Slinky waves

Introduction

In this experiment you will look at the propagation of waves. The experimental system is designed to make the wave motion slow enough to allow you to view directly a number of features of wave motion. We hope that by the end of the experiment you will have developed a number of intuitions about how waves travel, that you will have learned some new properties of wave motion and that you will have some level of quantitative understanding of wave propagation.

Much of this investigation is qualitative. In introducing a number of phenomena we try to give a conceptual explanation. In this process we shall be quoting formulae and relations that you should be able to understand and the results of which you should be able to use. However, many of these formulae involve theory beyond the reach of students in an introductory physics course, so we do not expect that you will be able to follow the derivations, which we shall thus not detail.

Theory and background

You shall investigate the wave propagation in a dispersive medium. Non-dispersive propagation is simpler to deal with, but is probably less common in nature. Non-dispersive waves move at a constant speed, independent of the frequency of the wave disturbance. An example of this is wave motion on a stretched string. In a non-dispersive wave medium, waves can propagate without deformation. Electromagnetic waves in free space are nondispersive as well as nondissipative and thus can propagate over astronomical distances. Sound waves in air are also nearly nondispersive even in the ultrasonic frequency range.
Most waves in material media are dispersive. Dispersive propagation involves a variation of wave speed with frequency. Examples of this are waves on water surface and light waves moving through glass.

Our dispersive system

You shall investigate dispersive wave propagation using a suspended slinky system. This consists of a coil spring which can viewed as a series of coil loops coupled together by the spring constants of the coils. When one loop is displaced by a distance y along the axis of the spring, the elastic force on its neighbours will determine an acceleration. A wave motion results because the displacement of one coil loop produces a delayed displacement of its adjacent loops. The delay depends on the mass of the loop and the spring constant of the joining components between the loops. The slinky would be non-dispersive, but this would be correct only if the slinky were resting on a friction-free skating rink. However, our slinky is suspended by strings of a fixed length, so that each coil loop thinks itself to be a pendulum of length equal to the string length and mass equal to the loop mass. Thus, two sets of forces act on each loop, one as described above which produces wave motion, and one from gravity which provides a restoring force to make the loop swing back to its centre position. In this experiment,we shall find out that the pendulum component makes the slinky waves dispersive.

It can be shown that, for this suspended slinky system, ω and k are related in a slightly more complicated form than that of equations 1, and 2:

$$\omega = \sqrt{\omega_o^2 + c_o^2 k^2} \tag{1}$$

$$k^2 = \frac{\omega^2 - \omega_o^2}{c_o^2} \tag{2}$$

Recall: $k = \frac{2\pi}{\lambda}$ where k is the wave number and λ is the wavelength. Also: ω_o is the angular frequency for a single pendulum of the slinky suspension string system and c_o is, the constant wave speed of a non dispersive system, in this case consisting of the slinky resting on a frictionless surface.

Note that since the slinky can be driven by a motor at any frequency, ω can assume a variety of values. Equation (1) leads us to investigate three ranges of values of for ω.

Range 1: $\omega > \omega_o$

In this case $k^2 > 0$, so k is a real number. Equations (3) and (4) below fully describe the wave motion of the slinky under these conditions:

$$y_r = A_r cos(\omega t - kx + \phi_r) \tag{3}$$

$$y_l = A_l cos(\omega t + kx + \phi_l) \tag{4}$$

where y_r and y_l represent waves moving to the right and left respectively and y represents the general case: $y = y_r + y_l$. $\omega = 2\pi f$ is the angular frequency and f is the frequency of oscillations as seen from a position in space. The speed of phase propagation of these waves is given by:

$$c = \frac{\omega}{k} = \frac{\omega c_o}{\sqrt{\omega^2 - \omega_o^2}} \tag{5}$$

At higher frequencies, this case produces wave motion because the high acceleration of the coil loops makes the inertial effect of each loop predominate over the pendulum effect due to the strings.

In your experiment with the slinky, one end of the slinky is held fixed and the other may be driven sinusoidally. For convenience take the fixed end to be $x \equiv 0$; the fact that it is fixed implies $y = 0$ at $x = 0$ for all times t. From (4) and (5) you can derive, with proper choice of the origin of time, the following solution:

$$y = y_o sin(\omega t) sin(kx) \tag{6}$$

where y_o is a constant. The wave in this case is a standing wave rather than a propagating wave: it is a superposition of two waves of equal amplitudes propagating in opposite directions with nodes of zero displacement separated by distances $\frac{\pi}{k}$, half the wavelength $\frac{2\pi}{k}$ of the propagating waves.

The amplitude y_o is determined by the driving motor at the end of the slinky where $x = L$. If the motor enforces a sine wave motion with amplitude y_d we obtain: $y = y_d sin\omega t$ at $x = L$. From equation (6) we get:

$$y_d sin(\omega t) = y_o sin(\omega t) sin(kL) \tag{7}$$

and so:

$$y_o = \frac{y_d}{sinkL} \tag{8}$$

y_o will tend toward infinity, being limited only by dissipation and non-linear effects, for any k such that:

$$k = k_n \equiv \frac{n\pi}{L}, n = 1, 2, 3... \tag{9}$$

This represents resonant behaviour: the different natural modes of oscillation represented by (9), are called the normal modes or the eigenmodes. Between the two ends, they exhibit one loop, two loops, three loops, etc. They may be thought of as: fundamental (or first harmonic), first overtone (second harmonic), second overtone (third harmonic), etc. Note, however, that the harmonics are characterized by k being some integer multiple of the fundamental value $\frac{\pi}{L}$; the associated ω, from (1) does not progress in steps of integer multiples, but instead goes as:

$$\omega = \sqrt{\omega_o^2 + c_o^2 k_n^2} = \sqrt{\omega_o^2 + \frac{n^2 c_o^2 \pi^2}{L^2}} \tag{10}$$

This is different from a non-dispersive violin string in which: $\omega_n = \frac{n c_o \pi}{L}$

It might be pointed out that the human ear finds pleasure in combinations of notes whose frequencies are integer multiples of a fundamental frequency; octaves have a frequency ratio of 2:1, fifths have a 3:2 frequency ratio, major thirds a 5:4 ratio, etc. Thus non-dispersive violin strings can produce pleasant sounds. However, this writer would hate to listen to sounds produced by a high frequency dispersive suspended slinky.

Range 2: $\omega < \omega_o$

In this case $k^2 < 0$, so k is an imaginary number. This would imply that no real wave could be propagated. Thus, solutions (1) and (2) do not work. The solutions for this case are:

$$y_r = A_r cos(\omega t + \phi_r) e^{-kr} \tag{11}$$

$$y_l = A_l cos(\omega t + \phi_l) e^{+kr} \tag{12}$$

$$y = y_r + y_l \tag{13}$$

where:

$$k^2 = \frac{\omega_o^2 - \omega^2}{c_o^2} \tag{14}$$

Now we have $\omega = 2\pi f$ where f is the frequency of oscillation as seen from one position in space and $k = \frac{1}{x_o}$ where x_o is the exponential decay distance to $\frac{1}{e}$ of the initial amplitude.

This case produces no wave motion at lower amplitudes because the pendulum effect due to the restoring force of gravity on the coil loops predominates over inertial effects. Thus each coil sees its adjacent coil pulling it to one side, while gravity tries to pull it back to its central position. The whole slinky moves in unison, the amplitude of motion decreasing exponentially with distance along the slinky. This solution of equations 10-12 produces two waves which are called *evanescent*: they are stationary, rather than propagating in the x direction, one growing and the other decaying exponentially with increase of x, the *e-folding length* being $\frac{1}{k}$.

With an infinitely long sinusoidally driven spring, the relevant solution would of course be the one that decayed exponentially away from the driven point (y_r being 0 on one side, and y_l being 0 on the other side). In the present experiment, with the fixed end taken as $x \equiv 0$ so that $y = 0$ at $x = 0$, we have the solution:

$$y = y_o sin\omega t (e^{+kx} - e^{-kx}) \tag{15}$$

As before, the amplitude y_0 is determined by the driving motor at $x = L$ where it enforces a sine wave motion with amplitude y_d: $y = y_d sin\omega t$ at $x = L$, so:

$$y_o = \frac{y_d}{e^{+kL} - e^{-kL}} \tag{16}$$

Note that there are no resonances available here: y is necessarily a monotonic function of x, so there are no nodes between the end points; the wave is a standing or stationary wave, but one without the familiar node-and-loop pattern.

The limiting case of $\omega \to 0$, approximated by taking one end of the suspended slinky and holding it with a fixed displacement shows the exponential decay of the displacement along the slinky. We have:

$$k = \frac{\sqrt{\omega_o^2 - \omega^2}}{c_o} \to \frac{\omega_o}{c_o} \tag{17}$$

In this case, if the pendulum restoring force is much weaker than the spring restoring force, as is the case in our slinky, so that $\frac{c_o}{\omega_o}$ is very small compared to the apparatus dimensions. For reasonable distances away from the fixed end $kL \geq kx \gg 1$ so that $y \approx y_o e^{kx} sin\omega t$ and the displacement of the coils is exponential.

Range 3: $\omega = \omega_o$

This is the limiting case between the previous two ranges, with $k_{1,2} = 0$

For our already stated boundary conditions of $y = 0$ at $x = 0$ and $y = y_d sin\omega t$ at $x = L$ we can obtain the solution:

$$y = y_d \frac{x}{L} sin\omega t \tag{18}$$

with the amplitude diminishing linearly from y_d at the driven end to 0 at x = 0.

The Apparatus

The slinky is fixed at one end and has a high precision sinusoidal drive at the other. You should inspect the drive to assure yourself that the motion is truly sinusoidal. The drive motor speed is continuously variable and is adjusted from a control box. The drive has an electronic timer associated with it which measures the time between successive oscillations. The timer has an accuracy of $\pm 0.001s$. It should be noted that the slinky system has quite a high Q value, which means that little energy gets lost from the slinky spring in each oscillation. A result of this is that modes of motion that get started take a long time to decay away and it takes a long time for new modes to start up. Thus in varying the frequency (speed) of the drive it is important to make the changes slowly or at least to wait a long time after a sudden change is made to observe the steady state result of the change.

You will note the three sets of *combs* on the apparatus. These may be raised or lowered to change the effective string length and thus change ω_o. A 3 mm diameter steel rod is provided to facilitate the lowering or raising of the combs: the rod gets inserted in the hole near the edge of the comb.

The Experiment

The following suggests a number of things you can do to check out the properties of the slinky. Investigate the wave characteristics according to your interest.

1. Start a pulse off at one end and time it over the round trip to the other end and back. This is done by giving one lateral shake to a coil near one end of the slinky. Note that the pulse that returns is not as sharply defined as the one you send; this is to a lesser part the result of dissipation (whose effects have not been discussed) but principally is non-dissipative dispersion caused by the fact that c in equation (5) is not precisely constant for the range of ω values of your pulse. Find an approximate value for c_0 for the suspended slinky.

2. Deduce ω_o for the suspended slinky by direct measurement of the vertical length of the strings, with filtering *combs* out of the way.

3. Drive the slinky at a variety of ω values in the propagating range ($\omega > \omega_o$). Measure the distance between adjacent nodes and determine the corresponding k. Plot ω^2

against k^2 and deduce ω_0^2 and c_o^2 from a linear fit according to equation (1). After each measurement, turn the driving motor off suddenly; note that, in general, the oscillations persist in a haphazard fashion, back and forth and gradually dissipate. They clearly not continue as simple oscillations (at a given point) at the original ω. This haphazardness does not exist for normal modes as you will find below.

4. Find the family of resonant ω values within the range of values provided by your driving motor. *Note: do not destroy the suspension in the process!* Are the frequencies integral multiples of some fundamental angular frequency, or is the more general relation of equation (10) required to describe them? For each resonant mode, turn off the motor abruptly. Note that the mode remains in situ, decaying gradually and not haphazardly.

5. Drive the slinky at an $\omega \ll \omega_o$. Does the amplitude fall off exponentially with distance from the driven end? Or are both exponentials in equation (15) required to describe it? Collect some data and perform a nonlinear fit, trying one exponential or a sum of two exponentials as your model. Can you pick in advance an ω that will lead to $kL \gg 1$ and hence a simple exponential decay near the driven end? Try the extreme of $\omega \to 0$ by merely displacing and holding one end stationary and noting the displacements y of the coils down the apparatus.

6. Having determined ω_o before, drive the slinky at $\omega = \omega_o$. Collect some data and perform a linear fit to test how accurately the linear decay law of equation (18) is obeyed. Discrepancies will indicate imperfections of suspension, or non-uniformities in the slinky, or perhaps other dissipative effects. Can you distinguish between them?

7. When a filtering comb is lowered and meshed with the supporting strings, it effectively shortens the string length ℓ and so raises the ω_o of that portion of the slinky. Let ω_ℓ be the ω_o for the original long strings and ω_s be the ω_o for the short strings, with $\omega_\ell < \omega_s$. There are ω values that satisfy $\omega_\ell < \omega < \omega_s$, such that corresponding waves may propagate through the long-string ℓ section of the slinky but are prevented from propagating through the short s section. The s section acts as a filter for the removal of propagating waves. As you will see in the next exercise, wave energy can actually *tunnel* through the s section, particularly if k_sL_s is not $\gg 1$ where k_s and L_s are the wave number and length of the s sections.

8. Try lowering combs number 2 and number 3, away from the driven end. Now you can establish a resonance belonging essentially to section number 1, nearest the drive; since sections number 2 and number 3 will not support propagation, they behave somewhat as if they moved the fixed point $x = 0$ toward the driven end (in the sense that they produce only small amplitudes of oscillation), and the loopa are confined primarily to section number 1.

9. Now lower combs number 1 (nearest to the driven end) and number 2, raising comb number 3. Drive at the same frequency. Does section number 3 exhibit resonance? Can the energy from the driving motor indeed *tunnel* through sections number 1 and number 2, to establish a resonant response in section number 3? Does section number 3 exhibit larger amplitudes than sections number 1 and number 2, despite being farther from the driven end?

10. With comb number 1 raised and with section number 1 resonating, the combs in the other two sections are lowered. Determine the resonance wave period. Turn off the driving motor abruptly and simultaneously raise the comb in section number 3 so that section 3 could also resonate. Resonance happens only if section 3 receives energy at

the oscillation frequency of the other section. Watch (in fascination) as the energy tunnels through the intervening section number 2, and then tunnels back again, in a fashion reminiscent of loosely coupled pendula, though now more dissipative. Section number 2 provides the coupling, and the fact that it is a non-propagating region is what makes the coupling loose. Can you determine the beat period of tunnelling?

11. The section before suggests that the apparently resonant mode with which you started has become a superposition of two resonant modes, beating together with the lifting of the comb number 3. Find the resonant modes, at frequencies slightly higher and lower than the original frequency. Note that one is a symmetric mode $y(x) = y(-x)$ when x is measured from the centre, and the other is an antisymmetric mode $y(x) = -y(-x)$ when x is measured from the centre. Compare the difference between the resonance frequencies and the beat frequency of tunnelling.

For those who did not intend to do all three weights of this experiment, we would suggest that parts 1 and 2 are fast and useful, and parts 5, 7, 10, 11 are probably most interesting.

REFERENCES

A.P. French, Vibrations and Waves, Norton 1971, Chapters 5, 6, 7, 8

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