Motion in fluids

- **Introduction**

In swimming bacteria or diffusing proteins, viscous rather than inertial forces dominate the dynamics of motion. A common measure of the ratio of the inertial to viscous forces is known as the Reynolds number:

$$R_e = \frac{\rho \ell v}{\eta} \tag{1}$$

Where ρ is the fluid density, v is the velocity of the object, ℓ is a characteristic length of the object, and η is the fluid viscosity.

Our everyday experience is mostly with high Reynolds number environments where inertial forces dominate. Swimming, for example, is a high Reynolds number activity. We propel ourselves through the water by accelerating the fluid behind us; the inertial force from a single stroke lets us glide meters before we come to a stop. Low Reynolds number activities are less common, but stirring a jar of honey with a spoon is one example. It is the viscosity of the honey and not the mass of the honey that makes the stirring difficult. When you let go of the spoon, does it continue to swirl around the jar? No, the spoon stops moving fairly quickly. The viscous force dominates the inertial force.

Swimming can be a low Reynolds number activity when the length scale of the swimmer is small. Microorganisms fit this category. A bacterium such as *E. coli*, is about one micron (10^{-6} meters) in diameter and travels around $20\mu m$ per second, so swimming bacteria have a Reynolds number much less than one and the viscous forces dominate inertial forces. To us, this is a very alien hydrodynamic world. For you to swim at an equivalent Reynolds number, you would need to "swim" in something viscous like honey, at speeds of about a foot a day, while cycling our arms at about 1 stroke per hour.

- **Theoretical background**

$F = ma$ forces are familiar, but what are viscous forces? Imagine you have a fluid between two plates. Intuitively you know that as the viscosity of the fluid increases, it requires more force to slide the plates apart (think water versus honey). Now assume that the bottom plate is fixed while the top plate, at some distance ℓ, is free to move parallel to the fixed plate (see Figure 1).

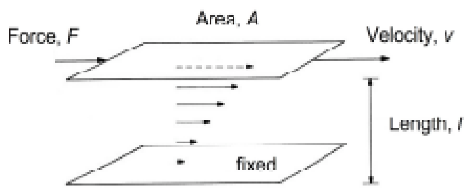

Figure 1: Fluid motion between two plates

If a force F is applied to the top plate, it will move at some velocity , forming a velocity gradient between the top and bottom plates. As the viscosity increases, it will take a larger force to form the same velocity gradient. It is this proportionality between the force per

area (also known as shear stress) and the velocity per length (shear rate) that is known as viscosity.
From Figure 1, we can define the viscous force F_v and the "inertial" force $F_i = ma$:

$$F_v = \frac{\eta A v}{\ell} \tag{2}$$

$$F_i = \rho V a = \frac{\rho \ell A v}{t} \tag{3}$$

Taking the ratio gives the Reynolds number:

$$R_e = \frac{F_i}{F_v} = \frac{\rho \ell v}{\eta} \tag{4}$$

Equations of motion
In this experiment, you will be following the motion of objects falling in fluids. Check out the force diagram for a sphere falling in a fluid (see Figure 2).

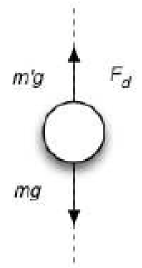

Figure 2: Force diagram

The equation of motion is:

$$m\frac{dv}{dt} = mg - F_d \tag{5}$$

Where F_d is the drag force and m is the mass of the object corrected for buoyancy: $m = (\rho_{sphere} - \rho_{fluid})V$. When the object reaches its terminal velocity: $a = \frac{dv}{dt} = 0$ and $mg = F_d$. The form of F_d depends on the Reynolds number.
At a **high Reynolds number**, the drag force is commonly written as:

$$F_d = \frac{1}{2}\rho C_d A v^2 \tag{6}$$

The drag coefficient C_d is measured empirically, ρ is density of fluid and A is cross-sectional area of the object, perpendicular to the direction of motion. Terminal velocity is this regime can be obtained from the equation of motion (5) with the appropriate substitution for F_d:

Using the terminal condition $mg = F_d$, we can define the *terminal velocity*, given by:
$v_{term} = \left(\frac{2mg}{\rho C_d A}\right)^{1/2}$
By separating the variables, we obtain the following solution:

$$v(t) = v_{term}\tanh\left(\frac{gt}{v_{term}}\right) \tag{7}$$

The terminal velocity at high Reynolds number depends on the bead radius as $v_{term} \simeq r^{1/2}$.

The drag force at **low Reynolds number** is directly proportional to velocity. For the particular case of a spherical particle of radius r moving at velocity v in a fluid with viscosity η we can write:

$$F_d = 6\pi\eta v r \tag{8}$$

The equation of motion in this case is written as:

$$m\frac{dv}{dt} = mg - 6\pi\eta r v \tag{9}$$

From the terminal condition at low Reynolds number, we can determine the terminal velocity: $v_{term} = \frac{mg}{6\pi\eta r}$.
Separating variables and integrating, we obtain:

$$v(t) = \frac{mg}{6\pi\eta r}(1 - e^{-\frac{t}{\tau}}) = v_{term}(1 - e^{-\frac{t}{\tau}}) \tag{10}$$

where $\tau = \frac{m}{6\pi r\eta}$ is a time constant.
Qualitatively, we can see that the solution from equation (12) has the correct initial and asymptotic behavior: at $t = 0$, $v = 0$ and at $t \to \infty$, $v = \frac{mg}{6\pi\eta r}$, which is the expected terminal velocity.
Numerical exercise. The time constant is a good estimate for the time needed to reach the terminal velocity: when $\tau = t$, $v(\tau) = \frac{mg}{6\pi\eta r}(1 - \frac{1}{e}) = 0.634 v_{term}$ which means that at $t = 3\tau$ the object reaches 98% of terminal velocity.
Estimate τ for an aluminum sphere of radius $r = 0.5 \times 10^{-3}$m falling in glycerine where viscosity is $\eta = 1500 \times 10^{-2}$Poise. Density of aluminum is $\rho = 2.7\frac{g}{cm^3}$

The terminal velocity in the low Reynolds number regime depends on the particle's radius as: $v_{term} \simeq r^2$

- **Notes about the experimental procedure**

  The experiment uses a video tracking method. Open the LabView application "Motion through Fluids" (shortcut on computer's desktop). Confirm the default camera. The program asks you to select the number of frames to be recorded. Do some trial experiments with 120 frames, adjust the number later on, if needed.

  The frame rate means how many frames per second. Try 20, this will cover 6 seconds when combined with 120 frames, adjust it later if needed.

  You will have to provide a location for saving the ∗.avi file (movie of the falling bead).

  When starting the experiment, submerge the tweezers + bead in order to avoid surface tension problems. Click on "Start the avi movie capture". You'll hear warning beeps; release the bead at the long beep. The application will output a text file with time (in seconds) and position (in mm) of the falling bead.

- **Correction due to the wall effect**

  An object falling near a boundary (like the wall of the container) moves more slowly than an object falling far from a wall. *Try this:* drop simultaneously two identical spheres: one near the wall the other one at the center of the container.

  To account for the effect of the container wall, a correction has to be calculated (*Metrologia* 2001, **38** (531-534)):

$$v_{corr} = \frac{v_m}{1 - 2.104\frac{d}{D} + 2.089(\frac{d}{D})^2} \tag{11}$$

  In equation (13), v_m is the experimental mean velocity, d is the diameter of the bead, D is the dimension of the container perpendicular to the direction of the fall and v_{corr} is the expected velocity of the bead as if it were falling in an unbounded fluid.

- **Exercise 1: Low Reynolds number**

  Use the box marked "Pure Glycerine" with five different sizes of spherical beads made of Teflon. Use the container marked "Glycerol", place it in front of the dark chamber. The video camera is located at the rear of the chamber.

  Test the setup by dropping beads and timing the fall using a stopwatch. This would give you a hint about the number of frames and the frame rate.

  Take 2-3 measurements for each bead size. Measure the diameters using the caliper provided.

  Obtain the mean terminal velocity (with uncertainty) for each bead size from the collected data.

  Correct the data for the wall effect.

  Calculate the Reynolds number. Is the value a "low R_e"?

  Python: plot the mean terminal velocity as a function of bead radius. Fit the data using: $v_{term} = ar^2$. Does the fit work well? How does the value of parameter a compare with the theoretical value?

- **Exercise 2: High Reynolds number**
  Use the box marked "Water" with five sizes of Nylon beads. Measure the diameter of each bead.
  Replace the "Pure Glycerine" container by the "Water" container (be careful not to spill glycerine).
  Remove all the air bubbles.
  Test your setup; you may notice that larger particles do not always fall in straight lines: sometimes they wobble due to water turbulence. Practice until you get the best trajectory.
  Take 5 measurements for each bead size using the tracking program.
  Obtain the mean velocity (with uncertainty) for each bead size from the collected data. Can this be interpreted as "terminal velocity"?
  Correct the data for the wall effect.
  Calculate the Reynolds number. Is the value a "high R_e"? This would be $R_e > 200$. Is there a critical bead size or a critical velocity that confirms the "high R_e"?
  Python: plot the mean terminal velocity as a function of bead radius. Fit the data using: $v_{term} = br^{1/2}$. Do you notice any discrepancy with the theory?
  For intermediate Reynolds number situations, both linear and quadratic terms may be relevant. The dependence of v_{term} on r is more complicated.
  Try to come up with a fit function to describe this case. Does it improve the fit?

- **Physical reference data**
  Glycerol density: $\rho_g = 1.26\frac{g}{cm^3}$
  Glycerol viscosity: $\eta_g = 934$ centipoises (cp) or $9.34\frac{g}{cm \cdot s}$ at 25^oC.
  Water viscosity: $\eta_w = 1$ cp at 25^oC.
  Water density: $\rho_w = 1\frac{g}{cm^3}$ at 25^oC.
  Teflon density: $\rho_t = 2.2\frac{g}{cm^3}$.
  Nylon density: $\rho_n = 1.12\frac{g}{cm^3}$

The experimental setup and the LabView tracking program were made by Larry Avramidis.
This guide was written by Ruxandra Serbanescu in 2013 and revised in 2019. Thanks to Michael Bartram for suggestions.