Thermodynamics of 2D Quantum Gravity

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Thermal flow in 2D Quantum Gravity

- Thermal perturbations of minimal CFT's
  - Generated by the thermal operator $\Phi_{1,3}$
    [Nienhuis'84, Al. Zamolodchikov"86, 91, Ludwig-Cardy'91, Fendley-Saleur-Zamolodchikov'93]
    [UV] $\mathcal{M}_{p,p+1} \longrightarrow \mathcal{M}_{p-1,p}$ [IR]
  - Microscopic realization: Loop gas [Nienhuis'84]
    [UV] Dense loops $\longrightarrow$ Dilute loops [IR]
- Thermal flow in 2D QG:
  - Generated by the Liouville-dressed thermal operator $\Phi_{1,3}\, e^{\alpha\phi}$
  - Marginal perturbation, changes simultaneously Liouville and matter central charges so that their sum remains 26.
  - Microscopic realization: loop gas on planar graphs [I.K.'88]

Thermal perturbation of critical 2D QG

Effective action of perturbed $(p, p+1)$- critical QG:

$$\mathcal{L}(\mu, t) = \mathcal{L}^{\mathrm{UV}}_{\mathrm{matter}} + \frac{1}{4\pi}(\partial_a \phi)^2 + Q\phi \hat{R}$$

$$+\mu e^{2b\phi} \; + \; t\, \Phi_{1,3}\, e^{2(1/b-b)\phi}$$

μ – cosmological constant, t – temperature

$$c^{\mathrm{UV}}_{\mathrm{matter}} = 1 - \tfrac{6}{p(p+1)}, \quad b = \sqrt{\tfrac{p}{p+1}}, \quad Q = b + 1/b$$

$\Phi_{1,3}$ – thermal operator with dimensions $\Delta_{1,3} = \bar{\Delta}_{1,3} = \frac{p+1}{p-1}$

Partition function: $\mathcal{F}(\mu, t) = \left\langle e^{-\int \mathcal{L}_{\mathrm{pert}}(\mu,t)} \right\rangle_{\mathrm{sphere}}$

- Expected critical regimes:
  - $t > 0$: flow to pure QG: $c^{\rm IR}_{\rm matter} = 0$ (massive matter)
  - $t < 0$: "massless flow" to QG with $c^{\rm IR}_{\rm matter} = 1 - \frac{6}{p(p-1)}$
- CFT approach: The first 4 terms in the t-expansion of the partition function were calculated by [Belavin and Zamolodchikov'05]
- Discrete (microscopic) approach:
  - Ising model ($p = 3$) – [Bulatov - Kazakov'86]

$$\mu = u^3 - \tfrac{3}{4} t u^2, \qquad u \equiv \partial_\mu^2 \mathcal{F}$$

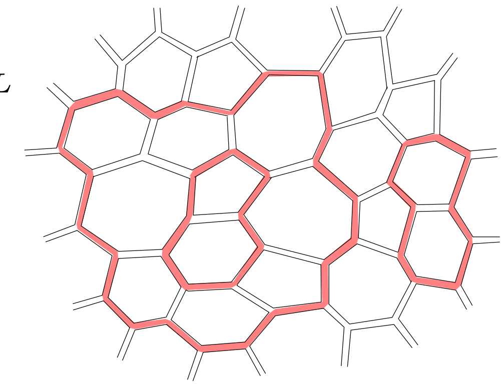
  - ADE string theories, $p \in \mathbb{N}$ [IK'88]
  - $O(n)$ model, $n = 2\cos(\pi/p)$, $p \in \mathbb{R}_+$ [IK'89]:

    In all these theories the thermal operator $\Phi_{1,3}$ is well defined microscopically

Microscopic representation in terms of gas of loops

- Partition function on the sphere = loop gas on planar graphs

$$\mathcal{F}(\tau_0, \mu_0) = \sum_{A=0}^{\infty} e^{-\mu_0 A} \sum_{\mathcal{G}_A} \sum_{\text{loops}} e^{-t_0 L} \left[2\cos\left(\frac{\pi}{p}\right)\right]^{N_L}$$



$\mathcal{G}_A$ - fat graph of genus 0 and volume A,
$N_L = \#$ loops, $L = \#$ occupied lines,

- Disk partition function

$$\Phi(\mu_0, \lambda_0, \mu_B) = \sum_{L_B, A \geq 1} e^{-\mu_0 A - \mu_M L_B} \sum_{\mathcal{G}_{A, L_B}} \sum_{\text{loops}} e^{-t_0 L} \left[2\cos\left(\frac{\pi}{p}\right)\right]^{N_L}$$

μ_0, μ_B – bulk and boundary cosmological constants,
t_0 - temperature

Continuum limit

- Critical lines:
  - $\mu_0 = \mu_0^c(t_0)$: the volume of the graph diverges
  - $t_0 = t_0^c(\mu_0)$: length of the loops diverges.
- Critical point: $t^* = t_0^c(\mu^*),\ \mu^* = \mu_0^c(t^*)$.
  The two couplings in the continuum limit:

$$\mu = \mu_0 - \mu^*, \quad t = t_0 - t^*.$$

- All the information about the continuum limit is contained in the boundary entropy:

$$M(\mu, t) = -\lim_{\ell \to \infty} \frac{\log \tilde{\Phi}(\mu, t, \ell)}{\ell}$$

where $\tilde{\Phi}(\mu, t, \lambda)$ is the disk partition function for fixed boundary length $\Phi(\mu, \lambda, \mu_B) = \int_0^\infty d\ell\, e^{-\mu_B \ell}\, \tilde{\Phi}(\mu, \lambda, \ell)$

Explicit expressions in the continuum limit [IK-hep/th-0602075]

- Disk partition function:

$$\begin{aligned} \mu_B &= M\cosh\tau, \\ -\partial_{\mu_B}\Phi|_{\mu} &= \tfrac{2p}{p+1}\, M^{1+1/p}\, \cosh(\tfrac{p+1}{p}\tau) + \tfrac{2p}{p-1}\, t\, M^{1-\nu} \cosh(1-\nu)\tau \\ \partial_{\mu}\Phi|_{\mu_B} &= \nu^{-1} M^{\nu} \cosh \nu\tau. \end{aligned}$$

- Compatibility of the two expressions $\Rightarrow$ equation for the boundary entropy :

$$M^2 - \frac{p}{p+1} t\, M^{2-2/p} = \mu$$

- Susceptibility $u \equiv -\partial_{\mu}^2 \mathcal{F}$:

$$u = M^{2/p} \quad \Rightarrow \quad u^p - \frac{p}{p+1}\, t\, u^{p-1} = \mu$$

Small t expansion

- Low temperature regime: $t/\mu^{1/p} << 1$
  - Small t expansion around the UV critical point (dilute phase) :

$$
\begin{aligned}
\mathcal{F}(\mu,t) &= -\tfrac{p}{p+1} t\,\mu^2 \\
&+ \mu^{\frac{2p+1}{p}} \left(\frac{-p^3}{(p+1)(2p+1)} + \sum_{n=2}^{\infty} \frac{\Gamma\left(\frac{n-1}{p}\right)}{\Gamma\left((1-p)\frac{n-1}{p}\right)} \frac{\left(t\,\mu^{-1/p}\right)^n}{n!} \right)
\end{aligned}
$$

  - $\Rightarrow$ n-point functions of the thermal operator $\Phi_{1,3}\ e^{2(1/b-b)\phi}$ in Liouville gravity.
  - Up to t^5 matches with recent result of [A. Belavin-Al. Zamolodchikov'05].

Large t expansion

- High temperature regime: $t/\mu^{1/p} \to -\infty$
  - Expansion about the IR critical point (dense phase) with cosmological constant $\mu_{\rm IR} = -\mu/t$

$$u(\mu,t) = \mu_{\rm IR}{}^{\frac{1}{p-1}} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n+1}{p-1}+n\right)}{n!\;\Gamma\left(\frac{n+1}{p-1}\right)} \left(t^{-1}\mu_{\rm IR}{}^{\frac{1}{p-1}}\right)^{n}$$

  - Generating function for the correlators of the dual operator $\Phi_{3,1}$

Analogy with sine-Liouville perturbation of 2D QG

- sine-Liouville perturbation of linear dilaton background:

$$\mathcal{L}_{\rm SL}(\mu, t) = \frac{1}{4\pi}\left[(\partial_a \sigma)^2 + (\partial_a \phi)^2 + 2\phi\hat{R}\right]$$

$$+\mu\, e^{2\phi} + \lambda\, \cos R\sigma\; e^{(2-R)\phi}$$

- Equation for the susceptibility $\chi = \partial_\mu^2 \mathcal{F}$ [KKK'01]:

$$\mu = e^{-\chi/R} - (R-1)\,\lambda^2\, e^{\chi(1-R)/R}$$

- Identical for the equation for thermal perturbation of the $(p, p+1)$ model
$p = (2-R)^{-1}, \quad u \equiv e^{-\chi(2-R)/R}, \quad t = (R-1)(3-R)\lambda$

- Dilute phase $\leftrightarrow$ Weak SL perturbation:

$$\chi(\mu, \lambda) = -R \log \mu + f(\lambda/\mu^{2-R})$$

- Dense phase $\leftrightarrow$ Strong SL perturbation:

$$\chi(\tilde{\mu}, \lambda) = -\tilde{R} \log \tilde{\mu} \ + \tilde{f} \left(\tilde{\mu}^{\tilde{R}-2}/\lambda \right);$$

$$\tilde{R} = \frac{R}{R-1} > 2, \quad \tilde{\mu} = \mu/\lambda$$

Unsolved Problem:

- The meaning of the critical point $\mu = 0$

$$\mathcal{F}(t) = t^{2p+1} \hat{f}\left(\mu t^{-p}\right)$$

  - This is not the phase of dense loops.
  - A new kind of a theory of 2D gravity, similar to the sine-Liouville theory.
- In Sine-Liouville gravity this is the point expected to describe the 2D Black hole