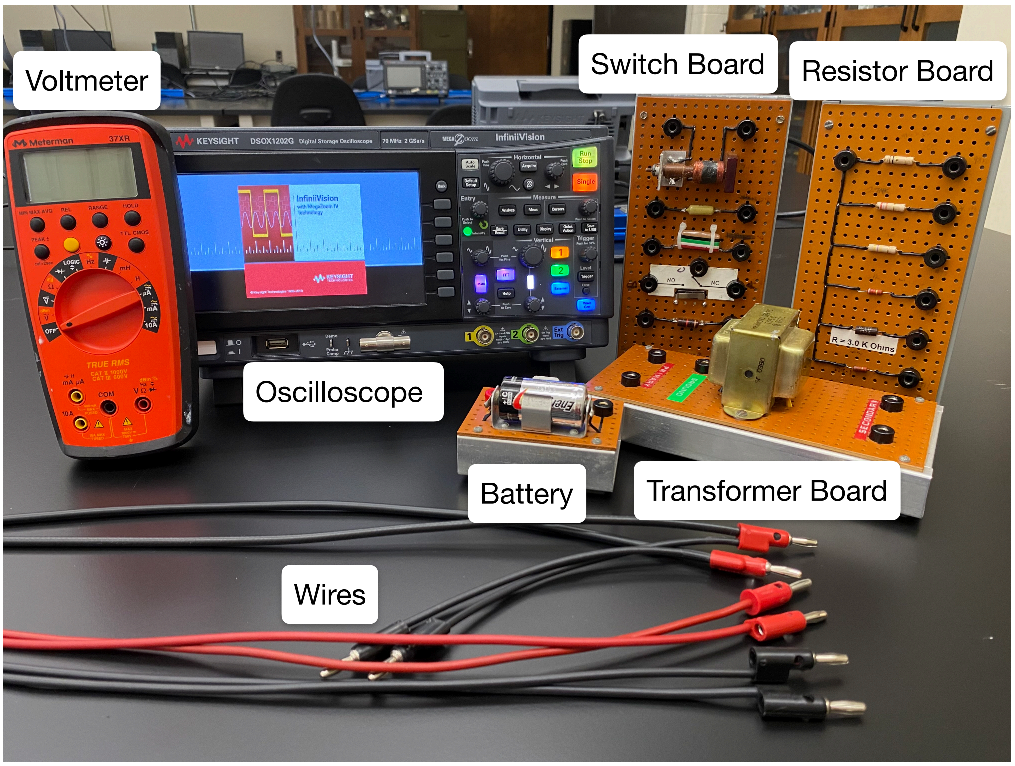
Physics
UNIVERSITY OF TORONTO

Inductors and Capacitors in AC Circuits



Revisions

2023	C. Lee
2022	T. Vahabi, E. Horsley, C. Dale, A. Harlick
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2006	R. M. Serbanescu
2005	R. M. Serbanescu

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A USB flash drive is useful for saving data from the oscilloscope in this lab. Make sure to bring one with you!

Introduction

The goal of this lab is to look at the behaviour of inductors and capacitors. In AC circuits currents vary in time, therefore we have to consider variations in the energy stored in electric and magnetic fields of capacitors and inductors, respectively.

You are already familiar with **resistors**, where the voltage-current relation is given by Ohm's law:

$$V_R(t) = RI(t)\,, \tag{1}$$

In an **inductor**, the voltage is proportional to the rate of change of the current.

$$V_L(t) = L\frac{dI(t)}{dt}\,, \tag{2}$$

where the inductance L is the measure of the components ability to resist current changes, in units of *henries*. A coil of wire is an example of an inductor, where the current flowing through the cause induces a magnetic field and voltage in the opposite direction (Lenz's law).

In a **capacitor**, the voltage is proportional to the charge difference across the component.

$$V_C(t) = \frac{Q(t)}{C} = \frac{1}{C}\int_0^t I(t')dt'\,, \tag{3}$$

where the capacitance C is the measure of the components ability to store electric charge, in units of *Farads*. A simple example of a capacitor is a pair of parallel plates separated by a small distance, where a charge is applied to one of the plates causing a charge difference and inducing a voltage across the gap between the plates. The potential difference between the plates depends on the charge difference Q, which can also be written as the integral over time of the current flowing into/out of the capacitor.

Transient Behaviour

In this first section, you will look at how circuits with these components behave when an applied DC voltage is switched from one value to another.

Background

The transient behaviour can be derived by using Kirchhoff's law and solving the resulting differential equation. For example, in a circuit with a capacitor and a resistor (known as an RC circuit), with some constant voltage V applied, we obtain

$$V - RI(t) - \frac{1}{C}Q(t) = 0\,. \tag{4}$$

Since we know that $I(t) = dQ(t)/dt$, this can be rewritten as

$$\frac{dQ(t)}{dt} + \frac{Q(t)}{RC} = \frac{V}{R}\,. \tag{5}$$

The solution to this equation is given by

$$Q(t) = Ae^{-t/RC} + VC\,, \tag{6}$$

where A is a constant determined by the initial conditions. The first term is a transient term that represents charging or discharging the capacitor. After a long enough time ($t \to \infty$) this term will disappear. The second term is the steady state solution that will be reached once the capacitor is fully charged or discharged.

Derive an equation for the voltage and current through a circuit made of a resistor and inductor in series (an RL circuit), and for a circuit made of a capacitor and inductor in series (an LC circuit).

Both circuits should give time dependent equations. What are the time constants for these circuits and what do they describe?

Experiments

Use the oscilloscope to see the voltage changing with time for these experiments. There are (at least) two channels, so you can measure the applied voltage and the voltage across a component of interest at the same time. You can also *Trigger* the oscilloscope from a separate source if needed. This data can then be saved onto a USB flash drive for analysis on the computer, or measured on the oscilloscope using the `Measure` tools.

Remember that both of the oscilloscope channels share a common ground. That means that no matter what circuit you build, the negative sides (typically the black banana plugs) are connected. If you put them in different locations in the circuit, that's like wiring those locations together, shorting your circuit!

Experiment 1

For the first experiment, we will use a manual switch to change the applied voltage between a battery (1.5 V) and no applied voltage (0 V). Construct an RC circuit using the 1 μF capacitor and the 470 kΩ resistor, as shown in Fig. 1. Measure the applied voltage V and the voltage across the capacitor V_C. Connect the oscilloscope across the resistance R. You will want to increase the time range of the oscilloscope so that it goes into scanning mode.).

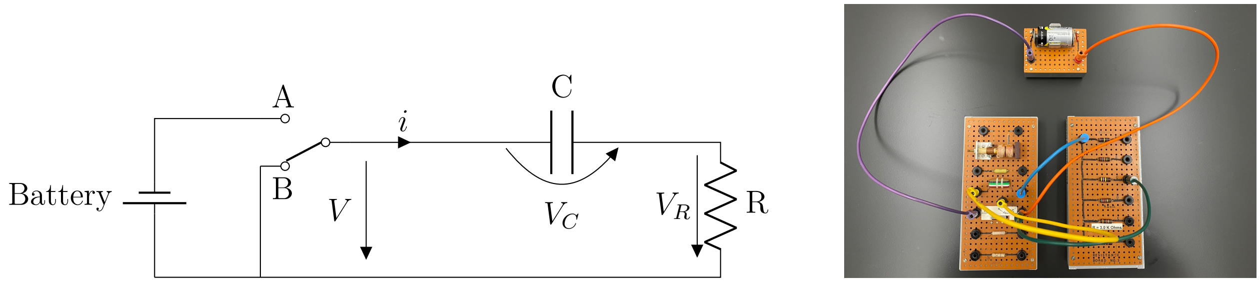

Figure 1: Setup for studying slow transient voltages. Voltages can be measured with the oscilloscope to look at the transient behaviour.

Measure the voltages as a function of time after the switch opens or closes. You should see voltages that look like those in Fig. 6 depending on your measurement configuration. You can measure the time constant τ from the graph using the `Measure` tools, or by fitting equation 6.

With a 470 kΩ resistor and a 1 μF capacitor, the expected time constant is about 0.47 s. You might measure a different time constant because of other resistance or capacitance in the circuit (for example, in the oscilloscope itself Use the voltages and the time dependent equations for each circuit (e.g. equation 6) to find the time constants of the circuit.

Plot your collected dataset, along with the best parameter fits to your model equations, and the time constant you calculate using the values of each component. Make sure to comment on any differences from theory to measurement, and identify possible causes.

Experiment 2

The manual switch circuit can be used to look at slow changing voltages, but for faster circuits it is inconvenient.

Instead, build the circuit shown in Fig. 2. Replace the battery and switch with the signal generator set to generate a square wave. This square wave will behave as if the switch in Fig. 1 is opened and closed at a steady pace. If you have difficulty finding a good setup to show the voltage changes, consider the relative sizes of the time constant of the circuit and the period of the signal generator.

1. With this new method, construct an RC circuit using the 22 nF capacitor and a suitable resistor and again measure the applied voltage V and the voltage across the resistor V_R. **Plot and fit** this dataset and find the time constants.

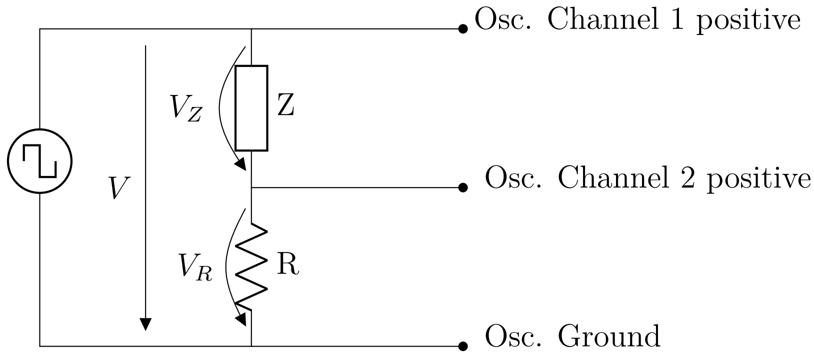

Figure 2: Setup for studying fast transient voltages.

2. Construct a series LR circuit using the smallest resistor available and the inductor. Again measure V and V_R as a function of time, using the function generator to make a square wave. **Plot and fit** this dataset and find the time constants.

3. Construct a series LC circuit using the 22 nF capacitor and the inductor. First, measure both V and V_L, then measure both V and V_C. **Plot and fit** this dataset and find the time constants.

Compare each of the circuits with the theoretical response, and comment on any differences you observed.

Impedance

In the previous section, the circuit was setup as a switching circuit to measure the transient behaviour of the circuit, but was essentially a DC circuit. This section deals with **impedance**, which extends the concept of resistance to AC circuits. In a DC circuit, the current and voltage are constants:

$$I(t) = I_0\,,$$
$$V(t) = V_0\,.$$

If the circuit is linear then it can be described by its resistance

$$R = \frac{V_0}{I_0}\,. \tag{7}$$

In an AC circuit, however, the current and voltage oscillate at an angular frequency ω (or cycle frequency $f = \frac{\omega}{2\pi}$) and can also differ by a phase shift:

$$I(t) = I_0 \sin(\omega t)\,, \tag{8}$$
$$V(t) = V_0 \sin(\omega t + \phi)\,. \tag{9}$$

In this case, we need both the ratio of the amplitudes R and the phase shift ϕ to describe the circuit. These can be combined into a single number using **phasors**. Writing the current and voltage as

$$I(t) = I_0 e^{j(\omega t)}\,, \tag{10}$$

$$V(t) = V_0 e^{j(\omega t + \phi)}\,, \tag{11}$$

where j is the imaginary number (this is used in electronics because i is often the current). Note that the circuit is assumed to be linear, so we can always recombine these complex exponential functions back into sine or cosine functions (so that the current and voltage are real numbers).

With this formulation, the ratio of voltage to current is now a complex number that contains both the ratio of amplitudes and the phase shift

$$Z = \frac{V}{I} = \frac{V_0}{I_0} e^{j\phi} = R e^{j\phi}\,. \tag{12}$$

This value Z is what is known as the impedance and uses the same units as resistance (Ohms).

The impedance of a circuit element can be found using Eq. (10) for current I in the relevant equation. The equation for a resistor is already $V = IR$, so $Z_R = R$. For the inductor, we have

$$\begin{aligned} V &= L\frac{dI}{dt} = L\frac{d}{dt}(I_0 e^{j\omega t})\,, \\ &= j\omega L(I_0 e^{j\omega t})\,, \\ &= j\omega L I\,, \end{aligned}$$

and so

$$Z_L = \frac{V}{L} = i\omega L\,. \tag{13}$$

Similarly, for the capacitor,

$$\begin{aligned} V &= \frac{1}{C}\int I(t')dt' = \frac{1}{C}\int I_0 e^{j\omega t'} dt'\,, \\ &= \frac{1}{j\omega C} I_0 e^{j\omega t}\,, \\ &= \frac{1}{j\omega C} I\,, \end{aligned}$$

and

$$Z_C = \frac{1}{j\omega C}\,. \tag{14}$$

Notice that the impedance of the inductor and capacitor depends on the frequency of the current ω. The impedance of an inductor increases with frequency (it resists fast changes more strongly than slow changes), while the impedance of a capacitor decreases with increasing frequency.

The equivalent impedance of a circuit follows the same rules as resistance. Thus, for an LCR circuit (a resistor, inductor and capacitor in series) the impedance should look like

$$Z_{LCR} = R + j\Big(\omega L - \frac{1}{\omega C}\Big), \tag{15}$$

and the phase difference ϕ between voltage and current is

$$\phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right). \tag{16}$$

The phasor representation of Eq. (15) is presented in Fig. 3.

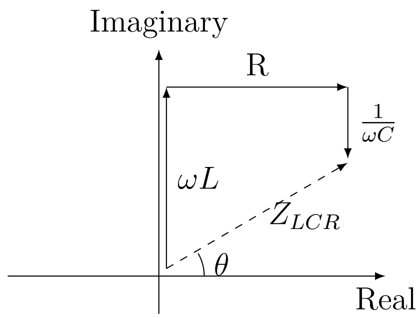

Figure 3: Phasor representation of Eq. (15), shown as sums of imaginary and real components.

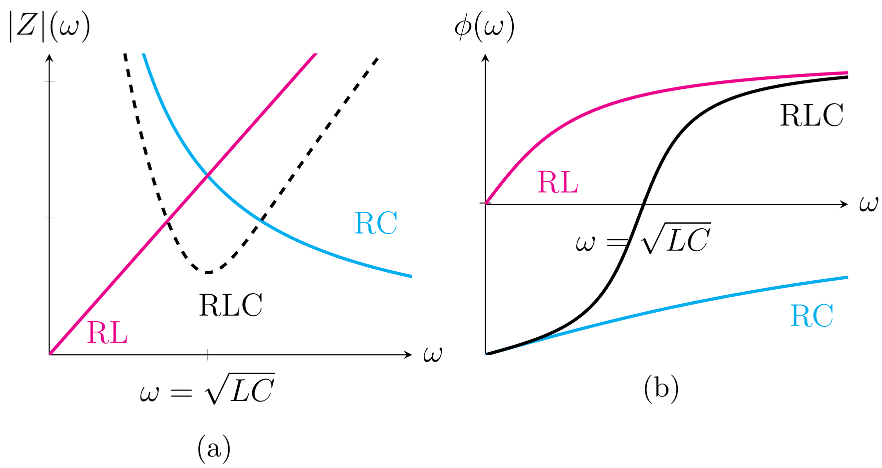

Figure 4: (a) Theoretical impedance plotted as a function of frequency for an RC,RL,and LCR circuit. (b) Theoretical phase plotted as a function of frequency for and RC, RL, and and LCR circuit.

Experiment 3

In this experiment you will directly measure impedance of components in a circuit. A typical setup for measuring impedance is shown in Fig. 5, with a known resistor R in series with a device of unknown impedance, and a sinusoidal driving voltage at some frequency f. Measuring the voltage across the resistor essentially allows us to measure the current going into our circuit, since $I = I_R = V_R/R$. With a simultaneous measurement of V, the amplitude and phase difference of V and I can be used to calculate the impedance Z as a function of frequency.

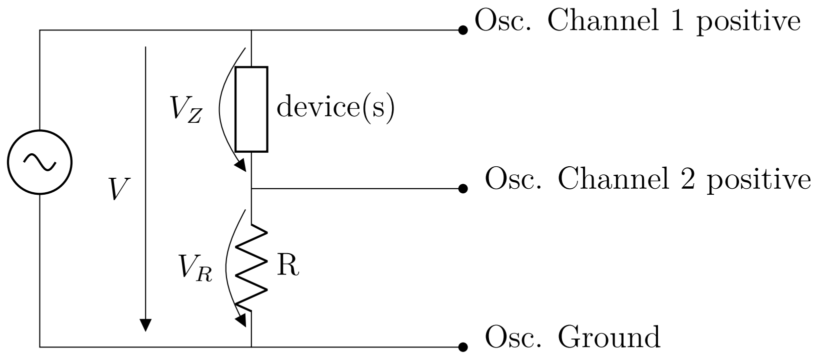

Figure 5: Setup for measuring impedances

Collecting this data is straightforward using the two channels of an oscilloscope. The oscilloscope has `Measure` functions that you can use to read off the amplitude of each wave and the phase shift between them. By varying the output frequency of the function generator, you should be able to obtain Z across a wide frequency range.

- In this circuit, it's necessary to use the transformer to decouple the function generator from the rest of the circuit. Connect the signal generator to the `Primary` side of the transformer, and the circuit to the `Secondary` side.
- If you have difficulty triggering of the measured voltages, you can also trigger on the wave generator within the oscilloscope to get a consistent trigger signal.
- In some configurations and frequencies the voltages you measure will be small and/or noisy. The `Acquire` menu allows you to average the measured data to reduce the noise.

Using an appropriate circuit, measure the impedance of the 22 nF capacitor as a function of frequency. Make sure the frequency range is wide enough to changes in the circuit. It is a good idea to plot the theoretical curve of circuit impedance against frequency so that you know what to expect.

Repeat the experiment using the inductor instead of the capacitor (a resistor of 500 Ω is recommended for this).

Plot the data you collect and find the best fit model for the impedance of the circuit (e.g., equation 15). Since the components are not ideal (they all have some resistance) you will need to consider an equivalent circuit of ideal components to fit the data properly.

Appendix

Voltage plots for RC and RL

The following figures show the theoretical response of an RC circuit (figure 6) and an RL circuit (figure 7). For the RC circuit, when a voltage change occurs the capacitor will charge or discharge, depending on the direction of the voltage change. The voltage across the capacitor will exponentially approach the new voltage value with a time constant $\tau = RC$, and the voltage across the resistor will exponentially decay to zero at the same rate.

For the RL circuit after a voltage change, the current change induces a reverse voltage response in the inductor. In this case the voltage across the resistor gradually increases to the new voltage value with a time constant $\tau = L/R$, and the voltage across the inductor will exponentially decay to zero at the same rate.

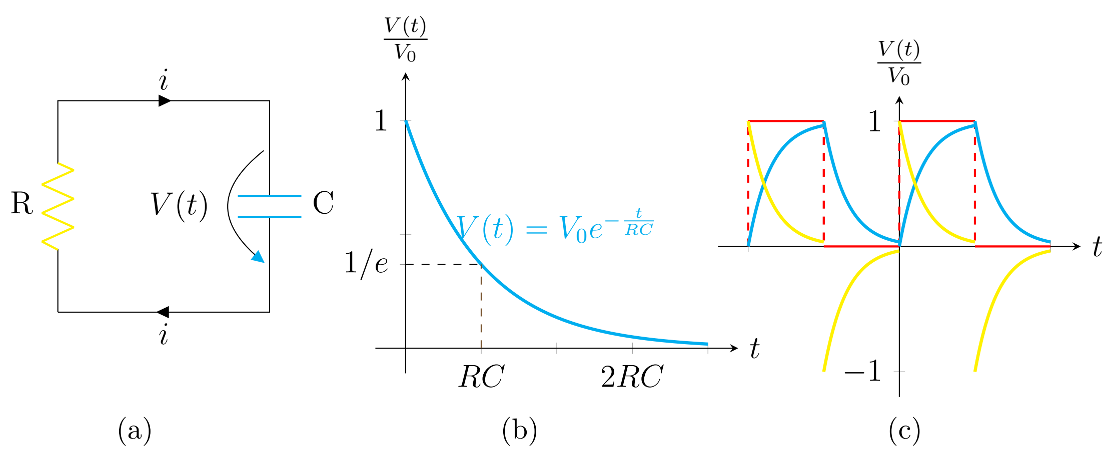

Figure 6: (a) RC Circuit. (b) Voltage as a function of time after connecting the circuit with a charged capacitor.(c) Voltage across the capacitor and resistor as a function of time driven by a square wave.

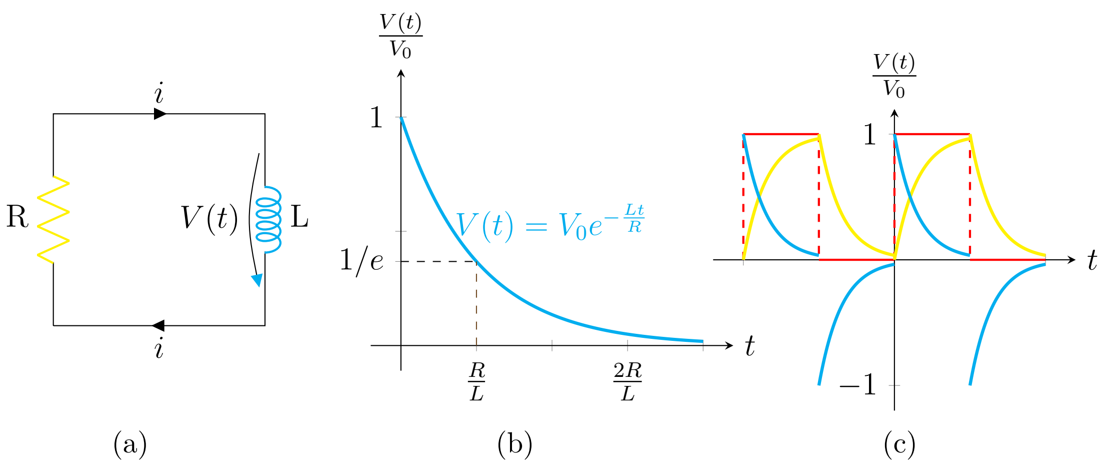

Figure 7: (a) RL Circuit. (b) Voltage across the inductor as a function of time after connecting the circuit to a DC power source. (c) Voltage across the inductor and resistor as a function of time driven by a square wave.

Voltage response of an RLC circuit

When the inductor, capacitor, and resistor are connected in series the inductor and capacitor behave as an oscillator, while the resistor (along with the other components) damps out the oscillations. The relative contribution of each component depends on their size, and in particular the response timescales they represent.

For the real circuit, where there is always some resistance, there are three possible behaviours:

> **underdamped**: When $R < 2\sqrt{\frac{L}{C}}$, the voltage in the circuit oscillates with a decaying amplitude and appears as the product of a sinusoidal function and a decaying exponential.
>
> **critically damped**: When $R = 2\sqrt{\frac{L}{C}}$, the voltage in the voltage in the circuit decays with the smallest possible timescale, with (**just**) no oscillatory response.
>
> **overdamped**: When $R > 2\sqrt{\frac{L}{C}}$, the voltage in the circuit decays exponentially with no oscillatory response.

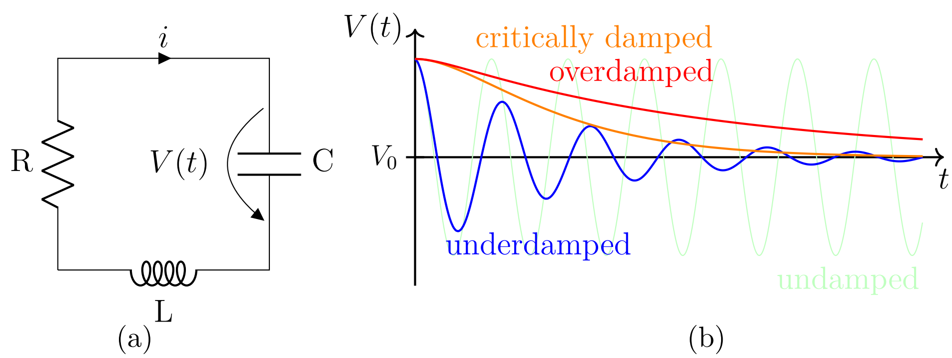

Figure 8: (a) RLC Circuit. (b) Voltage across the capacitor as a function of time after a change in voltage to V_0. Oscillator plot from Neutelings [2021].

References

I. Neutelings. Harmonic oscillator plots, July 2021. URL https://tikz.net/dynamics_oscillator/.