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Adjoint sector of c=1 MQM

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Introduction

c=1 Quantum Gravity

Liouville Theory

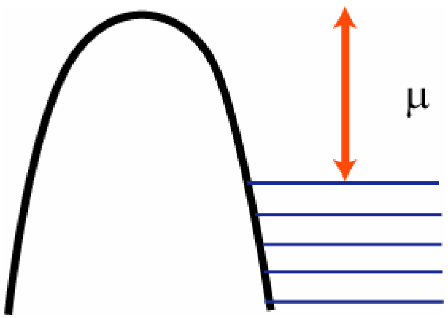
$$L = \frac{1}{2}\partial\phi\bar{\partial}\phi + Q\phi R + \mu e^{-\phi}$$

μ : Cosmological const.

short string

Matrix Quantum Mechanics

$$L = \frac{1}{2}\mathrm{Tr}\dot{M}^2 + \frac{1}{2}\mathrm{Tr}M^2$$



up-side down potential

singlet sector = free fermion

Representation of wave function in MQM

Action is invariant under $M \to UMU^{\dagger} \; : \; U \in U(N)$

Conserved charge : $J = i[M, \dot{M}]$

Wave function transforms as

$$\Psi(UMU^{\dagger})_a = \rho(U)_{ab}\Psi_b(M)$$

$$\rho \;\; : \;\; \text{Some irrep. of } U(N)$$

Usually we consider only singlet sector:

the dynamics reduces to free fermion which corresponds to short strings of Liouville theory

Possible representations

Since we have to construct states from M (adjoint), the representation that wave function can take is limited.

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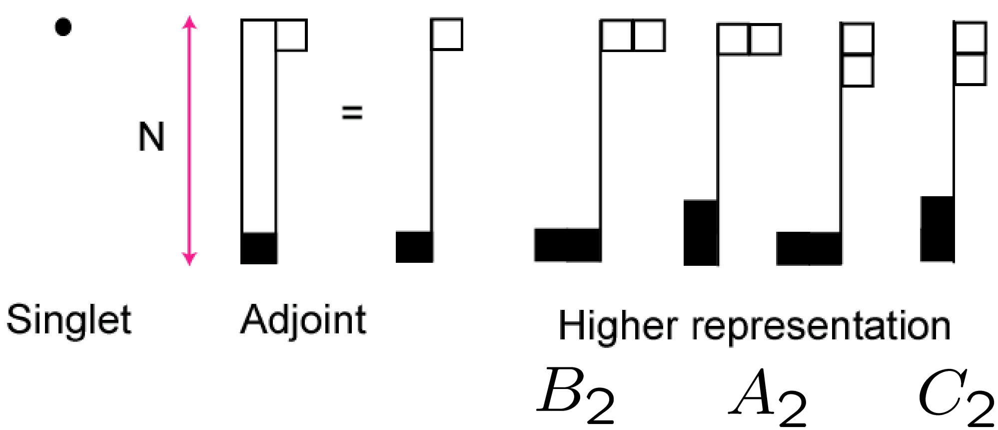


Boulatov-Kazakov

Role of non-singlet sector

- Vortex configuration (KT phase transition)

  insertion of $e^{\pm i n \phi}$

- 2D Black hole (cf. Kazakov, Kostov, Kutasov)

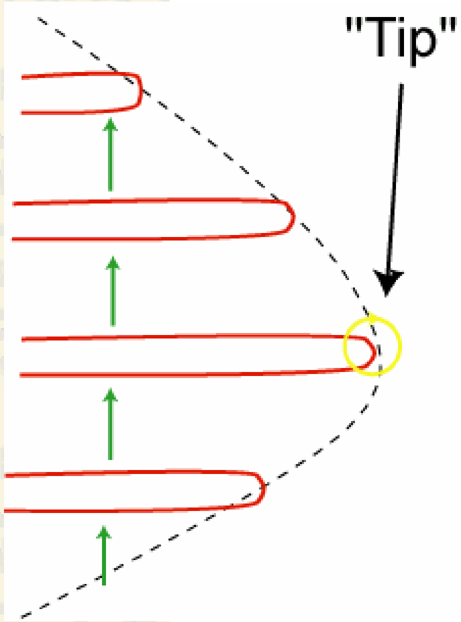
- Long string with tips (Maldacena)

Maldacena's long string

hep-th/0503112

Virasoro constraint $(\partial_\pm\phi)^2 - Q\partial_\pm^2\phi - 1 = 0$

$$\Rightarrow \phi = \phi_0 - Q\log\left(\cosh\frac{\tau}{Q} + \cosh\frac{\sigma}{Q}\right)$$



Motion of "tip"

$$\phi \sim -\tau^2/4Q$$

= massive particle motion
with constant force from string tension

What is the corresponding object in matrix model ?

⟶ ***non-singlet sector in MQM?***

Correspondence with Liouville

Correspondence between tip of long string and adjoint sector of MQM is established by Maldacena (0503112) and Fidkowski (0506132) by comparing its scattering phase

With some simplifying assumption on large N limit of Calogero equation and with fixed background fermion

$$\sum_k h_k w_i + \sum_j \frac{w_j - w_i}{(x_j - x_i)^2} = E w_i$$

$$h_k = -\frac{1}{2}\frac{\partial^2}{\partial x_k^2} - \frac{1}{2}x_k^2$$

$$\rho(x) = \sum_i \delta(x - x_i)$$

: distribution of singlet sector

Similar to quenched approximation in QCD

Link with finite N is missing

I. For singlet sector, finite N theory is completely known as well as large N limit

II. 1/N correction has physical significance such as stringy higher loop correction

III. Study of finite N case is also essential to understand the *back reaction* in the presence of vortex

IV. The quantities which we studied

    I. exact eigenfunction of Hamiltonian

    II. scattering phase

§ 2 Exact solutions of adjoint sector

We start from one body problem
(QM with upside down potential) $L = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2$

Canonical quantization

$p = \dot{x}$

$[p, x] = i$

$-\frac{1}{2}\partial_x^2\psi - \frac{x^2}{2}\psi = E\psi$

$\psi(E)$ parabolic cylinder ft.

Chiral (lightcone) quantization

$x^{\pm} = \frac{1}{\sqrt{2}}(p \pm x)$

$\left[x^+, x^-\right] = i$

$H = \frac{1}{2}\left(2x^-x^+ + i\right) = \frac{i}{2}\left(2x^-\partial_- + 1\right)$

$H\tilde{\psi}(x^-) = E\tilde{\psi}(x^-)$

$\tilde{\psi}(x^-) = \frac{1}{\sqrt{2\pi}}(x^-)^{-iE-1/2}$

Relation between two basis

Integral transformation

$$\langle x|\psi\rangle = \int dx^- \langle x|x^-\rangle \langle x^-|\psi\rangle$$

$$\langle x|x^\pm\rangle \propto \exp\left(\frac{\pm i}{2}x^2 + i\sqrt{2}x^\pm x \pm \frac{i}{2}(x^\pm)^2\right)$$

(analog of generating functional of Hermite polynomial)

Generalization to MQM is much easier in chiral basis.
Expression for canonical basis is obtained by analog of integral transformation

Generalization to MQM

Canonical

$$X, \Pi = \dot{X}$$

$$H = \frac{1}{2}\mathrm{Tr}\dot{X}^2 + \frac{1}{2}\mathrm{Tr}X^2$$

Chiral

$$X^{\pm} = \frac{1}{\sqrt{2}}(\Pi \pm M)$$

$$\left[(X^{+})_{ij}, (X^{-})_{kl}\right] = -i\delta_{il}\delta_{jk}$$

$$H = \frac{i}{2}\mathrm{Tr}\left(2X^{-}\partial_{X^{-}} + N\right)$$

Transformation between two basis

$$\langle X|\psi\rangle = \int d^{N^2}X^{-}\langle X|X^{-}\rangle\langle X^{-}|\psi\rangle$$

$$\langle X|X^{\pm}\rangle \propto \exp\mathrm{Tr}\left(\frac{\pm i}{2}X^2 + i\sqrt{2}X^{\pm}X \pm \frac{i}{2}(X^{\pm})^2\right)$$

Reduction to eigenvalue dynamics

In canonical basis, the dynamics of *eigenvalue* is **Calogero** system

$$H = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_i (x_i)^2 + \frac{1}{2}\sum_{i\neq j}\frac{K_{ij}K_{ji}}{(x_i - x_j)^2}$$

$$K_{ij} \equiv \rho(E_{ij})$$

Boulatov-Kazakov

However, the dynamics for *chiral basis* remains the same !

$$\tilde{H} = \frac{i}{2}\left(2\sum_i x_i^- \partial_{x_i^-} + N\right)$$

Origin of the simplicity in the chiral basis is that differentiation w.r.t. matrix is *first order* and does not contain any nontrivial *off-diagonal* component

Partition function for upside-up case

We use the partition function (Boulatov-Kazakov) to guess the eigenfunctions in the chiral basis

$$
\begin{aligned}
Z_{sing}(q) &= \frac{q^{N^2/2}}{(1-q)(1-q^2)\cdots(1-q^N)} \\
Z_\bullet(q) &= P_\bullet(q) Z_{sing}(q) \\
P_{adj}(q) &= \frac{q-q^N}{1-q} = \sum_{r=1}^{N-1} q^r \\
P_{A_2}(q) &= \frac{q^3(1-q^{N-2})(1-q^{N-1})}{(1-q)(1-q^2)} = \sum_{0<r<s<N} q^{r+s} \\
P_{B_2}(q) &= \frac{q^2(1-q^{N-1})(1-q^N)}{(1-q)(1-q^2)} = \sum_{0<r\leq s<N} q^{r+s} \\
P_{C_2}(q) &= \frac{q^2(1-q^{N-3})(1-q^N)}{(1-q)(1-q^2)}
\end{aligned}
$$

Wave function in chiral basis

Counting of the states is consistent with the basis of solutions

$$\Psi^{(sing)}(x^-) = \mathrm{det}_{rs}(x_r^-)^{n_s}$$

$$\Psi_{ii}^{(adj)}(x^-) = (x_i^-)^n \Psi^{(sing)}(x^-) \quad (0 < n < N)$$

$$\Psi_{ij,ij}^{(A_2,B_2,C_2)}(x^-) = (x_i^-)^n (x_j^-)^m \Psi^{(sing)}(x^-)$$

$$(0 < n \leq m < N)$$

Correction

- For adjoint, we need to subtract the trace part
- For A_2,B_2,C_2, we need to take appropriate (anti-) symmetrization of indices
- The wave function for upside-down case is obtained by replacing *n* by *(ie-1/2)*

Transformation to canonical basis

The matrix integration in the integral transformation from chiral to canonical basis become nontrivial due to the *integration over angular variables*

$$\tilde{\Psi}_a(x) = e^{\pm\frac{i}{2}\sum_i x_i^2} \sum_b \int d^N x K_{ab}(\sqrt{2}ix, x^{\pm}) e^{\pm\frac{i}{2}\sum_i (x_i^{\pm})^2} \tilde{\Psi}_b^{\pm}(x^{\pm})$$

$$K_{ab}(x,y) = \langle \rho_{ab}(U) \rangle = \Delta(x)\Delta(y) \int dU \rho_{ab}(U) e^{Tr(xUyU^{-1})}$$

where (with appropriate symmetry factor again)

$$\text{singlet} \;:\; \rho(U) = 1$$

$$\text{adjoint} \;:\; \rho(U)_{ij} = U_{ij} U^{\dagger}_{ij}$$

$$A_2, B_2, C_2 \;:\; \rho(U)_{i_1 i_2 j_1 j_2} = U_{i_1 j_1} U_{i_2 j_2} U^{\dagger}_{i_1 j_1} U^{\dagger}_{i_2 j_2}$$

Unitary matrix integration

singlet: $K^{(sing)}(x,y) = \det{}_{rs} e^{x_r y_s}$ *Itzykson-Zuber formula*

adjoint: *(Morozov, Bertola-Eynard)*

$$K_{ij}^{(adj)}(x,y) = -\frac{\det{}_{kl}\left[\left((x_k - x_i)(y_\ell - y_j) - 1\right) e^{x_k y_\ell}\right]}{\prod_{k(\neq i)}(x_k - x_i)\prod_{\ell(\neq j)}(y_\ell - y_j)}$$

2n point function <UUUUUUUU> *(Eynard 0502041)*

closed form has not been obtained yet

formal expression is given as gaussian integration over triangular matrices

These *correlation functions* plays the role of transformation kernel from chiral basis to canonical basis and essentially solves the dynamics of Calogero system

Use of generating functional

Does two point function really generate solutions of adjoint Calogero? To check it, it is more convenient to recombine the wave function

$$w(\xi,\zeta;m) = \sum_i \frac{w_i(\zeta,m)}{\xi - m_i}$$

$$w_i(\zeta,m) = e^{-\frac{i}{2}\sum_i m_i^2} \int dx^- K(2im, x^-)_{ij} e^{-\frac{i}{2}\sum_i (x_i^-)^2} \frac{1}{\zeta - x_j^-} \Psi^{(sing)}(x^-)$$

It is equivalent to recombining the kernel:

$$K^{(adj)}(\xi,\zeta : x,y) = \sum_{i,j} \frac{1}{\xi - x_i} \frac{1}{\zeta - y_j} K_{ij}^{(adj)}(x,y)$$

$$= -\det{}_{k\ell}\left(\left(1 - \frac{1}{(\xi - x_k)(\zeta - y_\ell)}\right) e^{x_k y_\ell}\right) + \det_{k\ell} e^{x_k y_\ell}$$

Derivation of adjoint Calogero

We rewrite adjoint Calogero eq. in terms of $w(\xi, \zeta; m)$

$$\mathrm{Res}_{\xi=x_i}\left(H_2(\xi,m)w - H_1(\zeta)w - Ew\right) = 0$$

$$H_2(\xi,m) = \sum_k h_k - \sum_j \frac{1}{(\xi - m_j)^2}$$

$$H_1(\zeta) = i\zeta\partial_\zeta\,, \qquad h_k = -\frac{1}{2}(\partial_{m_k}^2 + m_k^2)$$

This computation is doable and having been checked

It gives all the exact energy eigenfunctions for adjoint sector.

Scattering Phase

Previous computation does not solve the problem completely.
Inner product between incoming & outgoing wave
= scattering phase
For general representation, it is written as,

$$\langle \Psi^+(X^+)|\Psi^-(X^-)\rangle = \int dX^+ dX^- e^{iTr(X^+X^-)} \left\langle \Psi^+(X^+), \Psi^-(X^-) \right\rangle_\rho$$
$$= \int d^N x^+ d^N x^- K_{ab}(ix^+, x^-) \overline{\tilde{\Psi}_a^+(x^+)} \tilde{\Psi}_b^-(x^-)$$

For singlet state, the Slater determinant state is diagonal with respect to inner product

Fermionic representation

In order to give a compact notation for the inner product, it is useful to introduce the fermion representation.

$$\det \psi_r(x_s^{\pm}) \;\rightarrow\; |\Psi\rangle = \psi_1 \wedge \cdots \wedge \psi_N$$

$$\langle \Phi | \Psi \rangle = \det{}_{rs} \left(\langle \phi_r | \psi_s \rangle \right)$$

$$\langle \phi | \psi \rangle = \int dx^+ dx^- e^{ix^+ x^-} \phi(x^+)^* \psi(x^-)$$

$$W(\mathcal{O}) |\Psi\rangle = \sum_{i=1}^{N} \psi_1 \wedge \cdots (\mathcal{O}\psi_i) \cdots \wedge \psi_N$$

It describes the singlet part of the wave function compactly and adjoint part is given as “operator” acting on it

Inner product formula for adjoint

The following choice of the wave function simplifies the formula

$$\tilde{\Psi}(x^{\pm})_i = \frac{1}{\zeta^{\pm} - x_i^{\pm}} \psi_1^{\pm} \wedge \cdots \wedge \psi_N^{\pm}$$

Inner product for the adjoint wave function becomes,

$$\langle \Psi^+(\zeta^+) | \Psi^-(\zeta^-) \rangle^{(adj)} = \langle \Psi^+ | \mathcal{O}(\zeta^+, \zeta^-) | \Psi^- \rangle$$

$$\mathcal{O}(\zeta^+, \zeta^-) = e^{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \widehat{W}(\mathcal{D}^n)}$$

$$\mathcal{D} = \sum_{n,m=0}^{\infty} (\zeta^+)^{-(n+1)} (\zeta^-)^{-(m+1)} (-i\partial_-)^n (x_-)^m$$

$$= \frac{1}{\zeta^+ + i\partial_-} \cdot \frac{1}{\zeta^- - x^-}$$

Mixing in adjoint sector

$\mathcal{O}_{m,n} \neq 0$ for $m \neq n$ implies that there is mixing between

$$(x^+)^n \det \psi \leftrightarrow (x^-)^m \det \psi'$$
$$\det \psi \ : \ \text{level } L-n$$
$$\det \psi' \ : \ \text{level } L-m$$

To obtain scattering phase of solitons, it is not sufficient to obtain eigenfunction of Hamiltonian.
Inner product is off-diagonal because of the degeneration of energy level.
It implies nontrivial interaction between tip and background fermion

Higher conserved charges?

Unfortunately this problem has not been solved.

Usually, however, there is an infinite number of conserved charges in the *solvable system*. Calogero system is certainly of that type. Such higher charges may be written as higher order differential equations of the adjoint kernel.

$$\mathcal{H}_n(x)K^{(adj)}(x,y) - \mathcal{H}_n(y)K^{(adj)}(x,y) = 0$$

It may be better to come back to the angular integration.

§ 3 Large N limit

Maldacena's reduced equation

$$\sum_{j(\neq i)} \frac{w_i - w_j}{(m_i - m_j)^2} = E w_i$$

Dropping the kinetic term

Large N variable

$$\rho(m) = \sum_i^N \delta(m - m_i)\,, \qquad h(m) = \sum_i^N w_i \delta(m - m_i)$$

e.o.m reduces to

$$K[h](m) + v(m)h(m) = Eh(m)$$

$$v(m) = \int dm' \frac{\rho(m)}{(m - m')^2}\,, \; K[h](m) = -\rho(m) \int dm' \frac{h(m')}{(m - m')^2}$$

v(m) : potential energy *K[h]* : Kinetic energy

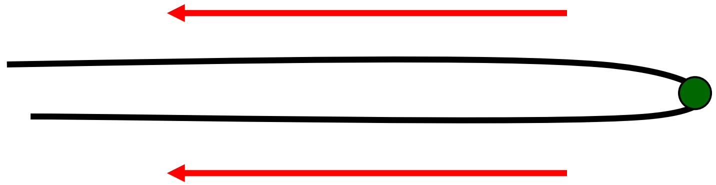
Computation with fixed background

If we use the density function of free fermion,

$$\rho(\lambda) = \frac{1}{\pi}\lambda\,, \quad \tau = \log\lambda$$

$$\to v(\lambda) = \frac{1}{\pi}(\tau_c - \tau)$$

Linear potential : constant force by string tension



Fidkowski's exact solution

$$\lambda = \sqrt{2\mu}\cosh\tau\,, \qquad \rho(\lambda) = \sqrt{2\mu}\sinh\tau$$

$$\widehat{\epsilon} h(\tau) = -\frac{1}{\pi}\int d\tau' \frac{h(\tau')}{4\sinh^2\frac{\tau-\tau'}{2}} + v(\tau)h(\tau)$$

$$v(\tau) = -\frac{1}{\pi}\frac{\tau}{\tanh\tau}$$

After Fourier transformation, *Maldacena*

$$\pi\widehat{\epsilon} h(k) = \left(\frac{\pi k}{\tanh\pi k} - \frac{\partial_k}{\tan\partial_k}\right) h(k) = \pi\widehat{\epsilon} h(k)$$

Fidkowski solved this equation exactly and reproduced the scattering phase from Liouville theory exactly

$$h(k) = \frac{\sinh\pi k}{\sqrt{\sinh^2\pi k + e^{2\pi\epsilon}}}\exp\left(\pi i\int^k dk' \frac{\sinh\pi k'}{\sqrt{\sinh^2\pi k' + e^{2\pi\epsilon}}}\left(\epsilon - \frac{k'}{\tanh\pi k'}\right)\right)$$

Derivation of scattering phase from finite N result

Use of grand canonical ensemble

$$\frac{\langle \Psi^+ | W(e^{\mathcal{D}}) | \Psi^- \rangle}{\langle \Psi^+ | \Psi^- \rangle} \to \mathrm{Det}(W(e^{\mathcal{D}})) = e^{i\phi(\zeta^+,\zeta^-)}$$

$$i\phi = \mathrm{tr}\mathcal{D} = \mathrm{tr}\frac{1}{(\zeta^+ - \partial_-)(\zeta^- - x^-)}$$

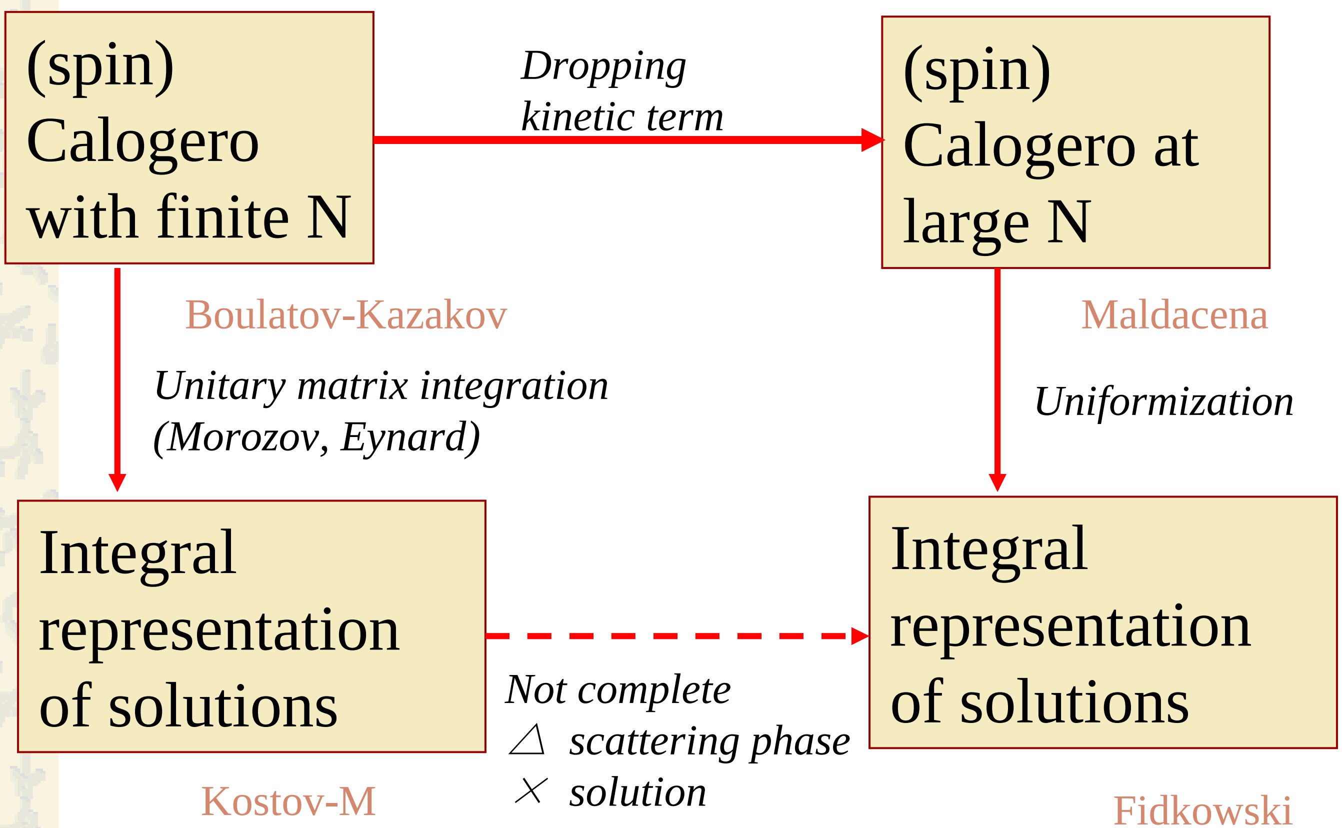
$$= \int_{\mathcal{C}} \frac{dx^+ dx^-}{2\pi} \frac{\theta(x^+ x^- - \mu)}{(\zeta^+ - x^+)(\zeta^- - x^-)}$$

$$= -\frac{1}{\pi}\int_{-\infty}^{y} dy' \left(\frac{y'}{\tanh y'} + y' \right) + \cdots$$

$$\zeta^{\pm} = -\sqrt{\mu} e^{-y \pm \sigma}$$

After Fourier transformation, it reproduces the scattering phase claimed by Maldacena (with some extra terms)

§ 4 Summary of the status



Summary and discussion

- At finite N, we can obtain the explicit form of the solutions of adjoint Calogero equation.
- It involves nontrivial integration and interaction between the singlet fermion and adjoint part can be seen.
- At the same time, we have met a tough problem:
  diagonalization of inner product
- Techniques of integrable system will be useful
  Higher conserved charges
  solution generating technique

- Taking large N limit has some problems
- Higher representation
    compact expression of 2n-point function is needed
    Maldacena conjectured that it corresponds to multiple tips