_1+\cdots+q_n\cdot y_n)} \langle 0|T\left(\phi_R^2(x)\phi'(y_1)\cdots\phi'(y_n)\right)|0\rangle$$

For $n=2$ the contributions to $O(\lambda)$ are

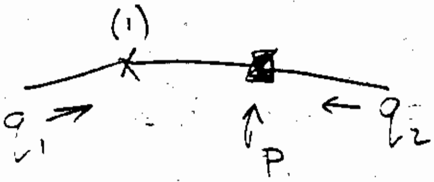


+ more graphs that aren't connected graphs.

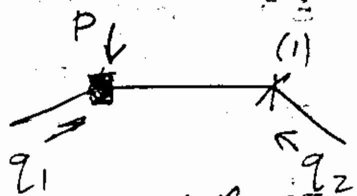
I'll discuss these contributions in reverse order. The disconnected graphs will be disposed of by only computing the connected part of $\tilde{G}$, denoted

$$\tilde{G}_c(p; q_1, \dots, q_n)$$

The fourth and third graphs will be exactly cancelled by the $O(\lambda)$ mass renormalization counterterm graphs



and



The second graph is moderately interesting. It is

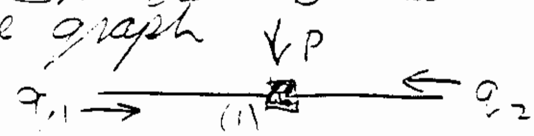
$$\frac{i}{q_1^2-\mu^2}\,\frac{i}{q_2^2-\mu^2}\,(2\pi)^4\delta^{(4)}(q_1+q_2-p)$$

$$\cdot\frac{(-i\lambda)}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{i}{(k-\frac{p}{2})^2-\mu^2}\,\frac{i}{(k+\frac{p}{2})^2-\mu^2}$$

This integral is logarithmically divergent which is why we need the $O(\lambda)$ graph coming from the $O(\lambda)$ part of

$$A\,\phi^2$$

in ϕ_R^2. I'll denote that $A^{(1)}\phi^2$ and we get one more graph



This graph and the first graph give

$$\left(1+A^{(1)}\right)\frac{i}{q_1^2-\mu^2}\,\frac{i}{q_2^2-\mu^2}\,(2\pi)^4\,\delta^{(4)}(q_1+q_2-p)$$

$A^{(1)}$ is a divergent constant chosen to cancel the logarithmically divergent part of the second graph. It is sufficient to do that. A superficially log divergent graph only needs one subtraction (in any order of perturbation theory) according to BPHZ.

We need a renormalization condition to determine the finite part of A. A logical one is that $\tilde{G}_c$ at zero momentum be given exactly by its lowest order contribution. This means

$$A^{(1)}+\frac{(-i\lambda)}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{(k^2-\mu^2)^2}=0$$

So finally to $O(\lambda)$

$$\tilde{G}^{(c)}(p;q_1,q_2)=(2\pi)^4\,\delta^{(4)}(p+q_1+q_2)\,\frac{i}{q_1^2-\mu^2}\,\frac{i}{q_2^2-\mu^2}$$

$$\cdot\left\{1-\frac{i\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\left[\frac{i}{(k-\frac{p}{2})^2-\mu^2}\,\frac{i}{(k+\frac{p}{2})^2-\mu^2}-\frac{i}{(k^2-\mu^2)^2}\right]\right\}$$

$B^{(1)}$ could be chosen so that

$$\langle 0|\,\tfrac{1}{2}\phi_R^2(x)\,|0\rangle=0$$

JAN. 15
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It is surprising to me that there is so much arbitrariness in the definition of Green's function with a composite operator that has to be fixed by renormalization conditions.

If someone hands you $T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$ you'd think something like

The energy-momentum tensor obtained thru Noether's theorem.

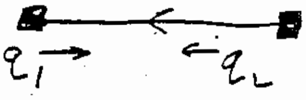
$$\langle 0 | T^{\mu\nu} | 0 \rangle$$

being such a physical thing would not be susceptible to redefinition. apparently the counterterms for a conserved current can usually be pinned down by calling upon cherished properties such as

$$\underline{T^{\mu\nu} = -T^{\nu\mu} \quad \text{and}} \quad \partial_\mu T^{\mu\nu} = 0 .$$

a stupid example to make sure I have my F.T. conventions right. Lets compute $\langle 0 | T(\psi(x)\bar{\psi}(y)) | 0 \rangle$ in free Dirac theory. according to page 27 this should be

$$\int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} e^{i q_1 \cdot x + i q_2 \cdot y}$$



$$= \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} e^{i q_1 \cdot x + i q_2 \cdot y} (2\pi)^4 \delta^{(4)}(q_1 + q_2) \cdot \frac{i}{\not{q}_2 - m + i\epsilon}$$

$$= \int \frac{d^4 q}{(2\pi)^4} e^{-i q \cdot (x-y)} \frac{i}{\not{q} - m + i\epsilon}$$

In agreement with Dec. 18 pp. 5-7