http://www.physics.utoronto.ca/~peet/online/pitp/

Also, refer to Clifford Johnson's D-book!

Lecture 2.

Supergravity and Quantum Numbers

Concepts introduced:

- supergravity as low-energy string theory
- Hodge duality and branes
- conserved quantum numbers
- supersymmetry algebra and central charges
- brane web: S, T and M/IIA duality
- dimensional reduction
- solution-generating
- the basic BPS M-branes and D-branes

Today is all classical physics. Tomorrow we worry about quantum.

Want to know which BH and black p-branes occur in string theories. Study low-energy string theory, a.k.a. supergravity (SUGRA).

How is SUGRA Lagrangian derived? Couple in SUGRA fields to string σ-model and demand conformal invariance. Two-dimensional field theory β-functions yield conditions on spacetime fields. Reconstruct action.

For simplicity, we discuss only Type IIA and IIB SUGRAs. These possess $\mathcal{N}$=2 supersymmetry in $d = 10$, i.e. 32 real supercharges. Type IIB is chiral: its two Majorana-Weyl 16-component spinors have the same chirality, while IIA is nonchiral as its two spinors have opposite chirality.

There are two sectors of massless modes of Type-II strings: NS-NS and R-R. In NS-NS sector we have string metric* $G_{\mu\nu}$, two-form potential B_2, and scalar dilaton Φ.

*Not to be confused with Einstein tensor, which we never use here.

In R-R sector we have n-form potentials C_n, n even for IIB and odd for IIA. For Type IIA, the independent R-R potentials can be chosen to be C_1, C_3.

Bosonic part of IIA SUGRA action is:

$$\begin{aligned} S_A \;=\; & \frac{1}{(2\pi)^7 l_s^8}\Big(\int d^{10}x\sqrt{-G}\left\{\frac{e^{-2\Phi}}{g_s{}^2}\left[R_G + 4\,(\partial\Phi)^2 - \frac{1}{2}\,|dB_2|^2\right]\right. \\ & \left. -\frac{1}{2}\,|dC_1|^2 - \frac{1}{2}\,|dC_3 - dB_2 \wedge C_1|^2\right\} - \frac{1}{2}\int dC_3 \wedge dC_3 \wedge B_2\Big) \end{aligned} \tag{1}$$

We have shifted dilaton field so that it is zero at infinity. We have used conventions of Polchinski.

Funny cross- and 'Chern-Simons' terms, such as $dC_3 \wedge dC_3 \wedge B_2$, are required by supersymmetry. In some simple cases there is a consistent truncation to an action without cross terms, but compatibility with field equations has to be checked in every case!

What kinds of objects naturally carry charges of NS-NS and R-R gauge fields (B_2, C_n)?

Recall $d = 4$ QED: electrically charged particle couples to A_1

$$\int_{\text{worldline}} d\tau A_\mu \frac{dx^\mu}{d\tau} = \int_{\text{worldline}} A_\mu dx^\mu = \int_{\text{worldline}} A_1 \tag{2}$$

which has field strength F_2.

Hodge dual field strength ${}^*F_2$ gives magnetic coupling to point particles.

By analogy, p-brane in d=10 couples electrically to $C_{n=p+1}$,

$$\int_{\text{worldline}} A_1 \longrightarrow \int_{\text{worldvolume}} C_{p+1} \tag{3}$$

or magnetically to C_{7-p}:

$$Dp : C_{p+1} \Rightarrow F_{p+2} \Rightarrow \tilde{F}_{8-p} \Rightarrow \tilde{C}_{7-p} \tag{4}$$

Get 1-branes "F1" and 5-branes "NS5" coupling to NS-NS potential B_2, and p-branes "Dp" coupling to R-R potentials.

Note: cannot allow both electric and magnetic potentials in same Lagrangian, as it would result in propagating ghosts.

For Type IIB string theory, R-R 5-form field strength is self-dual, and so there is no covariant action from which field equations can be derived. Can define a "pseudo-action" which can be used only if remember to impose self-duality condition as an equation of motion. See e.g. Myers hep-th/9910053; Kallosh et al. hep-th/0103233.

Not all aspects of physics of R-R gauge fields can be gleaned from action / equations of motion given for IIA and IIB above. D-branes aren't all cohomology! Subtle effects involve charge quantisation, self-duality, and K-theory. We will stick to putting branes on flat spaces, and for us these effects will not be noticeable.

We will however note one aspect of this beautiful story. It is that a Lorentz-invariant partition function (but not action) can be written down, for IIA and IIB - with no propagating ghosts! Leading term in partition function is $\sim$ theta-function, where the sum is done over K-theory classes... but only half end up contributing (self-duality condition).

See Diaconescu, Moore, Witten.

Conserved quantum numbers?

Energy: if there is a rest frame available, becomes mass M.

Angular momentum: in D-dim, have skew matrix $J^{[\mu\nu]}$ with $[\frac{1}{2}(D-1)]$ eigenvalues. These are independent angular momenta, J_i.

Last type of conserved quantity couples to long range R-R (or NS-NS) gauge field; it is gauge charge Q_p.

Low-energy approximation to string theory yielded supergravity actions. When a p-brane is present, it sources SUGRA fields. Get additional term in action, encoding collective modes of p-brane:

$$S = S_{\mathsf{SUGRA}} + S_{\mathsf{brane}}\,; \tag{5}$$

This combined action is well-defined for classical string theory. For fundamental quantum string theory, a different representation of degrees of freedom would be necessary.

S_{brane} is an integral over only $p{+}1$ dimensions of p-brane worldvolume, while first term is an integral over $d{=}10$ bulk. Varying action w.r.t. bulk SUGRA fields gives δ-function sources on RHS of SUGRA eqns of motion. Varying w.r.t. brane fields gives brane equation of motion.

Let us consider mass and angular momenta first. In $d{=}10$, p-branes of codimension smaller than 3 give rise to spacetimes which are not asymptotically flat; there are not enough space dimensions to allow fields to have Coulomb tails. We will cover only $p < 7$ for lack of time.

Mass for an isolated gravitating system can be defined by referring its spacetime to one which is nonrelativistic and weakly gravitating. This is always done in Einstein frame,

$$S = \int d^D x \left(\frac{\sqrt{-g} R_g}{16\pi G_D} + \mathcal{L}_{\text{matter}} \right) \tag{6}$$

where Einstein metric g is given in terms of string metric G as

$$g_{\mu\nu} = e^{-4\Phi/(D-2)} G_{\mu\nu} \tag{7}$$

In string frame IIA action, dilaton field had the "wrong-sign" kinetic term! Don't panic: it becomes "right-sign" in Einstein frame.

Field equation for Einstein metric is in D dimensions

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G_D T^{(\mathrm{m})}_{\mu\nu} \tag{8}$$

where $R_{\mu\nu}$ is Ricci tensor and $T^{(\mathrm{m})}_{\mu\nu}$ is energy-momentum tensor. Far away, metric becomes flat. Let us linearise about Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{9}$$

i.e. consider only first order terms in deviation h. (To this order, raise and lower indices with Minkowski metric.)

We also impose condition that system be non-relativistic, so time derivatives can be neglected and $T_{00} \gg T_{0i} \gg T_{ij}$.

Under coordinate transformations $\delta x^\mu = \xi^\mu$, metric deviation h still transforms: $\delta h_{\mu\nu} = -2\partial_{(\mu}\xi_{\nu)}$. (Partially) fix this symmetry:

$$\partial_\nu \left(h^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} h^\lambda_\lambda \right) = 0 \tag{10}$$

Harmonic gauge condition. Field equation for h becomes

$$\left(\partial^i \partial_i\right) h_{\mu\nu} = 16\pi G_D \left[T^{(\mathrm{m})}_{\mu\nu} - \frac{1}{(D-2)} \eta_{\mu\nu} T^{(\mathrm{m})\ \lambda}_{\lambda} \right] \equiv -16\pi G_D \tilde{T}_{\mu\nu} \tag{11}$$

This is a Laplace equation, with solution

$$h_{\mu\nu}(x) = \frac{16\pi G_D}{(D-3)\Omega_{D-2}} \int d^{D-1}\vec{y}\ \frac{\tilde{T}_{\mu\nu}\left(|\vec{x}-\vec{y}|\right)}{|\vec{x}-\vec{y}|^{D-3}} \tag{12}$$

where prefactor comes from Green's function and $\Omega_n =$area(S^n).
Now let us expand this in moments,

$$\begin{aligned} h_{\mu\nu}(x) = \quad & \frac{16\pi G_D}{(D-3)\Omega_{D-2}} \left[\frac{1}{r^{D-3}} \int d^{D-1} y \tilde{T}_{\mu\nu}(y) \right. \\ & \left. + \frac{x^j}{r^{D-1}} \int d^{D-1} y y^j \tilde{T}_{\mu\nu}(y) + \cdots \right] \end{aligned} \tag{13}$$

On other hand, definitions of ADM linear and angular momenta are

$$P^\mu = \int d^{D-1} y T^{\mu 0} \qquad\qquad J^{\mu\nu} = \int d^{D-1} y \left(y^\mu T^{\nu 0} - y^\nu T^{\mu 0} \right) \tag{14}$$

(No tildes here.)
Simplest in rest frame; read off mass and angular momenta

$$
\begin{aligned}
g_{tt} &\longrightarrow -1 + \frac{16\pi G_D}{(D-2)\Omega_{D-2}} \frac{M}{r^{D-3}} + \cdots; \\
g_{ij} &\longrightarrow 1 + \frac{16\pi G_D}{(D-2)(D-3)\Omega_{D-2}} \frac{M}{r^{D-3}} + \cdots; \\
g_{ti} &\longrightarrow \frac{16\pi G_D}{\Omega_{D-2}} \frac{x^j J^{ji}}{r^{D-1}} + \cdots
\end{aligned}
\tag{15}
$$

What about gauge charges? Need bulk+brane action. For Dp-branes,

$$
S_{\mathsf{brane}} = -\frac{1}{(2\pi)^p {\ell_{\mathsf{s}}}^{p+1}} \int C_{p+1} + \ldots \tag{16}
$$

For a single type of brane, consistent to ignore funny cross-terms

$$
S_{\mathsf{SUGRA}} = \frac{-1}{2} \frac{1}{(2\pi)^7 {\ell_{\mathsf{s}}}^8} \int d^{10}x \sqrt{-G} \left| dC_{p+2} \right|^2 + \ldots \tag{17}
$$

Hence field equation for potential C is

$$
d^* \left(dC_{p+1} \right) = (2\pi)^7 {\ell_{\mathsf{s}}}^8 {}^* \left(J_{p+1} \right) \tag{18}
$$

where conserved $p+1$-form current J is

$$
J_{p+1}(x) = -\frac{1}{(2\pi)^p {\ell_{\mathsf{s}}}^{p+1}} \int dX^0 \ldots dX^p \delta^{10}(X - x) \tag{19}
$$

Physics is easiest to see in static gauge

$$X^{\mu_i}(\sigma) = \sigma^{\mu_i} \quad i = 0 \dots p \tag{20}$$

Noether charge is integral of current

$$Q_p \propto \int_{S^{8-p}} {}^*(dC)_{p+2} \tag{21}$$

(For NS-type branes, prefactor of $e^{-2\Phi}/g_s{}^2$ in integrand.)

In addition to field equation for C, there is Bianchi identity

$$d\left([dC]_{p+2}\right) = 0 \tag{22}$$

Deduce existence of a topological charge,

$$P_{7-p} \propto \int_{S^{p+2}} (dC)_{p+2} \tag{23}$$

Obey Dirac quantisation condition (see Polchinski for details)

$$Q_p P_{7-p} = 2\pi n \quad n \in \mathbb{Z} \tag{24}$$

Supersymmetry algebra is of central importance to a SUGRA theory.

Anti-commutators involving two supersymmetry generators Q are

$$\{Q_\alpha, Q_\beta\} \sim (\mathcal{C}\Gamma^\mu)_{\alpha\beta} P_\mu + a \sum_p (\mathcal{C}\Gamma^{\mu_1 \cdots \mu_p})_{\alpha\beta} Z_{[\mu_1 \ldots \mu_p]} \tag{25}$$

where $\mathcal{C}$ is charge conjugation matrix, Γ's are antisymmetrised products of gamma matrices, Z are central charges, and P^μ is momentum vector. If there is a rest frame, then for state carrying particular central charge,

$$\{Q_\alpha, Q_\beta\} \sim \left(\mathcal{C}\Gamma^0\right)_{\alpha\beta} M + a \left(\mathcal{C}\Gamma^{1\ldots p}\right)_{\alpha\beta} Z_{[1\ldots p]} \tag{26}$$

Sandwich a physical states $|\text{phys}\rangle$ around this algebra relation. State $Q|\text{phys}\rangle$ has nonnegative norm, and this leads to *Bogomolnyi bound*

$$M \geq a|Z| \tag{27}$$

This bound can also be derived by analysing supergravity Lagrangian, via Nester procedure (boundary conditions for bulk fields at infinity important).

Constant a in Bogomolnyi bound depends on SUGRA theory via gauge field couplings.

Special states saturating bound are *BPS states*.

$M = a|Z|$ not renormalized by quantum corrections (although generically both M and Z may be renormalised.

Statistical degeneracy of states is also unrenormalised.

SUSY transformations of fields have a spinorial parameter ϵ. For preserved supersymmetries, SUSY relation gives projection condition (again schematic)

$$\left(1 + [\mathrm{sgn}(Z)]\ \Gamma^{01\cdots p}\right) \epsilon = 0 \tag{28}$$

$\underline{d = 11 \text{ SUGRA}}$

Matrix $\left\{Q_\alpha, Q_\beta\right\}$ is real and symmetric, so it has $(32 \times 33)/2 = 528$ components. Belongs to adjoint representation of group $Sp(32; \mathbb{R})$. Decompose this under $d = 11$ Lorentz group $SO(1, 10)$:

$$\underline{528} \to \underline{11} \oplus \underline{55} \oplus \underline{462} \tag{29}$$

$\underline{11}$ is momentum, $\underline{55}$ is $Z_{[\mu\nu]}$, $\underline{462}$ is 5-index $Z_{[\mu\nu\lambda\sigma\rho]}$.

Separate out spatial indices i and temporal index 0.

$Z_{[ij]}$	45	M2-brane
$Z_{[ijklm]}$	252	M5-brane
$Z_{[0j]}$	10	HW domain walls (codim. 1)
$Z_{[0ijkl]}$	210	KK monopole (codim. 4)
P_i	10	M-Wave (moves at c)

Using this and 3 dualities to be introduced, can work out

$$a_{F1} \sim 1 \quad , \qquad a_{Dp} \sim \frac{1}{g_{\mathrm{s}}} \quad , \qquad a_{NS5} \sim \frac{1}{{g_{\mathrm{s}}}^2} \tag{30}$$

F1 are lightest states at $g_s \ll 1$; Dp and NS5 are solitonic.

In other regions of parameter space (such as $g_s \gg 1$), F1 will no longer be elementary $\Rightarrow$ p-*brane democracy*.

Charges $Z \in \mathbb{R}$ in SUGRA, but $Z \in \mathbb{Z}$ in string theory.

Unit conventions on tension: $\alpha' \equiv l_s^2$ and

$$\tau_{\mathrm{F1}} = \frac{1}{2\pi {\ell_{\mathrm{s}}}^2} \quad \tau_{\mathrm{D}p} = \frac{1}{g_{\mathrm{s}}(2\pi)^p {\ell_{\mathrm{s}}}^{p+1}} \quad \tau_{\mathrm{NS5}} = \frac{1}{{g_{\mathrm{s}}}^2 (2\pi)^5 {\ell_{\mathrm{s}}}^6} \tag{31}$$

In $d = 10$, Newton constant related to $g_{\mathrm{s}}, \ell_{\mathrm{s}}$ by

$$16\pi G_{10} \equiv 2\kappa_{10}^2 = (2\pi)^7 {g_{\mathrm{s}}}^2 {\ell_{\mathrm{s}}}^8 \tag{32}$$

Planck length in d dimensions, ℓ_d, is defined by

$$16\pi G_d \equiv (2\pi)^{d-3} \ell_d^{d-2} \tag{33}$$

For later convenience, define volume to have implicit 2π's in it.

If fields indep. of $(10-d)$ coords, $\int d^{10}x = \left[(2\pi)^{10-d} V_{10-d}\right] \int d^d x$. Hence

$$G_d = \frac{G_{10}}{(2\pi)^{10-d} V_{10-d}} \tag{34}$$

Reminder: Bekenstein-Hawking formula must always be computed in Einstein frame, where kinetic term for graviton is canonically normalized,

$$S_{\mathrm{grav}} = \frac{1}{16\pi G_d} \int \sqrt{-g} R_g \tag{35}$$

Figuring out constants is only one tiny part of mechanics of dimensional reduction. Consider simplest example: Kaluza-Klein story for fields in string frame, for reduction on circle of radius R.

Label d-dim system with no hats and $(d-1)$-dim system with hats. Split indices as $\{x^\mu\} = \{x^{\widehat{\mu}}, z\}$. Vielbeins decompose as

$$\left(E^a_\mu\right) = \begin{pmatrix} \widehat{E}^{\widehat{a}}_{\widehat{\mu}} & e^{\widehat{\chi}}\widehat{A}_{\widehat{\mu}} \\ 0 & e^{\widehat{\chi}} \end{pmatrix} \quad \Rightarrow \quad (G_{\mu\nu}) = \begin{pmatrix} \widehat{G}_{\widehat{\mu}\widehat{\nu}} + e^{2\widehat{\chi}}\widehat{A}_{\widehat{\mu}}\widehat{A}_{\widehat{\nu}} & e^{2\widehat{\chi}}\widehat{A}_{\widehat{\mu}} \\ \widehat{A}_{\widehat{\nu}}e^{2\widehat{\chi}} & e^{2\widehat{\chi}} \end{pmatrix} \tag{36}$$

and

$$\Phi = \widehat{\Phi} + \frac{1}{2}\widehat{\chi}\,; \tag{37}$$

which yield

$$\begin{aligned} &\frac{1}{16\pi G_d}\int d^dx\sqrt{-G}e^{-2\Phi}R_G = \\ &\frac{1}{16\pi G_{d-1}}\int d^{d-1}x\sqrt{-\widehat{G}}e^{-2\widehat{\Phi}}\left[R_{\widehat{G}} + 4(\partial\widehat{\Phi})^2 - (\partial\widehat{\chi})^2 - \frac{1}{2}e^{2\widehat{\chi}}\left|d\widehat{A}\right|^2\right] \end{aligned} \tag{38}$$

Reduction on larger tori or Calabi-Yau manifolds leads to big symmetries. e.g. $E_{(7,7)}$ for Type II on T^6, $E_{(6,6)}$ for Type II on T^5.

BPS objects are interrelated via dualities.

Type IIA $\leftrightarrow$ M-theory

11th coordinate $x^\natural$ is compactified on a circle of radius

$$R_\natural = g_\mathsf{s}\ell_\mathsf{s} \tag{39}$$

SUGRA fields decompose as

$$ds^2_{11} = e^{-2\Phi/3} dS^2_{10} + e^{4\Phi/3}\left(dx^\natural + C_{\mathit{1}\,\mu} dx^\mu\right)^2 \tag{40}$$

and

$$A_{\mathit{3}} = C_{\mathit{3}} + B_{\mathit{2}} \wedge dx^\natural \tag{41}$$

Units: $\ell_{11} = g_s^{1/3}\ell_s$.

We can turn M-theory objects into Type IIA objects by pointing them in 11th direction ($\swarrow$) or not ($\downarrow$).

$$\begin{array}{cccccccc} & \mathsf{W} & & \mathsf{M2} & & \mathsf{M5} & & \mathsf{KK} \\ \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ \mathsf{D0} & \mathsf{W} & \mathsf{F1} & \mathsf{D2} & \mathsf{D4} & \mathsf{NS5} & \mathsf{D6} & \mathsf{KK} \end{array} \tag{42}$$

S-duality of IIB

IIB SUGRA has SL(2,$\mathbb{R}$) symmetry; SL(2,$\mathbb{Z}$) in full string theory. Define

$$\lambda \equiv C_0 + ie^{-\Phi} \quad \text{and} \quad H_3 \equiv \begin{pmatrix} dB_2 \\ dC_2 \end{pmatrix} \tag{43}$$

$$\text{Under } U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \quad H \to U\,H \quad \lambda \to \frac{a\lambda + b}{c\lambda + d} \tag{44}$$

$d = 10$ Einstein metric and $F_5^{(+)}$ are invariant.

Commonly considered $\mathbb{Z}_2$ subgroup obtains when $C_0 = 0$.
$\mathbb{Z}_2$ flips sign of Φ, and exchanges B_2 and C_2. Result:

$$D1 \leftrightarrow F1 \qquad D5 \leftrightarrow NS5\,; \tag{45}$$

All others such as W and KK are unaffected, and D3 goes into itself. Effect of this $\mathbb{Z}_2$ on units is

$$\tilde{g}_s = \frac{1}{g_s} \quad , \qquad \tilde{g}_s^{\frac{1}{4}}\tilde{\ell}_s = g_s^{\frac{1}{4}}\ell_s \tag{46}$$

T-duality

Operation of T-duality on a circle switches winding and momentum modes of fundamental strings (F1) and exchanges Type IIA and IIB.

$$\frac{\tilde{R}}{\tilde{\ell}_{\mathsf{s}}} = \frac{\ell_{\mathsf{s}}}{R} \quad , \qquad \frac{\tilde{g}_{\mathsf{s}}}{\sqrt{\tilde{R}/\tilde{\ell}_{\mathsf{s}}}} = \frac{g_{\mathsf{s}}}{\sqrt{R/\ell_{\mathsf{s}}}} \quad , \qquad \tilde{\ell}_{\mathsf{s}} = \ell_{\mathsf{s}} \tag{47}$$

T-duality acting $\perp$ to Dp gives Dp+1, acting $\|$ gives D$p-1$. Acting on isometry direction of KK gives NS5. Everything else is unaffected.

Let z be isometry direction. Then T-duality acts on NS-NS fields as

$$\begin{aligned} e^{2\tilde{\Phi}} =&\ e^{2\Phi}/G_{zz} \quad \tilde{G}_{zz} = 1/G_{zz} \quad \tilde{G}_{\mu z} = B_{\mu z}/G_{zz} \quad \tilde{B}_{\mu z} = G_{\mu z}/G_{zz} \\ \tilde{G}_{\mu\nu} =&\ G_{\mu\nu} - \left(G_{\mu z}G_{\nu z} - B_{\mu z}B_{\nu z}\right)/G_{zz} \\ \tilde{B}_{\mu\nu} =&\ B_{\mu\nu} - \left(B_{\mu z}G_{\nu z} - G_{\mu z}B_{\nu z}\right)/G_{zz} \end{aligned} \tag{48}$$

T-duality also acts on R-R fields; see e.g. Myers.

In some configurations, string momentum or winding number may not be conserved, e.g. winding in KK background. Conserved quantities still transform as expected under T-duality, though.

Solution-generating

Consider neutral black hole in d dimensions

$$d\hat{S}_d^2 = -\left(1 - K(r)\right) dt^2 + \left(1 - K(r)\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \tag{49}$$

where

$$K(r) \equiv \left(\frac{r_H}{r}\right)^{d-3} \tag{50}$$

No gauge field(s) or dilaton; solution in string and Einstein frame. Mass

$$M_d = \frac{(d-2)\Omega_{d-2} {r_H}^{d-3}}{16\pi G_d} \tag{51}$$

Lift procedure: simply tensor this with $\mathbb{R}$ (z direction); automatically satisfies $d+1$ dimensional Einstein equations

$$\begin{aligned} dS_d^2 = &\ dz^2 - \left(1 - K(r)\right) dt^2 + \left(1 - K(r)\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \\ = &\ \left(-dt^2 + dz^2\right) + K(r) dt^2 + \left(1 - K(r)\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \end{aligned} \tag{52}$$

Now do a boost on this configuration:

$$\begin{pmatrix} dt \\ dz \end{pmatrix} \to \begin{pmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{pmatrix} \begin{pmatrix} dt \\ dz \end{pmatrix} \tag{53}$$

This boost transformation takes solutions to solutions (check by substituting into equations of motion). Metric here becomes

$$\begin{aligned} dS_d^{2\,\prime} = & \left(-dt^2 + dz^2\right) + K(r)\left(\cosh\gamma dt + \sinh\gamma dz\right)^2 \\ & + \left(1 - K(r)\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \\ = & -dt^2\left(1 - K(r)\cosh^2\gamma\right) + dz^2\left(1 + K(r)\sinh^2\gamma\right) \\ & + 2dtdz\cosh\gamma\sinh\gamma K(r) + \left(1 - K(r)\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \end{aligned} \tag{54}$$

Horizon, at $G^{rr} \to 0$, occurs when $K(r) = 1$ i.e. at $r = r_H$
(not when $G_{tt} = 0$)

Now let us KK down again to make new d dimensional black hole. Using our previous relations

$$\begin{aligned} dS_d^2 = & \; d\widehat{S}_d^2 + e^{2\widehat{\chi}}\left(dz + \widehat{A}_\mu dz^\mu\right)^2 \\ e^\Phi = & \; e^{\widehat{\Phi} + \frac{1}{2}\widehat{\chi}} \end{aligned}$$

so, for example,

$$\widehat{G}_{tt} = G_{tt} - G_{tz}^2/G_{zz} = -1 + K\cosh^2\gamma - \frac{(K\cosh\gamma\sinh\gamma)^2}{(1 + K\sinh^2\gamma)} \tag{55}$$

From this we obtain

$$d\widehat{S}_d^2\ ' = \frac{-\left(1-K(r)\right)}{\left(1+K(r)\,\mathrm{sinh}^2\gamma\right)}dt^2 + \frac{1}{\left(1-K(r)\right)}dr^2 + r^2 d\Omega_{d-2}^2 \tag{56}$$

and

$$\widehat{A}_t = \frac{K(r)\,\mathrm{cosh}\gamma\,\mathrm{sinh}\gamma}{\left(1+K(r)\,\mathrm{sinh}^2\gamma\right)} \tag{57}$$

and

$$e^{\widehat{\Phi}} = e^{-\frac{1}{2}\widehat{\chi}} = \left(1+K(r)\,\mathrm{sinh}^2\gamma\right)^{-\frac{1}{4}} \tag{58}$$

Conserved quantum numbers of this new spacetime are

$$\begin{aligned} M' &= \frac{\Omega_{d-2}{r_H}^{d-3}}{16\pi G_d}\left[(d-2)+(d-3)\,\mathrm{sinh}^2\gamma\right] \\ Q' &= R\frac{\Omega_{d-2}{r_H}^{d-3}}{16\pi G_d}\left[\tfrac{1}{2}\,\mathrm{sinh}(2\gamma)\right] \end{aligned} \tag{59}$$

To regain original neutral black hole, we simply take limit $\gamma \to 0$.

Note: boost parameter $\gamma \in [0, \infty)$. Continuous only in SUGRA.

Now consider taking opposite limit $\gamma \to \infty$. To keep stuff finite, $r_H \to 0$ such that

$$\frac{1}{2}{r_H}^{d-3} e^{2\gamma} \equiv k = \text{fixed} \qquad \text{so } K(r) = \frac{k}{r^{d-3}} \tag{60}$$

Lifting and defining light-cone coords $dz^{\pm} \equiv (t \pm z)/\sqrt{2}$

$$dS^2_{d+1} = -2dz^+ \left[dz^- - \frac{k}{r^{d-3}} dz^+ \right] + \left(dr^2 + r^2 d\Omega^2_{d-2} \right) \tag{61}$$

This is gravitational wave W, which has zero ADM mass in $d+1$ dimensions.

Taking same $\gamma \to \infty$ limit for d dimensional black hole gives extremal black hole, which has zero Hawking temperature. The connection to the Wave comes via

$$M^2_{d+1} = 0 = M^2_d - \frac{Q^2}{R^2} \tag{62}$$

d-dimensional charge is z-component of the d-dimensional momentum.

Wave W is one of purely gravitational BPS objects in string theory. other is KK monopole.

Labelling five longitudinal directions $y_{1\cdots5}$, and four transverse directions $x_i, i = 1,2,3$, and z; KK metric is

$$\begin{aligned} &ds^2 = -dt^2 + dy_{1\cdots5}^2 + H^{-1}(x)\left(dz + A_i dx^i\right)^2 + H(x)dx_{1\cdots3}^2 \\ &2\partial_{[i}A_{j]}(x) = \epsilon_{ijk}\partial_k H(x) \end{aligned} \tag{63}$$

A_i can be found via curl equation, given that $H = 1 + k/|x|$. Periodicity of azimuthal angle must be 4π to avoid conical singularities.

Most bulk gauge fields in string theory are sourced by higher-d branes. Here, discuss objects with translational symmetry in p spatial directions. Thus, horizon (for zero angular momenta) is topologically $\mathbb{R}^p \times S^{q-1}$, where q is number of space dimensions transverse to p-brane.

Type IIA string theory in strong coupling limit is eleven-dimensional supergravity, which has only two fields in its bosonic sector, metric tensor and three-form gauge potential. We start our discussion of branes with BPS M-branes.

Since have A_3 expect M2 electric, M5 magnetic.

BPS M-brane and D-brane solutions

BPS M2-brane spacetime has worldvolume symmetry group $SO(1,2)$, and transverse symmetry group $SO(8)$. Let us define coords parallel and perpendicular to brane to be $(t, x_{\|})x_{\perp}$, respectively. Then, using these symmetries, spacetime metric depends only on $|x_{\perp}| \equiv r$, and has form

$$ds_{11}^2 = H_2^{-2/3} dx_{\|}^2 + H_2^{1/3} dx_{\perp}^2 \quad A_{012} = H_2^{-1} \tag{64}$$

Fact that same function appears in metric and gauge field is a consequence of supersymmetry. Note: metric is automatically in Einstein frame because there is no string frame in d=11.

Important: supersymmetry alone is not enough to determine H; rather, SUGRA equations of motion must be used. Find that H_2 must be harmonic; it satisfies a Laplace equation in $x_{\perp}$. Solution:

$$H_2 = 1 + \frac{r_2^6}{r^6} \quad \text{where } r_2^6 = 32\pi^2 N_2 \ell_{11}^6 \tag{65}$$

$\ell_{11} = g_s^{1/3} \ell_s$ is eleven-dimensional Planck length.

BPS M5-brane has symmetry group $SO(1,5) \times SO(5)$, and metric is

$$ds_{11}^2 = H_5^{-1/3} dx_{\|}^2 + H_5^{2/3} dx_{\perp}^2 \tag{66}$$

and harmonic function is this time

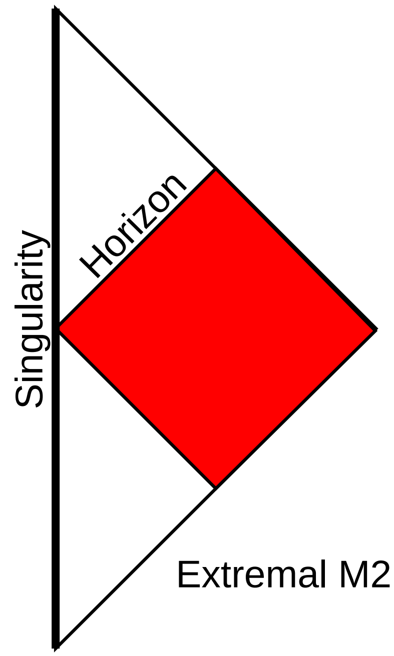
$$H_5 = 1 + \frac{r_2^3}{r^3} \quad \text{where } r_5^3 = \pi N_5 \ell_{11}^3 \tag{67}$$

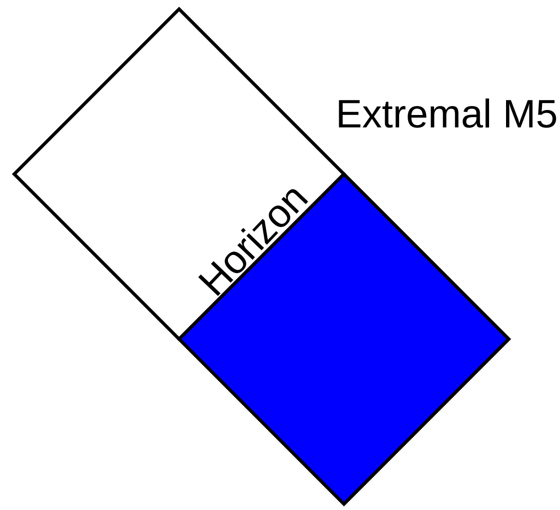
In this case, gauge field is magnetically coupled, F_4 is proportional to volume element on S^4 transverse to the M5-brane.

For M2, origin of coordinates $r = 0$ is singular and so there must be a δ-function source there, to wit fundamental M2-brane. This happens essentially because M2-brane is electrically coupled.

Magnetically coupled BPS M5-brane is 'solitonic' and nonsingular – that geometry admits maximal analytic extension without singularities. However, nonextremal version of M5 *does* have a singularity and needs a source.

Near-horizon, M2 spacetime is $AdS_4 \times S^7$ and the M5 is $AdS_7 \times S^4$. Since M2 and M5 are asymptotically flat, again we have interpolation between two maximally supersymmetric vacua as in case of Reissner-Nordström black hole.





Penroses ↑. Note that isotropic coordinates $x_\perp$ cover only shaded part of maximally extended spacetime.

Let us now move down to ten dimensions. Symmetry for BPS Dp-branes is $SO(1,p) \times SO(9-p)$. In string frame, solutions are

$$\begin{aligned} dS^2 &= H_p(r)^{-\frac{1}{2}} \left(-dt^2 + dx_\parallel^2\right) + H_p(r)^{+\frac{1}{2}} dx_\perp^2 \\ e^\Phi &= H_p(r)^{\frac{1}{4}(3-p)} \\ C_{01\cdots p} &= g_s{}^{-1} H_p(r)^{-1} \end{aligned} \tag{68}$$

Function $H_p(r)$ is harmonic; it satisfies $\partial_\perp^2 H_p(r) = 0$,

$$H_p = 1 + \frac{c_p g_{\mathrm{s}} N_p {\ell_{\mathrm{s}}}^{7-p}}{r^{7-p}} \qquad c_p \equiv (2\sqrt{\pi})^{(5-p)} \Gamma\left[\tfrac{1}{2}(7-p)\right] \tag{69}$$

Note that function H_p would still be harmonic if 1 were missing. Asymptotically flat part of geometry would be absent for this solution.

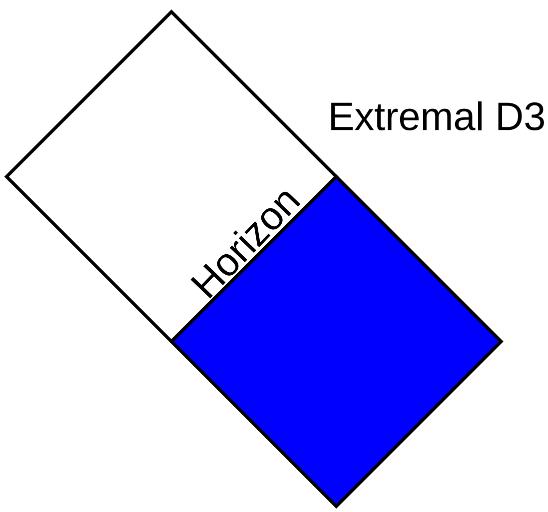
Double horizon of Dp-brane geometry occurs at $r = 0$ in isotropic coords. In every case except D3-branes, singularity at $r = 0$ as well. Hence, for Dp-branes with $p \neq 3$, singularity is null.

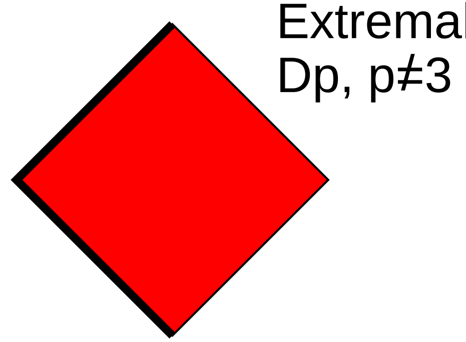
Since singularity and horizons coincide for BPS Dp-branes, may worry that singularity is not properly hidden behind an event horizon, in which case it should be classified as naked. We therefore demand that a timelike or null geodesic coming from infinity should not be able to bang into singularity in finite affine parameter.

Interestingly, this condition separates out D6-brane from others as being only one possessing a naked singularity.

For D3-brane dilaton is constant, and spacetime turns out to be totally nonsingular: all curvature invariants are finite everywhere. This allows a smooth analytic extension inside the horizon, like case of M5-brane. Near-horizon D3-brane spacetime is $AdS_5 \times S^5$. Penrose diagram for the D3 is like that of M5.

Causal structures of BPS Dp-branes are summarised in Penroses $\downarrow$; isotropic coordinates $x_\perp$ cover only shaded part of maximally extended spacetime.





F1 and NS5 spacetimes may be found by using T- and S-duality formulæ that we gave in last subsection. (*Try this!*) Ω