Bethe ansatz in Sigma Models

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AdS/CFT correspondence

Maldacena'97

$\mathcal{N}=4$ SYM	Strings on $AdS_5\times S^5$
't Hooft coupling: $\lambda = g_{YM}^2 N$	String tension: $T=\frac{\sqrt{\lambda}}{2\pi}$
Number of colors: N	String coupling: $g_s=\frac{\lambda}{4\pi N}$
Large-N limit	Free strings
Strong coupling	Classical strings
Local operators Scaling dimension: Δ	String states Energy: E

Gubser,Klebanov,Polyakov'98
Witten'98

Strings in AdS_5xS^5

Green-Schwarz-type coset sigma model
on SU(2,2|4)/SO(4,1)xSO(5).

Conformal gauge is problematic:
no kinetic term for fermions, no holomorphic
factorization for currents, …

Light-cone gauge is OK.
Still, the action is too complicated.

Integrability

But the model is integrable!

Bena,Polchinski,Roiban'03

Definition of integrability:

$$\begin{aligned} [H, Q_n] &= 0 \\ [Q_n, Q_m] &= 0 \end{aligned}$$

Usually integrability implies something more…

Yang-Baxter equation

$$R_{ab}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u-v)$$

$$\Downarrow$$

$$[\,\mathrm{tr}\,T(u),\,\mathrm{tr}\,T(v)] = 0$$

In the simplest case:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

$$A(u) + D(u) = \exp\left(iP + iHu + iQ_2u^2 + iQ_3u^3 \ldots\right)$$

Using RTT=TTR one can show that

$$B(u_1)\dots B(u_M)\,|0\rangle$$

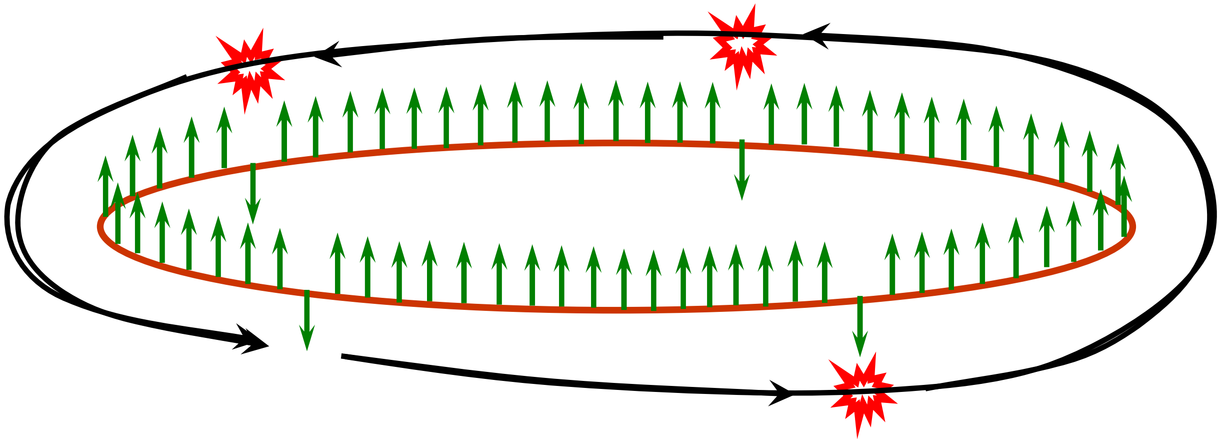
is an eigenstate of the Hamiltonian with an eigenvalue

$$E=\sum_{n=1}^{M}\varepsilon(u_n)$$

provided

$$\mathrm{e}^{\,ip(u_j)}=\prod_{k\neq j}\mathrm{e}^{\,i\Delta(u_j,u_k)}$$

Bethe equations



scattering phase shifts

momentum

$$e^{i\langle \text{total phase shift}\rangle} = e^{-ip_j L} \prod_{k \neq j} e^{i\Delta(p_j, p_k)}$$

periodicity of wave function

Exact periodicity condition:

$$e^{ip_j L} = \prod_{k \neq j} e^{i\Delta(p_j, p_k)}$$

Strategy:

- find the dispersion relation (solve the one-body problem):

$$\varepsilon = \varepsilon(p)$$

- find the S-matrix (solve the two-body problem):

$$S(p, p') = \mathrm{e}^{\,i\Delta(p,p')}$$

$\swarrow$

Bethe equations $\Rightarrow$ full spectrum

- find the true ground state

Successfully used on the gauge-theory side

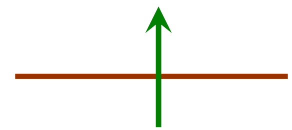
Staudacher'04; Beisert'05

Subsectors

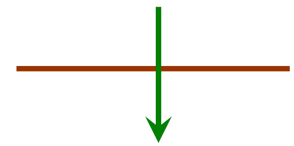
- Nonrelativistic su(2) sigma-model
- Fermionic su(1|1) sigma-model
- Faddeev-Reshetikhin model (su(2) sigma-model + Virasoro constraints)

su(2) sector in SYM

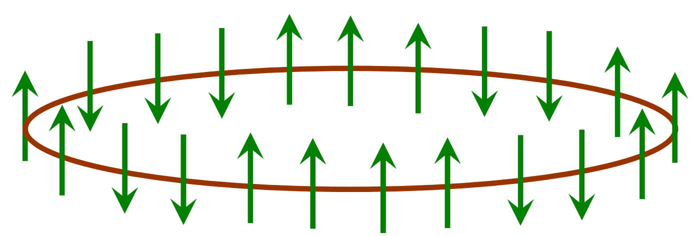
$Z = \Phi_1 + i\Phi_2 \quad \Longleftrightarrow$



$W = \Phi_3 + i\Phi_4 \quad \Longleftrightarrow$



$$\mathrm{tr}\, ZZZZWWZZZWWZZZWWWZZWW$$



Operators tr($Z^{L-M}W^{M}$+permutations) mix only among themselves

Beisert'03

Mixing matrix = Heisenberg Hamiltonian:

$$D = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}\right) + O(\lambda^2)$$

Minahan,Z.'02

Landau-Lifshitz model

Continuum+classical limit: $\sigma_l^i \to n_i(l), \qquad n^2 = 1$

$$S = \int dt \int_0^L dx \left[C_t[n] - \frac{1}{4} \left(\frac{\partial n}{\partial x} \right)^2 \right]$$

WZ term: $C_q[n] = -\frac{1}{2} \int_0^1 d\xi \, \varepsilon_{ijk} n_i \frac{\partial n_j}{\partial \xi} \frac{\partial n_k}{\partial q}$

Equivalent to the action for fast-moving strings on $S^3 x R^1$

Kruczenski'03

Perturbation theory

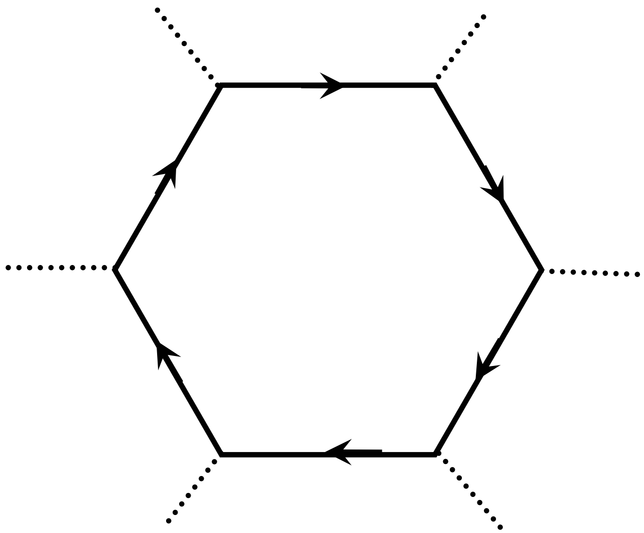
$$\phi = \frac{n_1 + i n_2}{\sqrt{2 + 2n_3}}$$

similar to Minahan,Tirziu,Tseytlin'04

Assume that fluctuations around n=(0,0,1) are small:

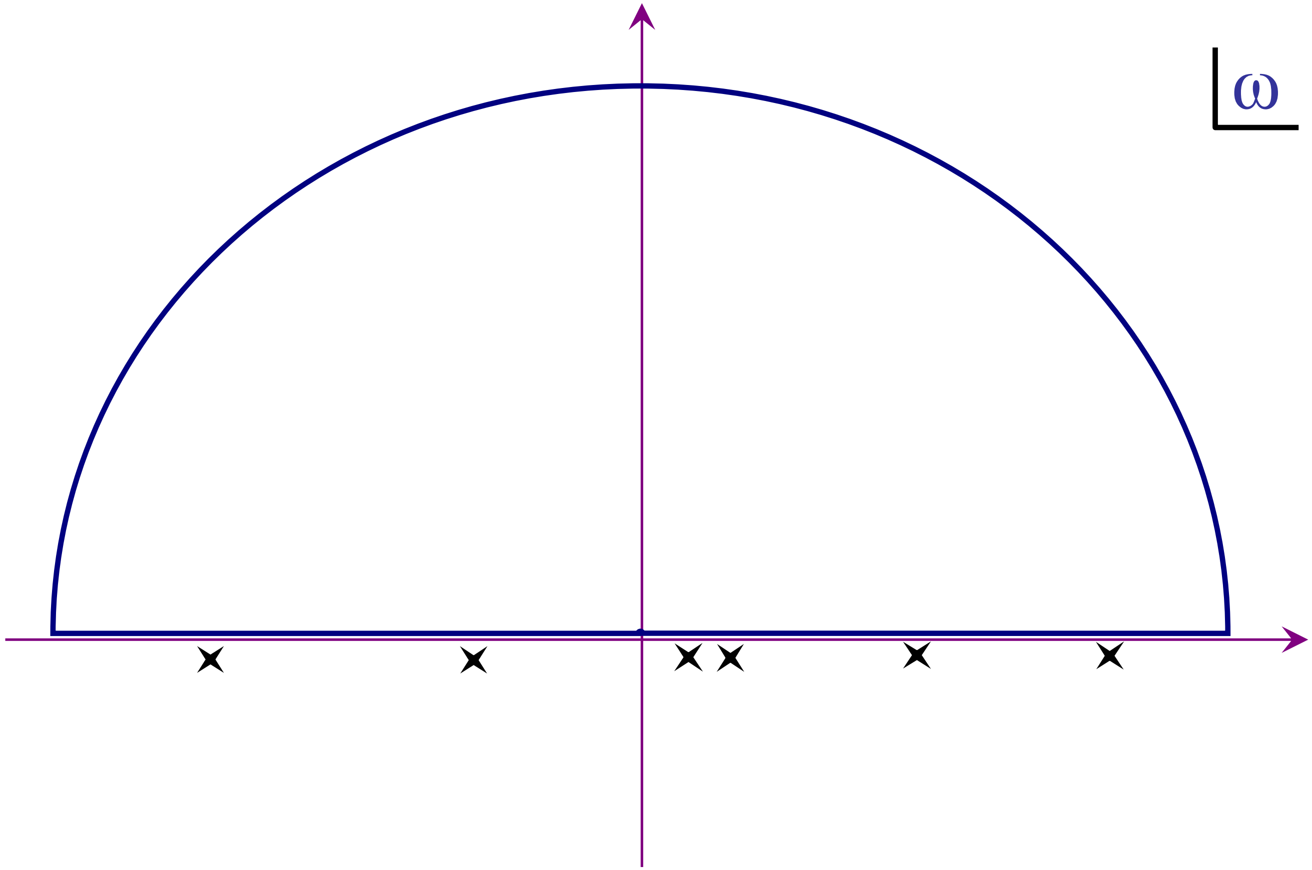
$$S = \int dt dx \left[\frac{i}{2} \left(\phi^* \dot{\phi} - \dot{\phi}^* \phi \right) - |\phi'|^2 - \frac{1}{4} \left(\phi^{*2} \phi'^2 + \phi^{*\prime 2} \phi \right) + O(\phi^6) \right]$$

"Non-renormalization theorem"



$$= \quad 0$$

$$= \quad \frac{i}{\omega - p^2 + i\epsilon}$$



$$\int d\omega \quad \frac{i}{\omega - p^2 + i\epsilon} \; \frac{i}{\omega + E_1 - (p+p_1)^2 + i\epsilon} \times$$
$$\times \ldots \frac{i}{\omega + E_1 + \ldots + E_n - (p + p_1 + \ldots + p_n)^2 + i\epsilon} = 0$$

Coordinate-space proof

$$D(t,x) = \langle \phi(t,x)\phi(0,0)\rangle = \theta(t)\sqrt{\frac{\pi}{it}}\,\mathrm{e}^{\frac{ix^2}{4t}}$$

The propagator is purely retarded, so

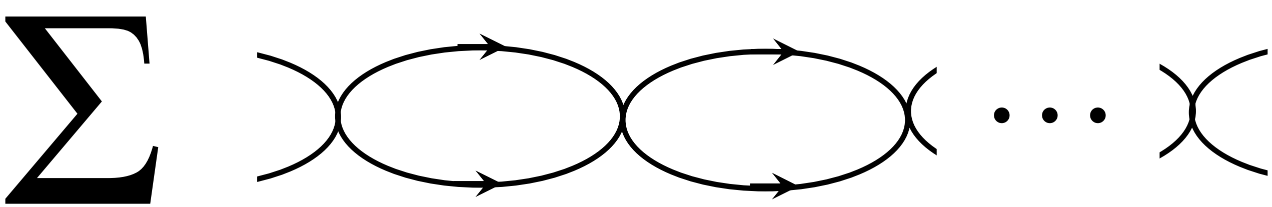
$$D(t_1-t_2,x_1-x_2)D(t_2-t_3,x_2-x_3)\ldots D(t_n-t_1,x_n-x_1)=0$$

Consequences

1. Ground state energy = 0
2. One-particle Green function is not renormalized

$$\varepsilon = \frac{\lambda}{8\pi^2}\, p^2$$

3. Exact two-particle S-matrix:



Summation of bubble diagrams yields

$$S(p, p') = \frac{p' - p + ipp'}{p' - p - ipp'}$$

Bethe equations

$$\mathrm{e}^{\,ip_j} = \prod_{k \neq j} \frac{p_k - p_j + ip_k p_j}{p_k - p_j - ip_k p_j}$$

$$e^{\frac{i}{u_j}} = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}$$

Sklyanin'88

$$P = \sum_j \frac{1}{u_j}, \qquad E = \frac{\lambda}{8\pi^2} \sum_j \frac{1}{u_j^2}$$

Heisenberg model: the same equations with

$$\frac{1}{u} \to \pi - 2\arctan u, \qquad \frac{1}{u^2} \to \frac{1}{u^2 + \frac{1}{4}}$$

The difference disappears
in the low-energy (u→∞) limit.

AAF model

SU(1|1) sector:

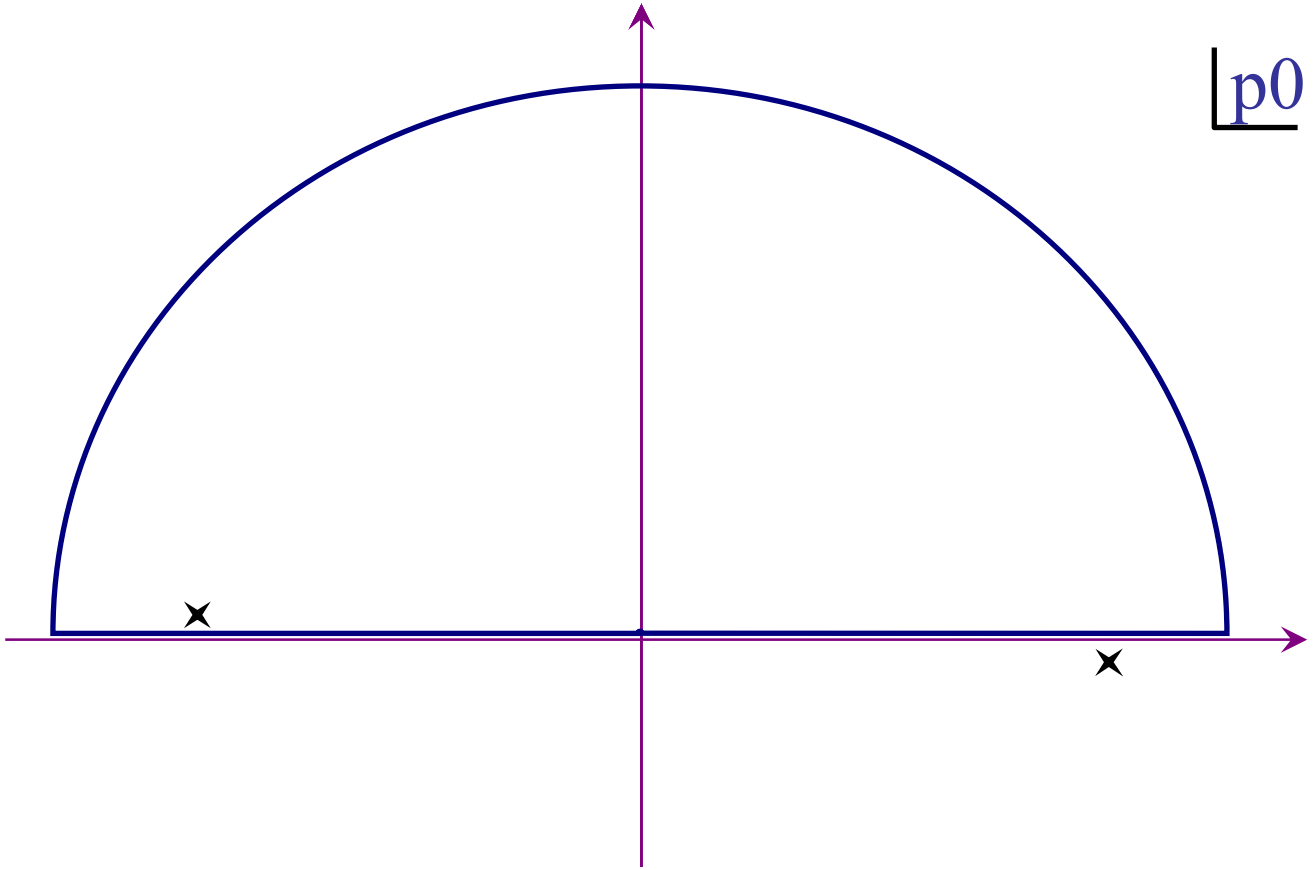
- in SYM: $\mathrm{tr}\, Z^{L-M/2}\Psi^{M}$

Callan,Heckman,McLoughlin,Swanson'04

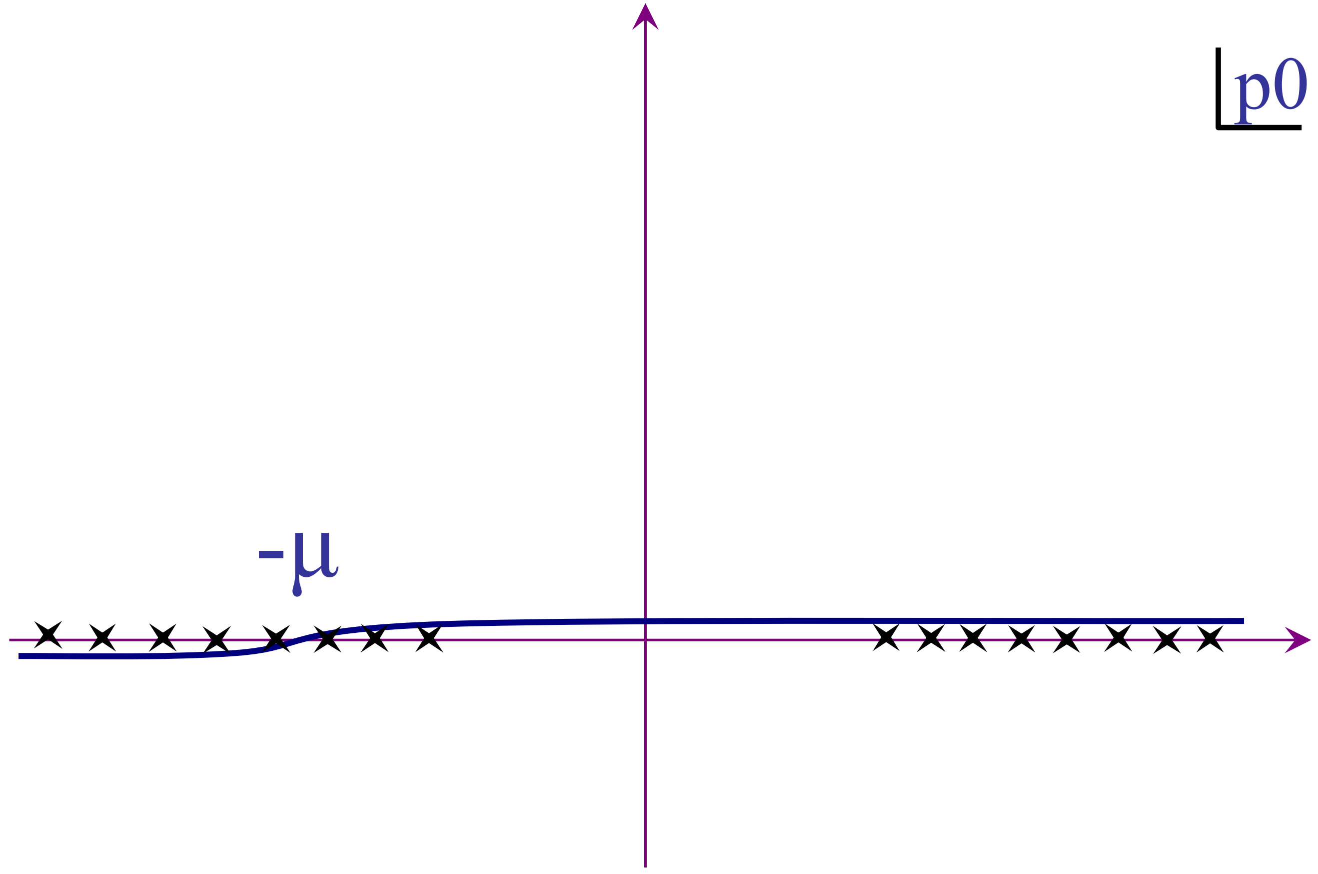
- in string theory:

$$S = \int d^2x \left[\bar{\psi}\left(i\gamma^a\partial_a - m\right)\psi + \frac{g}{4m^2}\,\varepsilon^{ab}\left(\bar{\psi}\partial_a\psi\,\bar{\psi}\gamma^3\partial_b\psi - \partial_a\bar{\psi}\psi\,\partial_b\bar{\psi}\gamma^3\psi\right) - \frac{g^2}{8m^4}\,\varepsilon^{ab}\left(\bar{\psi}\psi\right)^2\partial_a\bar{\psi}\gamma^3\partial_b\psi\right]$$

Alday,Arutyunov,Frolov'05



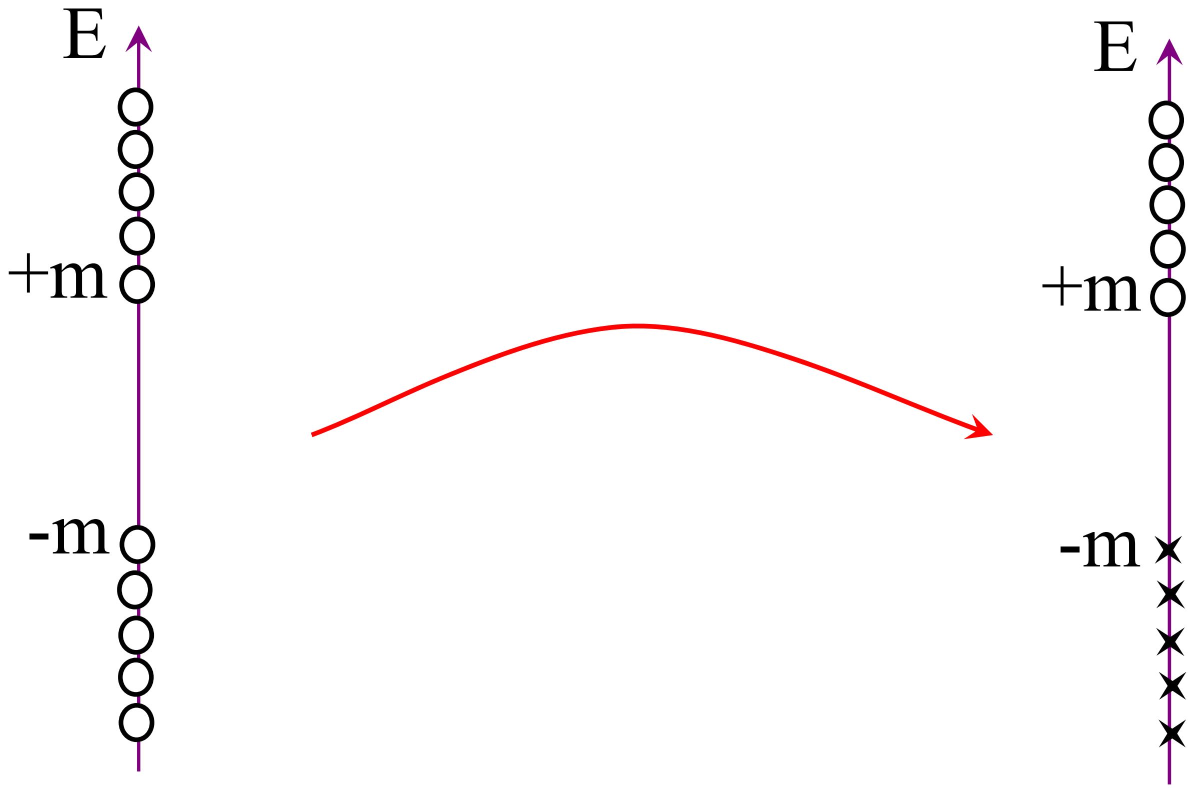
$$S(p) = \frac{\not{p} + m}{(p_0 + i\epsilon)^2 - p_1^2 - m^2}$$



μ – chemical potential

$\mu \to -\infty$ ⇨ All poles are below the real axis.

Empty Fermi sea: Physical vacuum:



$$\psi(x)\,|\,0\rangle = 0$$

Berezin,Sushko'65;Bergknoff,Thaker'79;Korepin'79

Lorentz invariance

$$S(p) = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon \,\mathrm{sgn}\, p_0}$$

$\mathrm{sgn}\, p_0$ is invariant under orthochronous Lorentz trnasformation

$\Downarrow$

S-matrix is Lorentz invariant

Exact S-matrix

$$S(\theta, \theta') = \frac{1 + ig \sinh(\theta - \theta')}{1 - ig \sinh(\theta - \theta')}$$

θ – rapidity:

$$\begin{aligned} p_1 &= m \sinh \theta \\ p_0 &= m \cosh \theta \end{aligned}$$

Bethe ansatz

$$e^{imL\sinh\theta_j} = \prod_{k\neq j} \frac{1 + ig\sinh(\theta_j - \theta_k)}{1 - ig\sinh(\theta_j - \theta_k)}$$

$$E = m\sum_j \cosh\theta_j$$

Im θ_j=0: positive-energy states

Im θ_j=π: negative-energy states

Naïve weak-coupling limit

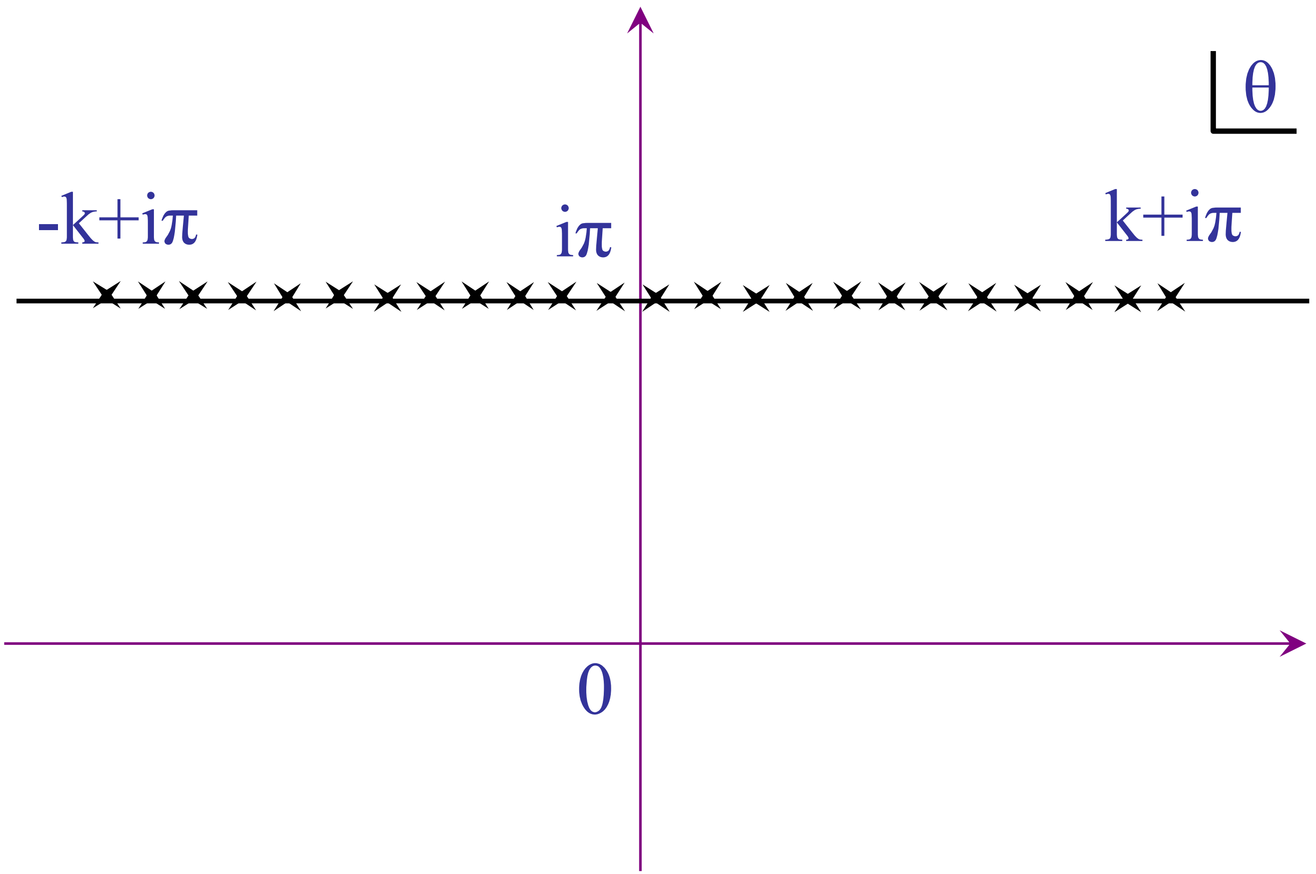
g⟶0:

$$e^{imL\sinh\theta_j} = \prod_{k\neq j} e^{2ig\sinh(\theta_j-\theta_k)}$$

Staudacher'04

Reproduces BMN (Berenstein,Maldacena,Nastase'02) and near-BMN (Callan,Lee,McLoughlin,Schwarz,Swanson,Wu'03) spectrum of the string.

Ground state



$\Lambda = m\,\mathrm{e}^{\,k}$ - UV cutoff

Mass renormalization: $M = m\Lambda^{\nu(g)}$

- Weak-coupling $(-1<g<1)$:

  Non-renormalizable

  (anomalous dimension of mass is complex)

- Strong attraction $(g>1)$

  Unstable

  (Energy unbounded below)

- Strong repulsion $(g<-1)$:

  Spectrum consists of fermions and anti-fermions with non-trivial scattering

Faddeev-Reshetikhin model

String on $S^3 x R^1$

$$S = \int d^2x \left[\bar{\varphi}\,(i\,\not{D} - m)\,\varphi - g\,\bar{\varphi}\frac{1+\gamma^3}{2}\varphi\,\bar{\varphi}\frac{1-\gamma^3}{2}\varphi + O(\varphi^6) \right]$$

$$\begin{aligned} D_0 &= \partial_0 + im - \frac{ig}{4}\,\bar{\varphi}\varphi \\ D_1 &= \partial_1 \end{aligned}$$

φ is a bosonic spinor

The S-matrix

$$S(\theta,\theta') = \frac{1 + ig\left(\frac{\cosh\frac{\theta+\theta'}{2}}{\sinh\frac{\theta-\theta'}{2}} - \coth\frac{\theta-\theta'}{2}\right)}{1 - ig\left(\frac{\cosh\frac{\theta+\theta'}{2}}{\sinh\frac{\theta-\theta'}{2}} - \coth\frac{\theta-\theta'}{2}\right)}$$

not Lorentz invariant

Conclusions

- Quantum corrections drastically affect the physics through dimensional transmutation
- Potentially, the weak-coupling and the strong-coupling regimes are separated by a phase transition
- What to do with negative-energy states?
    - ✓They do appear in the SYM in all-loop Bethe ansätze!

      Beisert'05;Rej,Serban,Staudacher'05

        - project them out Rej,Serban,Staudacher'05
        - fill negative levels → anti-particles