Inductors and Capacitors in AC Circuits

IMPORTANT NOTE: A USB flash drive is needed for the first section of this lab. Make sure to bring one with you!

Introduction

The goal of this lab is to look at the behaviour of inductors and capacitors - two circuit components which may be new to you. In AC circuits currents vary in time, therefore we have to consider variations in the energy stored in electric and magnetic fields of capacitors and inductors, respectively.
You are already familiar with resistors, where the voltage-current relation is given by Ohm's law:

$$V_R(t) = RI(t), \tag{1}$$

In an **inductor**, the voltage is proportional to the rate of change of the current. You may recall the example of a coil of wire, where changing the current changes the magnetic flux, creating a voltage in the opposite direction (Lenz's law).
A **capacitor** is a component where a charge difference builds up across the component. A simple example of this is a pair of parallel plates separated by a small distance, with a charge difference between them. The potential difference between the plates depends on the charge difference Q, which can also be written as the integral over time of the current flowing into/out of the capacitor.

Inductors and capacitors are characterized by their **inductance** L and **capacitance** C respectively, with the voltage difference across them given by

$$V_L(t) = L\frac{dI(t)}{dt}, \tag{2}$$

$$V_C(t) = \frac{Q(t)}{C} = \frac{1}{C}\int_0^t I(t')dt' \tag{3}$$

1 Transient Behaviour

In this first section, we'll look at how circuits with these components behave when an applied DC voltage is switched from one value to another.

1.1 Background

The transient behaviour can be derived by using Kirchhoff's law and solving the resulting differential equation. For example, in a circuit with a capacitor and a resistor (known as an RC circuit), with some constant voltage V applied, Kirchhoff's Law is written as:

$$\Sigma_i \frac{dV_i}{dt} = 0 \tag{4}$$

For a resistor: $\frac{dV}{dt} = R\frac{dI}{dt}$, but for a capacitor: $\frac{dV}{dt} = \frac{I}{C}$.
From an RC loop (see the following figure):

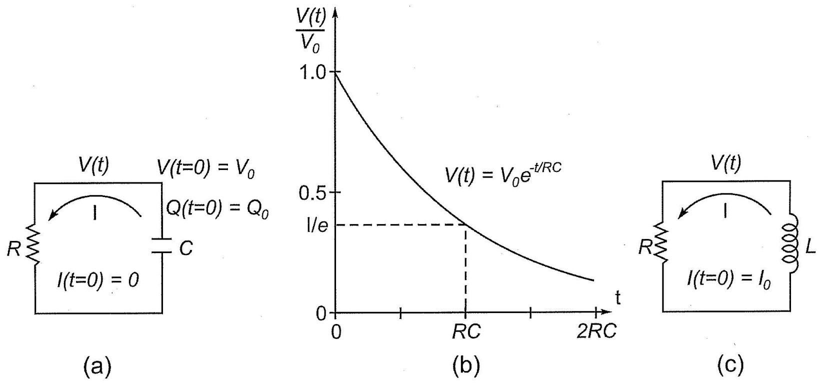

Figure 1: a), b) RC circuit, c) RL circuit

we can add resistor's and capacitor's contributions to Kirchhoff's Law as:

$$R\frac{dI}{dt} + \frac{I}{C} = 0 \tag{5}$$

Integrating from initial current I_o at $t = 0$ to current I at time t, we obtain:

$$\int_{I_o}^{I} \frac{dI}{I} = -\frac{1}{RC}\int_0^t dt \tag{6}$$

$$ln\frac{I}{I_o} = -\frac{t}{RC} \tag{7}$$

$$I(t) = I_o e^{-t/RC} \tag{8}$$

The solution to this equation is Ohm's Law

$$V(t) = RI(t) = V_o e^{-t/RC} = V_o e^{-t/\tau} \tag{9}$$

where $V_o = RI_o$ and $\tau = RC$.
You should work out the solutions to circuits with a resistor and inductor (an RL circuit) and a capacitor and inductor (an LC circuit). **The LCR Circuit**
The following figure presents the RLC circuit diagram, together with transient voltages in different damping conditions:

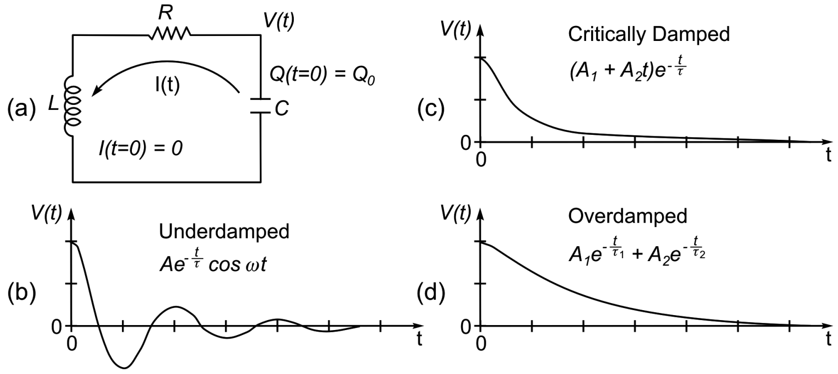

Figure 2: a) RLC circuit, b), c), d) Transient voltages

The circuit from Figure 2a) can be:

- *underdamped* (Figure 2b)) when $R < 2\sqrt{\frac{L}{C}}$, when the response function will be a product of a sinusoidal and an exponential,
- *critically damped* (Figure 2c)) for $R = 2\sqrt{\frac{L}{C}}$ when there is no oscillatory decay response,
- *overdamped* (Figure 2d)) for $R > 2\sqrt{\frac{L}{C}}$, when the transient response is given by the sum of two decaying exponentials.

1.2 Experiments

Use the oscilloscope to see the voltage changing with time. There are two channels, so you can measure the applied voltage and the voltage across a component of interest at the same time. This data can then be saved onto a USB flash drive for analysis on the computer.

> Note: Remember that both of the oscilloscope channels share a common ground. That means that no matter what, the negative sides (typically the black banana plugs) are connected. If you put them in different locations in the circuit, that's like wiring those locations together, and will probably ruin your data!

1. **Experiment 1**.
   For the first experiment, we'll use a manual switch to change the applied voltage between a battery (1.5 V) and no applied voltage (0 V). Construct an RC circuit using the 1 μF capacitor and the 470 kΩ resistor, as shown in Figure 3. Measure the applied voltage V_T and the voltage across the capacitor V_C. Connect the oscilloscope across the resistance R. You will want to increase the time range of the oscilloscope so that it goes into scanning mode.

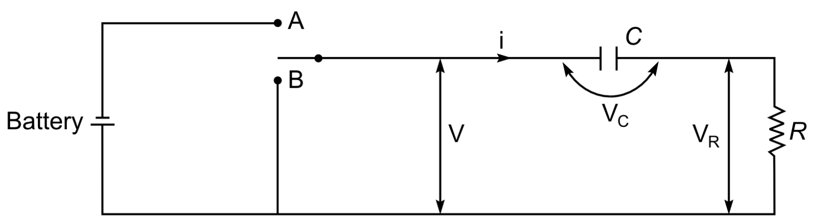

Figure 3: Setup for studying slow transient voltages

Note: This section of the experiment should be done quickly (in less than 15 minutes). Capacitors and resistors are not labeled; you may need to measure them.

The manual switch works when we have a slowly changing voltage, but for faster signals it is inconvenient.
Instead, as shown in Figure 4 below, we can use a function generator to switch the applied voltage V_T between $+V$ and $-V$, which can switch the voltage faster and more precisely. With this new method, construct an RC circuit using the 0.022μF capacitor and any resistor between 100Ω and 100kΩ. Measure the applied voltage V and the voltage across the resistor V_R.
What are the observed time constants? How do these compare with the value of RC?

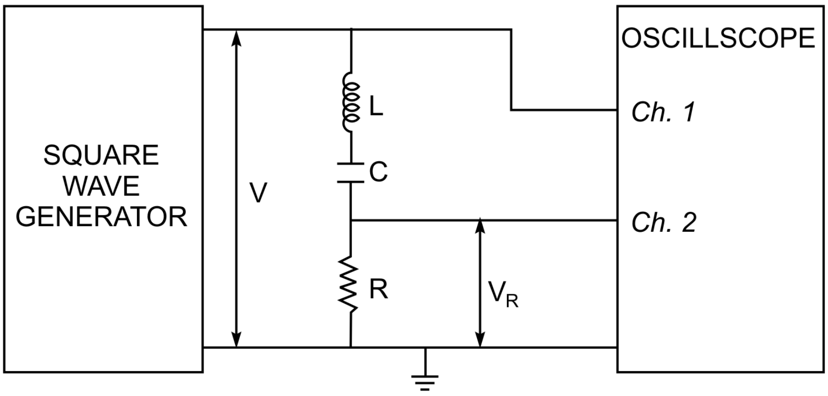

Figure 4: Setup for studying fast transient voltages

- Construct an LR circuit using the smallest resistor available and the coil provided.L for this coil is between 30 mH and 300 mH. Again measure V and V_R as a function of time, using the function generator. From the observed time constant, estimate the inductance of the coil.
Note: The coil is not a pure inductance, but acts as if there were a perfect inductance in series with a resistance. The effective series resistance is called the *internal resistance* of the coil.
- Construct an LC circuit using the 22 nF capacitor. First, measure both V and V_L, then measure both V and V_C.
Note: You will have to adjust the oscilloscope sweep frequency and the wave generator frequency in order to have adequate full display of the voltages being observed.

Plot and Fit each of these voltages using appropriate models. Compare the fit parameters between each data set and to those you would expect given how they were labeled and/or measured using a multimeter.

1.3 Impedance: theoretical background

This section deals with **impedance**, which extends the concept of resistance to AC circuits.
In a DC circuit, the current and voltage are constants:

$$I(t) = I_0$$
$$V(t) = V_0$$

If the circuit is linear (meaning changing the current will change the voltage proportionally) then it can be described by its resistance

$$R = \frac{V_0}{I_0} \tag{10}$$

In an AC circuit, however, the current and voltage oscillate at an angular frequency ω and thus can also differ by a phase shift:

$$I(t) = I_0 \sin(\omega t) \tag{11}$$
$$V(t) = V_0 \sin(\omega t + \phi) \tag{12}$$

In this case, we need both the ratio of the amplitudes R and the phase shift ϕ to describe the circuit. These can be combined into a single number using **phasors**. The idea here is to write the current and voltage as

$$I(t) = I_0 e^{i(\omega t)} \tag{13}$$
$$V(t) = V_0 e^{i(\omega t+\phi)} \tag{14}$$

Note that since the circuit is assumed to be linear, so we can always recombine these complex exponential functions back into sine or cosine functions (so that the current and voltage are real numbers). With this formulation the ratio of voltage to current is now a complex number that contains both the ratio of amplitudes and the phase shift

$$Z = \frac{V_0}{I_0} e^{i\phi} = R e^{i\phi} \tag{15}$$

This value Z is what is known as the impedance. Like resistance, it has units of Ohms. We can find the impedance of a circuit element by putting the phasor expression for I, equation (10), into the equation for V, equation (11). Note that for a resistor, the equation is already of this form, so $Z_R = R$. For the inductor, we have

$$\begin{aligned} V &= L\frac{dI}{dt} \\ &= L\frac{d}{dt}(I_0 e^{i\omega t}) \\ &= i\omega L(I_0 e^{i\omega t}) \end{aligned}$$

and so

$$Z_L = i\omega L \tag{16}$$

Similarly, for the capacitor,

$$\begin{aligned} V &= \frac{1}{C}\int I(t')dt' \\ &= \frac{1}{C}\int I_0 e^{i\omega t'} dt' \\ &= \frac{1}{i\omega C} I_0 e^{i\omega t} \end{aligned}$$

and we see that

$$Z_C = \frac{1}{i\omega C} \tag{17}$$

It's straightforward to see that calculating the equivalent impedance of a circuit follows the same rules as resistance. Thus, for an LCR circuit (a resistor, inductor and capacitor in series) the impedance should look like

$$Z_{LCR} = R + i\Big(\omega L - \frac{1}{\omega C}\Big) \tag{18}$$

The phasor representation of equation (18) is presented in Figure 5. Notation from the figure is j instead of i for the complex number:

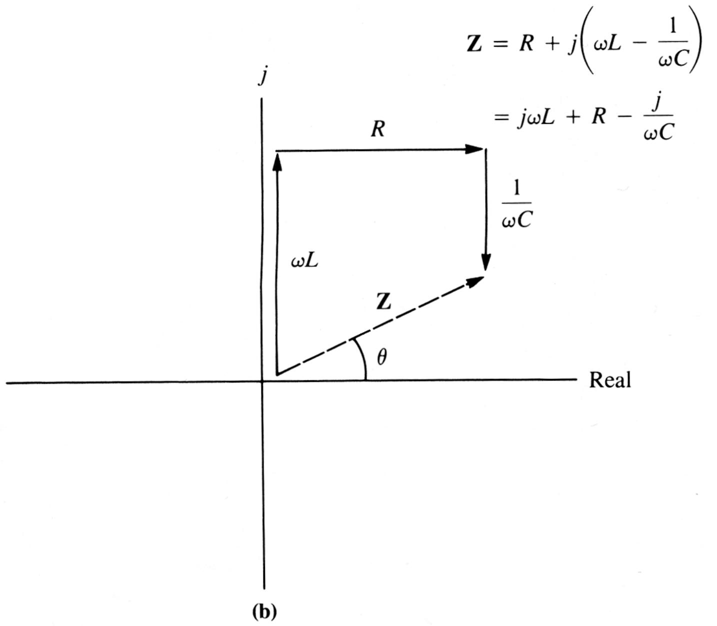

Figure 5: Phasor representation of equation (18)

1.4 Experiment 2

Now, we'll use our circuits to directly measure the impedance of some circuit elements. An RC circuit can be built as in Figure 6:

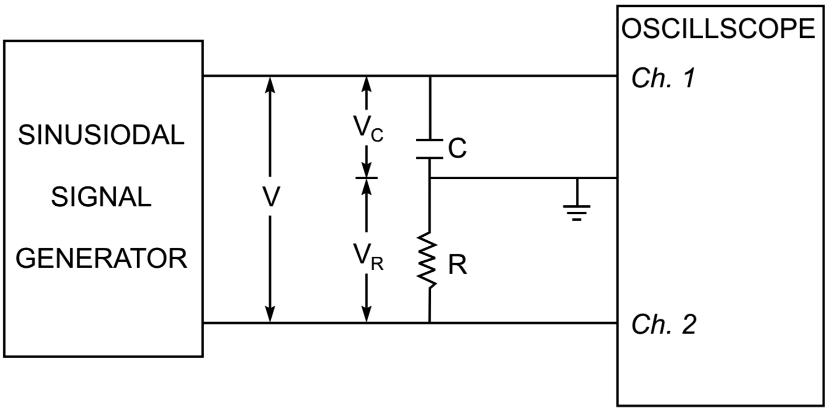

Figure 6: Setup for measuring impedance for a RC circuit

Note that this circuit compares $+V_C$ with $-V_R$ because both voltages are measured relative to oscilloscope's ground. Be careful in interpreting observed phase differences!
Useful Note: Instead of using the function generator directly, you should plug it into the `Primary` side of the transformer and use the `Secondary` side to drive the circuit. The transformer is used to decouple the signal generator from the circuit (it allows AC signals, but blocks DC and interference effects). It also provides the needed ground for Figures 6 and 7.
For comparing V with V_R in the LCR circuit, we use the diagram from Figure 7, where we have a known resistor R in series with our *LC circuit* of interest, with a sinusoidal driving voltage.
Measuring the voltage across the resistor essentially allows us to measure the current

going into our circuit, since $I_{circuit} = I_R = V_R/R$. If we simultaneously measure $V_{circuit}$, we can get the amplitude and phase difference between $V_{circuit}$ and $I_{circuit}$, which tells us $Z_{circuit}$ at this frequency.

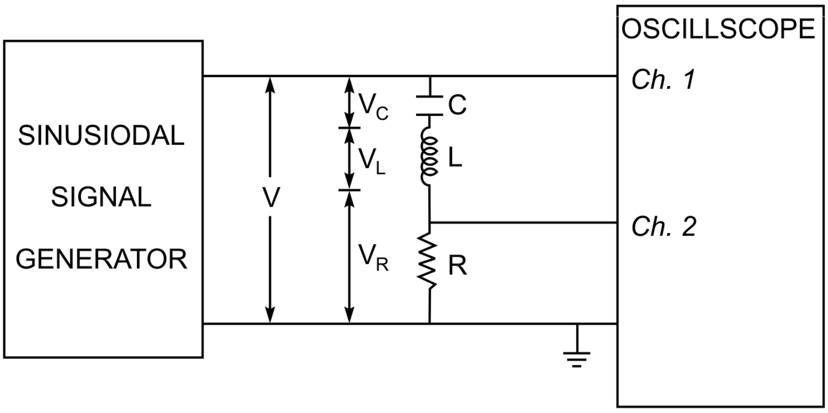

Figure 7: Setup for measuring impedances

Impedance measurements
For $C = 0.022\mu F$ and R between 100Ω and 100kΩ (Figure 6), measure $\frac{V_C}{V_R}$ in magnitude and relative phase for several frequencies between 10 Hz and 1.0 MHz. Observe the phase relation between V_C and V_R (whether V_C leads of lags behind V_R).
Repeat for the RL circuit (Figure 7) using the coil and the resistor with $R = 100\Omega$. Cover a selection of frequencies that shows the *resonance curve*.
Acquiring this data is straightforward using the two channels of the oscilloscope. The oscilloscope has `Measure` functions which you can use to read off the amplitude of each wave and the phase shift between them. By varying the output frequency of the function generator, you should be able to obtain $Z_{circuit}$ across a wide frequency range. Some other things to note:

- The auxiliary output of the function generator outputs a square wave at the same frequency as the main output. Plugging this into the external trigger port of the oscilloscope and using it for triggering can be helpful, as triggering off of a signal when it is small does not work very well.
- In the `Acquire` menu of the oscilloscope, you can switch between sample mode and average mode. You will often want to switch to averaging mode, particularly when the signal is small and/or noisy.

You should measure the impedance as a function of frequency for the following circuits:

1. The RC circuit with $C = 22nF$ capacitor (Figure 6)
2. The RL circuit. A resistor of $\approx 500\Omega$ is recommended for this.
3. The RCL circuit ($C = 22nF$ capacitor, inductor and a resistor in series) as in Figure 7.

Make sure you take data over a sufficiently wide range of frequencies. It may be a good idea to plot a theoretical curve in Python so that you know what to expect.

You should **plot and fit all of impedance data. Plot Z vs. frequency.**

ϕ is the phase dispacement of volage versus current:

$$\phi = \arctan\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \tag{19}$$

Plot phase vs. frequency and interpret the graphs.

Note that since the inductor and capacitor are not perfectly ideal, you will likely need to consider an equivalent circuit of ideal components in place of the non-ideal one in order to fit the data properly.

This guide sheet was revised in 2020 by Ruxandra M. Serbanescu with useful input from Michael Bartram