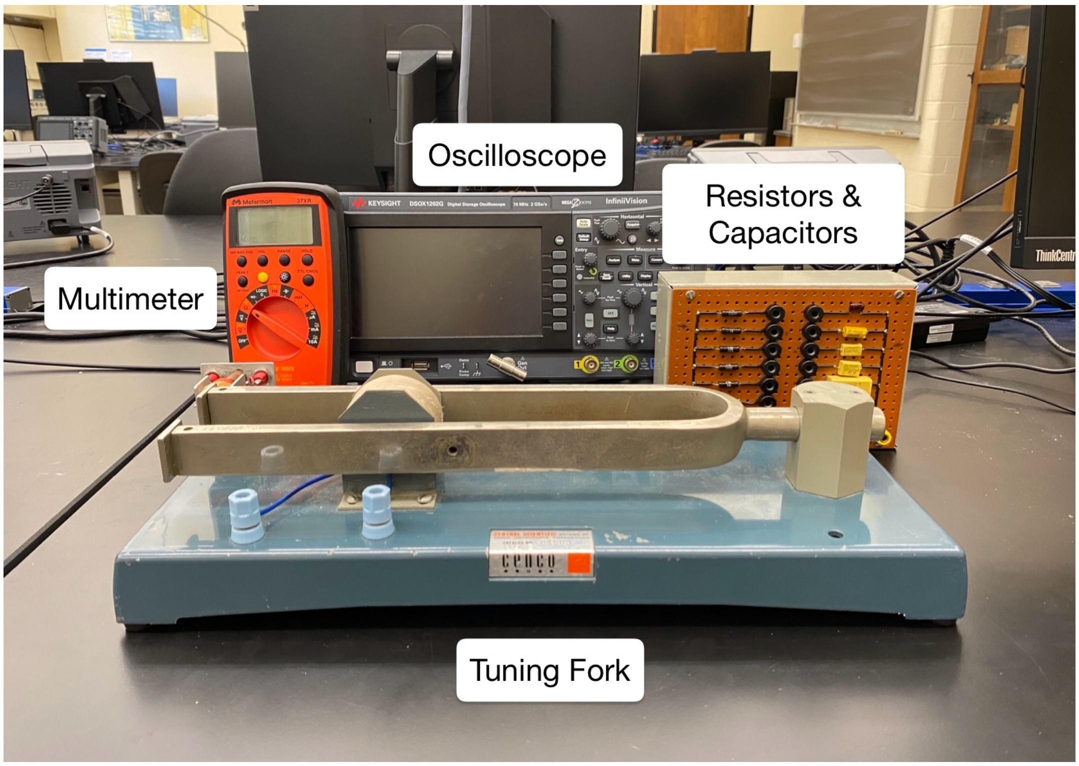
Physics
UNIVERSITY OF TORONTO

Q factor of oscillators



Revisions

2023	C. Lee
2009	R. M. Serbanescu
2007	R. M. Serbanescu, L. Helt
2006	R. M. Serbanescu

current revision: 3fcb776
date: March 17, 2025

© 1982-2022 University of Toronto
This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 Unported License (http://creativecommons.org/licenses/oby-nc-sa/4.0/)

A USB flash drive is useful for saving data from the oscilloscope in this lab. Make sure to bring one with you!

Introduction

In this experiment, you will study resonance in forced oscillators using an electronic circuit and a mechanical oscillator. You will first built an oscillator out of inductors (L), capacitors (C), and resistors (R), and study the response of the circuit as the frequency of an input sinusoid is varied.

You will then use the drive a mechanical oscillator and measure its response as the driving oscillator is adjusted around the resonance frequency.

Electrical Oscillators

In the *Prerequisite* experiment, you studied circuits built from inductors, capacitors, and resistors.

Capacitors in series circuits behave like high pass filters, resisting slow changes (e.g. DC currents) and allowing high frequency oscillations to pass through unchanged. Mathematically the current flow (I) through a capacitor depends on the changing voltage (V) across the circuit

$$I \propto \frac{dV}{dt}, \tag{1}$$

where t is time.

Similarly, inductors in series circuits behave like low pass filters, resisting rapid changes and allowing constant voltages or slow oscillations. The current through the inductor depends on the time-integrated voltage through the circuit

$$I \propto \int_0^t V dt', \tag{2}$$

or equivalently, the voltage across the inductor depends on the changing current through

$$V \propto \frac{dI}{dt}. \tag{3}$$

The two components can be combined in a circuit to build a circuit that oscillates as the components interact and exchange energy stored in the electrical field of the capacitor and magnetic field of the inductor.

If there is any resistance R in the circuit, then the current through the resistor depends on only on the voltage across it $V = IR$ and it acts as a damping on any oscillation between the inductor and capacitor components, converting electrical energy into thermal energy.

Consider the oscillator circuit in figure 1. The voltage across the circuit is

$$\frac{q}{C} - L\frac{dI}{dt} - IR = 0, \tag{4}$$

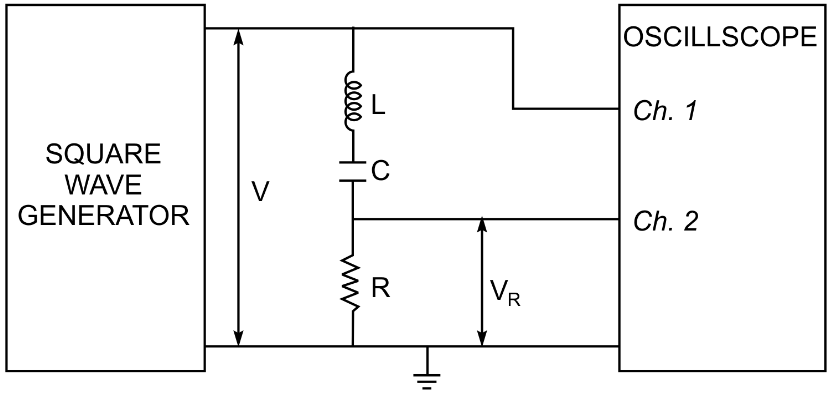


Figure 1: Setup for studying fast transient voltages.

where q is the charge across the capacitor ($q = V_c C$). Since the charge q is the (negative) rate of change of current I, $q = -\frac{dI}{dt}$ then the equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0. \tag{5}$$

Mechanical Oscillators

In the second part of this experiment you will use a large tuning fork as a mechanical oscillator. In this setup the fork behaves like a displaced spring with a restoring force proportional to the displacement x

$$F_r \propto -x, \tag{6}$$

a damping force that is proportional to the velocity of the mass

$$F_d \propto -\frac{dx}{dt}. \tag{7}$$

Combining these equations

$$F = F_r + F_d, \tag{8}$$

$$m\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \kappa x = 0, \tag{9}$$

where 2γ is the damping coefficient (the amplitude relaxation time $\tau = \frac{1}{\gamma}$), and κ the restoring force (the 'spring constant').

Solutions

Equations and describe different oscillations with equivalent equations, with equivalent solutions. For the idealized oscillation:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = F', \tag{10}$$

for arbitrary coefficiences, a, b, and c. For a damped system forced by an external oscillation, this can be written as

$$\frac{d^2x}{dt^2} + \frac{\omega_0}{Q}\frac{dx}{dt} + \omega_0^2 x = F\cos(\omega t), \tag{11}$$

where F is the amplitude of the external oscillation at angular frequency ω. $\omega_0^2 = \frac{c}{a}$ is the resonant frequency of the unforced oscillator, and $Q = \frac{a\omega_0}{b}$ is the *quality factor* (or **Q** factor) of the oscillation.

In the steady state, the solution for this equation is

$$x = A\cos(\omega t + \phi), \tag{12}$$

where the A and ϕ terms are given by

$$\begin{aligned} A &= \frac{F}{b\omega}\frac{1}{\sqrt{1 + Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2}}, \\ \tan\phi &= \left[Q\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)\right]^{-1} \end{aligned} \tag{13}$$

For both the electrical and mechanical setups in this experiment, you will not measure q (the electrical charge) or x (the mechanical position). Instead you will measure the rate of change of each, $I = \frac{dq}{dt}$ and $v = \frac{dx}{dt}$. As a result the amplitude ratio you measure will be different. For example,

$$v = \frac{dx}{dt} = -C\omega\sin(\omega t + \delta) \tag{14}$$

where the amplitude C and phase δ are

$$C = \frac{F}{b}\frac{1}{\sqrt{1 + Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2}}, \tag{15}$$

$$\tan\delta = \left[Q\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)\right]^{-1}. \tag{16}$$

The quality factor, Q can then be defined as the ratio of the resonant frequency ω_o to the *full width at half maximum* of the resonance peak $\omega_+ - \omega_-$. Consider the maximum amplitude of the oscillation (at $\omega = \omega_o$)

$$C^2_{\max} = \frac{F^2}{b^2}. \tag{17}$$

Then the frequency that gives half this value is

$$\frac{1}{2} = \frac{C^2}{C^2_{\max}} = \frac{1}{1 + Q^2\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2}. \tag{18}$$

Solving for ω gives two solutions

$$\omega = \omega_0 \pm \omega_0 \sqrt{1 - \frac{1}{4Q^2}}, \tag{19}$$

and for large enough values of Q the difference between the two solutions is

$$\Delta\omega = \omega_+ - \omega_- = \frac{\omega_0}{Q} \tag{20}$$

to give the mathematical definition of Q as

$$Q = \frac{\omega_0}{\Delta\omega_{\text{HM}}} \tag{21}$$

where $\Delta\omega_{\text{HM}}$ is the difference in frequency units between the ω_+ and ω_- frequency values that give half of the peak power in the oscillation.

Exercises

For these exercises you will have an inductor mounted near the tuning fork and a component board with capacitors and resistors.

Exercise 1: Electronic oscillator

> When setting up the oscilloscope, you may toggle the BW limit ON, to filter some of the noise. Connect channel 1 to the function generator output, using a Tee connector.

Using the inductor, a capacitor with a value of about 10nF, and a resistor with a value of about $1k\Omega$ construct the series LCR circuit in figure 1. You will drive this circuit with the signal generator and measure voltages in the circuit to calculate the Q factor.

First, measure the values of the L,C, and R components, and calculate the resonant frequency of the circuit. Drive the LCR circuit with a long period **square wave** (long compared to $2\pi/\omega_0$), and measure the oscillation frequency after each transition in the square wave. You should aim to have an underdamped circuit (figure 2) so that you can measure the oscillation inside the decaying signal. If the oscillation is hard to detect you might need to increase (underdamped) or decrease (overdamped) the value of the resistor.

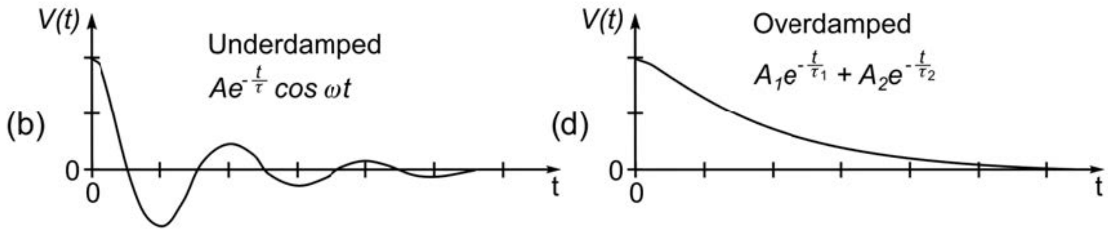


Figure 2: Transient response of and LCR circuit for underdamped and overdamped conditions.

Using the signal generator to generate sine waves, measure the input voltage, output voltage (e.g., across the resistor) and their phase difference as a function of the frequency of the input sine wave. Plot the phase shift and amplitude ratio (V_r/V_{in}) as a function of frequency and plot the theoretical equations (equations 13).

Calculate the Q factor of this circuit and compare it to the theoretical value. It is possible to fit the amplitude in equation 13 to a suitable dataset, or you can alternatively find the half power points and find the full-width and resonance values for the calculation.

Exercise 2: Mechanical Oscillator

The inductor used in exercise 1 is mounted near a large steel tuning fork with a resonance in the 40-70 Hz range. When the tuning fork oscillates the inductor behaves like a *pickup coil* and measures the magnetic field changes caused by the oscillating fork.

Connect the *pickup coil* directly to the oscilloscope (with a slow refresh rate) and gently squeeze and release the tuning fork. You should see an oscillation on the oscilloscope as measured by the *pickup coil*. Measure the natural oscillation frequency (ω_0) and decay rate ($\frac{1}{\gamma}$) of the system. You can rotate the *pickup coil* to improve the signal if needed.

The tuning fork surrounds a larger coil that can be used to drive the fork oscillation using the signal generator. Connect the *drive coil* to the signal generator and drive the tuning fork at a suitable frequency near resonance.

Repeat exercise 1 for the mechanical oscillator. Measure the input *pickup coil* voltage as a function of frequency. Plot the collected data with a theoretical curve on the same plot. Calculate the Q factor for this system and compare it to the theoretical value.

> The mechanical system should have a much larger Q factor than the electronic circuit. As a result the resonant peak and half-maximum points points will be much closer in frequency.

Appendix: LCR circuits and the transfer function

Ohm's Law applied can be applied to the series LCR circuit (Figure 1) can be written in complex notation as

$$I(\omega) = \frac{V(\omega)}{Z}, \tag{22}$$

where $I(\omega)$ and $V(\omega)$ are the complex instantaneous current and voltage in the circuit, ω is the angular frequency of the current and voltage ($\omega = 2\pi f$, where f is the frequency in cycles/second). Z is the complex impedance of the circuit:

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right), \tag{23}$$

where $j = \sqrt{-1}$ is the imaginary unit, R is the resistance of the ideal circuit,L the inductance, and C the capacitance. In reality the inductor and capacitor have an effective resistance that needs to be included in the total impedance.

The voltage across the resistor is then

$$V_R(\omega) = RI(\omega), \tag{24}$$

and combining equations 23 and 24 gives

$$V_R(\omega) = \frac{R}{R + j\left(\omega L - 1/\omega C\right)} V(\omega). \tag{25}$$

Equation 25 can be expressed in a general form as

$$V_r(\omega) = H(\omega)V(\omega), \tag{26}$$

where $H(\omega)$ is called a **transfer function** across the resistor in the frequency domain.

$$H(\omega) = \frac{R}{R + j\left(\omega L - 1/\omega C\right)}. \tag{27}$$

The transfer function, H, is an impedance ratio useful in describing the response of a driven LCR circuit. Since H is complex number, it can be converted into a phasor:

$$H(\omega) = H_a(\omega)e^{j\theta(\omega)}, \tag{28}$$

where $H_a(\omega)$ is the real amplitude (magnitude) of the complex number:

$$H_a(\omega) = |H(\omega)| = \frac{R}{\sqrt{R^2 + \left(\omega L - 1/\omega C\right)^2}}, \tag{29}$$

and $\theta(\omega)$ is the phase of the transfer function:

$$\theta(\omega) = \tan^{-1}\left(\frac{(1/\omega C) - \omega L}{R}\right). \tag{30}$$

The transfer function determines the phase and amplitude relationships between the output voltage across the resistor, $V_R(\omega)$ and the input applied voltage $V(\omega)$. We can characterize a circuit function by examining the magnitude and phase of its transfer function.

From this point, the analysis works the same as above. The resonant frequency maximizes equation 29. The quality factor can be derived from the the frequencies of half-maximum power.

However, this formulation makes it easier to incorporate additional components into the circuit. The non-ideal inductor and capacitor can be included using suitable idealized circuits.

References

[Fortney(1987)] L. R. Fortney. *Ch.2*. Harcourt Brace Jovanovic, 1987.

[Pain(2005)] H. J. Pain. *The Forced Oscillator*, chapter 3, pages 53–78. John Wiley & Sons, Ltd, 2005. ISBN 9780470016954. doi: https://doi.org/10.1002/0470016957.ch3. URL https://onlinelibrary.wiley.com/doi/abs/10.1002/0470016957.ch3.

Prerequisite: Currents through inductances, capacitances and resistances - 2nd year lab experiment, Department of Physics, University of Toronto, http://www.physics.utoronto.ca/~phy225h/currents-l-r-c/currents-l-c-r.pdf