The Q of oscillators

References:

L.R. Fortney – Principles of Electronics: Analog and Digital, Harcourt Brace Jovanovich 1987, Chapter 2 (AC Circuits)
H. J. Pain – The Physics of Vibrations and Waves, 5th edition, Wiley 1999, Chapter 3 (The forced oscillator).
Prerequisite: Currents through inductances, capacitances and resistances – 2nd year lab experiment, Department of Physics, University of Toronto,
http://www.physics.utoronto.ca/~phy225h/currents-l-r-c/currents-l-c-r.pdf

Introduction

In the *Prerequisite* experiment, you studied LCR circuits with different applied signals.
A loop with capacitance and inductance exhibits an oscillatory response to a disturbance, due to the oscillating energy exchange between the electric and magnetic fields of the circuit elements. If a resistor is added, the oscillation will become damped.

In this experiment, the LCR circuit will be **driven** at resonance frequency $\omega_r = \frac{1}{\sqrt{LC}}$, when the transfer of energy between the driving source and the circuit will be a maximum.

LCR circuits at resonance. The transfer function.

Ohm's Law applied to a LCR loop (Figure 1) can be written in complex notation (see Appendix from the *Prerequisite*)

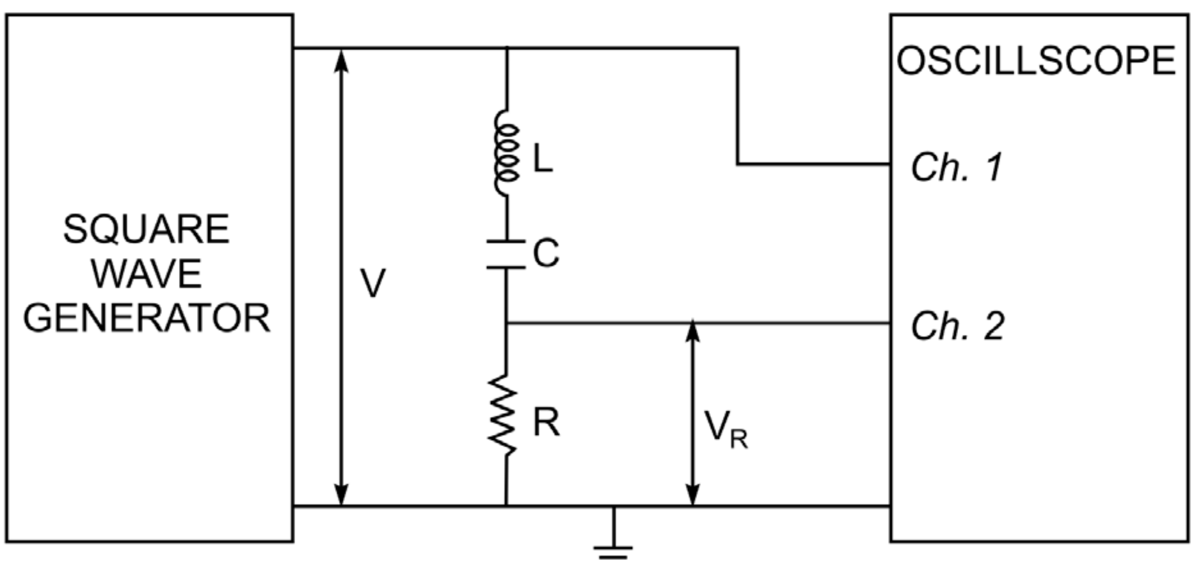

Figure 1 LCR circuit for resonance studies

Ohm's Law: $$i(j\omega) = \frac{v(j\omega)}{Z} \quad (1)$$

i(jω) and v(jω) are complex instantaneous values of current and voltage,
ω is the angular frequency $(\omega = 2\pi f)$,
Z is the complex impedance of the loop:

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{2}$$

$j = \sqrt{-1}$ is the complex number

The voltage across the resistor from Figure 1, as a result of current i can be expressed as:

$$v_R(j\omega) = R\,i(j\omega) \tag{3}$$

Eliminating i(jω) from Equations (1) and (3) results into:

$$v_R(j\omega) = \frac{R}{R + j(\omega L - 1/\omega C)} v(j\omega) \tag{4}$$

Equation (4) can be put into the general form:

$$\boxed{v_R(j\omega) = H(j\omega)\,v(j\omega)} \tag{5}$$

where H(jω) is called a **transfer function** across the resistor, in the frequency domain.

$$H(j\omega) = \frac{R}{R + j(\omega L - 1/\omega C)} \tag{6}$$

H is an impedance ratio, useful in describing the resonance of a driven LCR loop.
Any complex number such as H(jω) can be put in the form:

$$\boxed{H(j\omega) = H_A(\omega)\,e^{j\theta(\omega)}} \tag{7}$$

where $H_A(\omega)$ is the real amplitude (or magnitude) of the complex number:

$$|H_A(\omega)| = \frac{R}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \tag{8}$$

and θ(ω) is the phase, defined as: $\theta(\omega) = \tan^{-1}\left[\frac{(1/\omega C) - \omega L}{R}\right]$ (9)

The transfer function determines the phase and amplitude relationships between the voltage across resistor $v_R(j\omega)$ (output) and the applied voltage v(jω) (input).
The transfer function shows how a circuit modifies the input signal in creating the output.
Mathematically speaking, the transfer function completely describes how the circuit processes the input complex exponential to produce the output complex exponential.
We can characterize a circuit function by examining the magnitude and phase of its transfer function
From Equations (8) and (9), note that amplitude $H_A(\omega)$ is a maximum and the phase $\theta(\omega_r)$ is zero when $(\omega L - 1/\omega C) = 0$. This is called **resonance**. At resonance there is maximum of energy

transfer between the driving source and the circuit. The resonant frequency of the circuit is defined as:

$$\omega_r = \frac{1}{\sqrt{LC}} \qquad (10)$$

The Q factor

With reference to a specific LRC circuit, the Q factor measures the strength of a resonance.

For a series LCR loop, by definition: $Q = \frac{\omega_r L}{R} = \sqrt{\frac{L}{R^2 C}}$ (11)

More fundamentally, the Q factor of a resonance is 2π times the stored energy divided by the energy lost per oscillation cycle.

We can express now the complex transfer function with the Q factor, combining equations (6) and (11):

$$H(j\omega) = \frac{1}{1 + jQ\frac{\omega}{\omega_r}\left[1 - \left(\frac{\omega_r}{\omega}\right)^2\right]} \qquad (12)$$

A log-log plot of the magnitude of the transfer function $|H(j\omega)|$ as function of ω/ω_r for different values of Q would show the resonant behavior of a series LCR loop. Larger Q values correspond to narrower resonance curves (Figure 2):

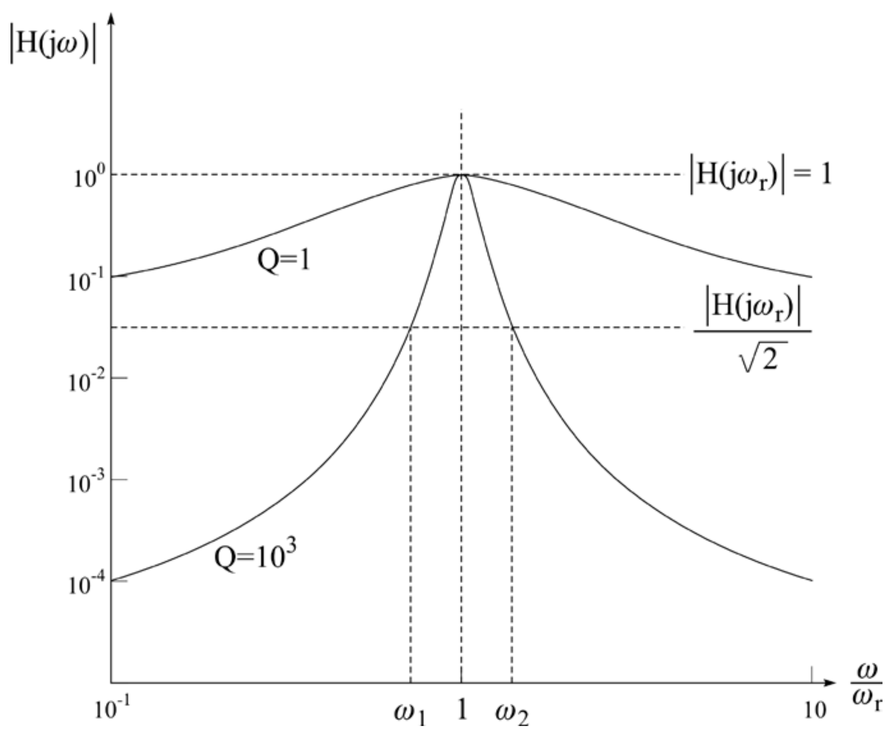

Figure 2 Resonant behavior of an LCR loop

Looking at Figure 2, we can define two frequencies ω_1 and ω_2 which satisfy: $\left|H(j\omega_{1,2})\right| = \frac{1}{\sqrt{2}}\left|H(j\omega_r)\right|$.

ω_1 and ω_2 are called "half-power frequencies". They allow rewriting the Q factor as:

$$Q = \frac{\omega_r}{\omega_2 - \omega_1} = \frac{\omega_r}{\Delta\omega} \qquad (13)$$

For practical applications, the ratio $\frac{V_{out}}{V_{in}}$ (where $V_{out} = V_R$ and V_{in} are voltage amplitude values you measure on oscilloscope) can be used to determine the resonance, instead of the transfer function.

A mechanical system at resonance (see References)

An oscillator with a linear restoring force and viscous damping γ obeys an equation of the form:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F_{driving} \qquad (14)$$

The damped oscillation is characterized by two time constants: the undamped period $T = \frac{2\pi}{\omega_0}$ and the amplitude relaxation time $\tau = \frac{1}{\gamma}$.

The Q_{mech} factor of the oscillator is defined as: $Q_{mech} = \frac{\omega_0}{2\gamma}$ (14)

Note that $\frac{Q_{mech}}{\pi}$ corresponds to the number of oscillations during a decay of amplitude to 1/e of its initial value.

Apparatus notes

Use the experimental arrangement from Figure 1. There will be only one inductor coil provided. It will be used in Exercises 1 and 2 as L in the LCR circuit, and in Exercise 3 as pickup coil (it has to be mounted close to the tuning fork arm).
When setting up the oscilloscope, you may toggle the BW limit ON, to filter some of the noise. Connect Ch. 1 to the function generator output, using a Tee connector.

Exercise 1: Free decay

Use L and a C of ~10000pF. You may use a GR bridge from the Resource Centre to accurately measure L and C. Calculate ω_r (the resonance frequency of the LC circuit).
Connect L-C in series with R~1000Ω and a signal generator.
Note that if the oscilloscope is connected across R, current i can be monitored. Estimate Q using Eq.11.
To observe the free decay, use a square wave with a long period compared to $2\pi/\omega_r$, and various values of R, including R = 0.

Exercise 2: Sine wave response

Observe the shift in phase of V_c relative to the generator voltage near the resonance.
Plot this phase shift vs. ω and also the magnitude of the circuit impedance vs. ω, using log-log coordinates to locate the resonance and the half-power frequency points.
Use the half-power points to recalculate Q.
➔ **Python Requirement 1 (PHY224/324 students only):** do the plots mentioned above using a Python program. Output Q.

Question: Why are Q values lower than the value calculated using Eq. (11)?

Optional: If L and C are connected in **parallel** instead of series the roles of i and v are interchanged: the current is a **minimum** at resonance. This is called a "current tank". The analysis is complicated by the coil resistance still being in series with L. It's not so simple to sort out the effective Q in this case unless it is very large. Try it if you wish.

Exercise 3: The tuning fork

In a low frequency circuit using coils, it is nearly impossible to achieve Q > 50 because of the coil electric resistance. A mechanical system can do much better: Q ~ 10^4 or more is feasible.

Set up the tuning fork without signal generator. Connect it to the oscilloscope using the pickup coil connectors. Note that the signal picked up is proportional to fork arm velocity. Pinch the fork. Using the RUN/STOP function, freeze the free decay and measure the average period $T \cong 2\pi/\omega_0$. In order to get the amplitude relaxation time $1/\gamma$, you have to switch the oscilloscope time base to seconds. Evaluate Q_{mech}.
Question: How can the drive coil pick up the fork oscillation?

Connect the generator to the drive coil and to Ch.1 of oscilloscope. Setup a sinusoidal wave in the range 50-100Hz. Mount the pickup coil, connect it to Ch. 2 and obtain the sinusoidal response of the fork (always allow for the fact that some of the output is direct pickup from the drive coil). Slowly rotate the pickup coil it until the signal/noise ratio is optimal.
Carefully tune the frequency until you reach the resonance. Please note that resonance is very narrow (occurs within 1-2 Hz). Within small limits, the fork will tend to "pull" the generator into the right relation. This is a primitive example of a "phase lock". Find the ½ power points and calculate another value for Q_{mech}.
Comment on differences and error sources.

➔ **Python Requirement 2 (PHY224/324 students only):** Evaluate Q_{mech}.and the ½ points as outputs of another Python program to fit the data from Exercise 3

Question: What is the role of the magnets mounted at the end of fork arms?
The long term stability of the fork means it can be used to set the frequency of an otherwise broad frequency system. This is the low frequency analogue to oscillators which use piezo-electric crystals with MHz resonant frequencies.
*Optional:*Try driving the fork with the square wave signal. The high Q ensures response only at certain Fourier components of the square wave.

This experiment was revised in 2007 by Ruxandra Serbanescu and Luke Helt. Revised in 2009 by RMS.