PHY 2404S Lecture Notes

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Winter, 2003

These notes are perpetually under construction. Please let me know of any typos or errors. Once again, large portions of these notes have been plagiarized from Sidney Coleman's field theory lectures from Harvard, written up by Brian Hill.

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1 Preliminaries

1.1 Counterterms and Divergences: A Simple Example

In the last semester we considered a variety of field theories, and learned to calculate scattering amplitudes at leading order in perturbation theory. Unfortunately, although things were fine as far as we went, the whole basis of our scattering theory had a gaping hole in it. Recall then when we used Wick's theorem to calculate S matrix elements between some initial state $|\, i\rangle$ and some final state $|\, f\rangle$,

$$S_{fi} = \langle f\,|S|\,i\rangle \tag{1.1}$$

the incoming and outgoing states $|\, i\rangle$ and $|\, f\rangle$ were considered to be eigenstates of the *free* Hamiltonian; that is, n particle Fock states. The idea was that far in the past or future when the colliding particles are widely separated, they don't feel the interactions between them. Thus, the incoming and outgoing states should be eigenstates of the free theory.

This is clearly nonsense. Even when an electrons is far away from all other electrons, it doesn't look anything like a single particle Fock state. It carries with it an electric field made up of photons: even when well-separated from other electrons, it is *always* in a complicated superposition of Fock states. It doesn't look anything like a free electron.

Nevertheless, we had a physical argument that suggested we could still calculate using free states in the distant past and future. The argument went as follows: suppose we replaced the interaction term in the Hamiltonian $\mathcal{H}_I$ by a modified interaction term,

$$\mathcal{H}_I \to f(t)\mathcal{H}_I \tag{1.2}$$

where $f(t)$ (the "turning on and off function") is some function which is one at $t = 0$ but which vanishes for large $|t|$. Since the interaction turns off in the far past and far future, we are justified in using free states in Eq. (1.1).

The question now is, can we do this without changing the physics? Clearly, if at $t = -T/2$ I suddenly turned the interaction on (that is, if $f(t)$ were a step function) all hell would break loose, and the scattering process would be drastically altered. Since a free electron is in a horribly complicated superposition of eigenstates of the full Hamiltonian (just as the electron with its electromagnetic field is in a horribly complicated superposition of eigenstates of the free Hamiltonian), as soon as I turned the interaction on I would be left in a superposition of states which looked nothing like a real electron. On the other hand, suppose I were to turn the interaction on slowly, very slowly (that is, adiabatically). Then maybe, just maybe, I would be ok, because if I took a very long time turning the interaction on, I would expect the free state to slowly acquire a photon field, and to smoothly, with probability 1, turn into an eigenstate of the full theory. In this case, $f(t)$ would look something like that shown in Fig. 1.1, in the combined limits $T \to \infty$ (so the interaction is on until the particles are arbitrarily far apart, $\Delta \to \infty$ (so the transition is adiabatic), and $\Delta/T \to 0$

(so the transition time is much less than the time the particle spends with the correct Hamiltonian).

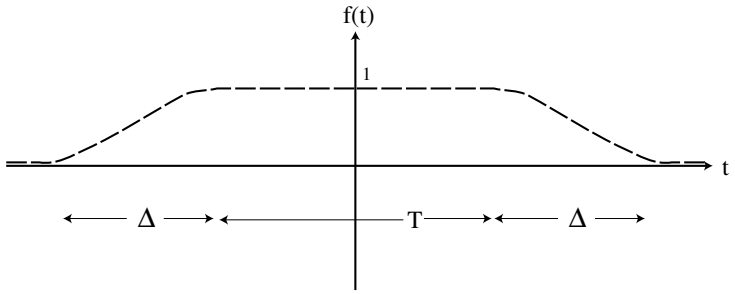


Figure 1.1: Schematic form of the "turning on and off" function $f(t)$, required to define scattering theory when the interaction doesn't vanish in the far past or future. In the limit $T \to \infty,\ \Delta \to \infty, \Delta/T \to 0$ the results for the original theory should be recovered.

In other words, the scattering process goes something like this: a billion years before the collision, its interaction with the photon field is turned off, and the electron is a free particle. Then, over a time of a million years, its charge is slowly turned on, and the electron picks up a photon cloud. Then at $t = 0$ the fully interacting electron collides with a target, and produces a bunch of particles. A billion years later, when they are all well separated, these particles slowly lose their photon clouds, and a million years later we have a bunch of free particles again. (An electron with its interactions turned off is usually known as a "bare" electron; when it comes along with its photon cloud it's known as a "dressed" electron). Now this approach clearly won't work for bound states, since no matter how far in the future you go the constituents never get far enough apart not to feel their interactions. But for a low-budget scattering theory it should work. Soon we will develop a hi-tech scattering theory without this kluge, but it will suffice for the moment.

To understand the importance of these considerations, it's instructive to go back to a problem we have looked at before, that of free field theory with a source, and see how the subtleties with the turning on and off function arise. Let us consider a theory with a *time-independent* source,

$$\mathcal{L}_I = -g\rho(\vec{x})\varphi(\vec{x},t), \quad \rho(\vec{x}) \to 0, |\vec{x}| \to \infty \tag{1.3}$$

(where g is a coupling constant). This looks just like a special case of the problem we considered before, but in fact it is much more subtle. The difference is that the source doesn't vanish in the far future or the far past, which was implicit in the exact solution we obtained for free field theory with a source. So we are going to have to implement our turning on and off function to make sense of it.

Let's see how this problem shows itself in perturbation theory. Recall from the perturbative solution to free field theory with a source that the Feynman

rule for the source term is that shown in Fig. 1.2, and that, diagrammatically,

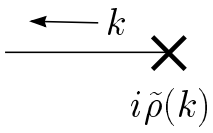


Figure 1.2: Feynman rule for a source term $\rho(x)$.

the amplitude for the vacuum to be unchanged in the far future has the perturbative expansion shown in Fig. 1.3. Defining α to be the subdiagram with a

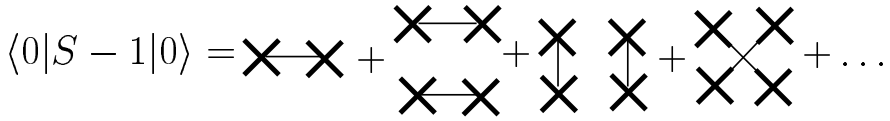


Figure 1.3: Perturbative expansion for $\langle 0\,|S-1|\,0\rangle$ in a theory with source.

two sources connected by a meson propagator (not including the factor of 1/2! coming from Wick's theorem),

$$\alpha = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{\tilde{\rho}(k)\tilde{\rho}(-k)}{k^2-\mu^2+i\epsilon} \tag{1.4}$$

and taking into account the combinatorics of connecting n points, it is simple to sum the series,

$$\begin{aligned} \langle 0\,|S|\,0\rangle &= 1 + \frac{1}{2!}\alpha + \frac{3\times 1}{4!}\alpha^2 + \frac{5\times 3\times 1}{6!}\alpha^3 + \ldots \\ &= 1 + \left(\frac{\alpha}{2}\right) + \frac{1}{2!}\left(\frac{\alpha}{2}\right)^2 + \frac{1}{3!}\left(\frac{\alpha}{2}\right)^3 + \ldots \\ &= e^{\frac{\alpha}{2}}. \end{aligned} \tag{1.5}$$

Now, what do we expect the result to be? Since this is just a theory of a static arrangement of charges, the system should just sit there. The source is time-independent, so it can't impart any energy to the system. Thus, it can't create mesons, so the vacuum state can't change, and we should find

$$\langle 0\,|S|\,0\rangle = 1 \tag{1.6}$$

or $\alpha = 0$. But this isn't what we get. Instead, from the expression above, and using the Fourier transform of a time-independent source (forgetting, for the moment, about $f(t)$)

$$\begin{aligned} \tilde{\rho}(k) &= \int \frac{d^3k}{(2\pi)^3}\rho(\vec{x}) \int \frac{dk_0}{2\pi} e^{ik_0 t - i\vec{k}\cdot\vec{x}} \\ &= \delta(k_0)\tilde{\rho}(\vec{k}) \end{aligned} \tag{1.7}$$

(where $\tilde{\rho}(\vec{k})$ denotes the usual three dimensional Fourier transform of $\rho(\vec{x})$) we find

$$\alpha = -ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{|\rho(\vec{k})|^2 |\delta(k_0)|^2}{k_0^2 - \vec{k}^2 - \mu^2 + i\epsilon}. \tag{1.8}$$

This is horribly divergent, since squaring a delta function is not a particularly well-defined thing to do. So instead of zero, we got a divergent imaginary result for α. Instead of being unity, the S matrix contains a divergent phase, $S \sim \exp(-i\infty)$.

Let's see what happens now if we carefully include our turning on and off function,

$$\mathcal{L}_I = -g\rho(\vec{x}) f(t). \tag{1.9}$$

In this modified theory, we find

$$\alpha = -ig^2 \int \frac{d^3k}{(2\pi)^3} |\tilde{\rho}(\vec{k})|^2 \int \frac{dk_0}{2\pi)} |\tilde{f}(k_0)|^2 \frac{1}{k_0^2 - \vec{k}^2 - \mu^2 + i\epsilon} \tag{1.10}$$

where $\tilde{f}(k_0)$ is the Fourier transform of $f(t)$. Now, as $T \to \infty$, $\tilde{f}(k_0)$ becomes sharply peaked about $k_0 = 0$, with a width proportional to $1/T$. Thus, we can drop the factor of k_0^2 in the propagator, and we obtain

$$\begin{aligned}
\alpha \overset{T\to\infty}{\to} \quad & ig^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{\rho}(\vec{k})|^2}{\vec{k}^2 + \mu^2} \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} |\tilde{f}(k_0)|^2 \\
= \quad & ig^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{\rho}(\vec{k})|^2}{\vec{k}^2 + \mu^2} \int_{-\infty}^{\infty} dt\, |f(t)|^2 \\
= \quad & ig^2 T \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{\rho}(\vec{k})|^2}{\vec{k}^2 + \mu^2} \times (1 + O(\Delta/T)) \\
\equiv \quad & -2iE_0 T\, (1 + O(\Delta/T))
\end{aligned} \tag{1.11}$$

where we have used Parseval's theorem (which is simple to prove just from the definition of the Fourier transform) between the first and second lines, and we have defined

$$E_0 \equiv \frac{-g^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{\rho}(\vec{k})|^2}{\vec{k}^2 + \mu^2}. \tag{1.12}$$

Thus, we find

$$\langle 0\,|S|\,0\rangle = \lim_{T\to\infty} e^{-iE_0 T}. \tag{1.13}$$

Aha! This is just the usual Schrödinger evolution of a state with energy E_0! The problem is that the vacuum state in the interacting theory isn't the same as the vacuum state in the free theory, and since by fiat we set the energy of the *free* vacuum to be zero, the energy of the *true* vacuum (the vacuum of the full, interacting theory) is not zero, but E_0. No wonder we got divergent nonsense.

The origin of this energy shift should be clear - it's just the energy stored in the meson field produced by the sources. We can make this more transparent

by expressing the potential in position, rather than momentum, space. We can write

$$\begin{aligned} V(\vec{x}) &= -g^2 \int \frac{d^3x}{(2\pi)^3} \frac{e^{i\vec{k}\cdot x}}{\vec{k}^2 + \mu^2} \\ &= -\frac{g^2}{4\pi|\vec{x}|} e^{-\mu|\vec{x}|} \end{aligned} \tag{1.14}$$

in terms of which we have

$$E_0 = \frac{1}{2} \int d^3x \, d^3y \, \rho(\vec{x})\rho(\vec{y})V(\vec{x} - \vec{y}). \tag{1.15}$$

In analogy with electrostatics, this is just the potential energy of a charge distribution $\rho(\vec{x})$, where the potential between two unit point charges a distance r apart is $V(r)$ (the factor of $1/2$ is there because by integrating over $\vec{x}$ and $\vec{y}$ you double count each pair of point charges). This is the Yukawa potential we discussed in the context of scattering, and provides another way of seeing the exchange of a scalar boson of mass μ produces an attractive Yukawa potential. In particular, if we consider the source to consist of two almost-point charges at points $\vec{y}_1$ and $\vec{y}_2$,

$$\rho(\vec{x}) = \Delta(\vec{x} - \vec{y}_1) + \Delta(\vec{x} - \vec{y}_2) \tag{1.16}$$

where $\Delta(\vec{x})$ approaches a δ function, we get

$$E_0 = (\text{something independent of } \vec{y}_1, \ \vec{y}_2) + V(\vec{y}_1 - \vec{y}_2) \tag{1.17}$$

The term independent of $\vec{y}_1$ and $\vec{y}_2$ is just the interaction of each "point" charge with itself, and diverges as $\Delta(\vec{x}) \to \delta^{(3)}(\vec{x})$. This is the same problem as the divergent energy stored in the field of a single point charge in classical electrodynamics. But since it's a constant, we don't care about it: the internucleon potential $V(r)$, which is the only measurable quantity, is perfectly well-defined.

So now that we understand the origin of the divergent phase, it's easy to see how to fix it. We should just define our zero of energy to be the energy of the *true* vacuum, not the free vacuum. Thus, we just add a constant term to the interaction Hamiltonian,

$$H_I \to H_I - E_0 \tag{1.18}$$

or, equivalently,

$$L_I \to L_I + E_0 \tag{1.19}$$

or in terms of the Lagrange density,

$$\mathcal{L}_I \to \mathcal{L}_I + a \tag{1.20}$$

where

$$a \equiv \frac{1}{2} \int d^3y \rho(\vec{x})\rho(\vec{y})V(\vec{x} - \vec{y}) \tag{1.21}$$

is called a *counterterm*. This is a term added to the Lagrangian which fixes up the fact that some property of the free theory (such as the vacuum energy)

is not the same in the interacting theory, and this must be corrected for. We will encounter more of these later on when we start looking at theories with dynamical sources. But for now, using the corrected Lagrangian, we have the new (rather trivial) interaction shown in Fig. 1.4, which once again exponenti-

Figure 1.4: Feynman rule corresponding to the vacuum energy counterterm.

ates, precisely cancelling each term of the previous series. Thus, in the modified theory, we find

$$\langle 0\,|S|\,0\rangle = 1 \tag{1.22}$$

as required.

A couple of comments:

1. In the case of a point source, $\Delta(\vec{x}) \to \delta^{(3)}(\vec{x})$, the counterterm a diverges. Nevertheless, the S matrix element is perfectly finite. The divergent counterterm is required to cancel the divergent energy of the field of a point source. While this infinity may be scary, it is not harmful. Since terms in the Lagrangian are not observable, they need not be finite.

2. There is a simpler way to deal with the vacuum energy shift than worrying about counterterms. Since the disconnected diagrams exponentiate, and every S-matrix element contains the same set of disconnected diagrams, they contribute a common phase to *every* S-matrix element. Thus, we can write any S-matrix element as

$$\langle k_1', \ldots, k_n'\,|S|\,k_1, \ldots, k_m\rangle = (\text{sum of connected diagrams}) \times \langle 0\,|S|\,0\rangle \tag{1.23}$$

where, when the counterterm is not included,

$$\langle 0\,|S|\,0\rangle = \exp\left(\text{sum of disconnected diagrams}\right) = \exp(-iE_0T). \tag{1.24}$$

Thus, if we adopt the rule of thumb that we only calculate *connected* diagrams, we can completely neglect the vacuum energy counterterm. Formally, this just means that we divide all amplitudes by $\langle 0\,|S|\,0\rangle$:

$$\langle k_1', \ldots k_n'\,|S|\,k_1, \ldots, k_m\rangle_R = \frac{\langle k_1', \ldots k_n'\,|S|\,k_1, \ldots, k_m\rangle}{\langle 0\,|S|\,0\rangle}. \tag{1.25}$$

where the subscript R indicates that the vacuum energy has been renormalized to 0. This will be the approach we take.

Let's push this model a bit harder. Having found the energy of the true vacuum, let's find the particle content of the ground state in terms of eigenstates of the

free theory. We can find it by using the results for the case of a time-dependent source, by considering the following form for the source:

$$\begin{aligned} \rho(x) &= \rho(\vec{x})e^{\epsilon t},\ t<0 \\ \rho(x) &= 0,\ t>0 \end{aligned} \tag{1.26}$$

and then taking the limit $\epsilon \to 0^+$. Physically, what we are doing is turning the interaction on very slowly, so that the free vacuum smoothly transforms into the true vacuum of the static theory. Then we turn the interaction abruptly off, so that in the interaction picture the state doesn't evolve at all after $t = 0$. This is not the same as the theory of a static source, since the interaction is not adiabatically turned off far in the future. Instead of evolving smoothly back into the free vacuum, when the source is suddenly turned off the system is left in the physical vacuum.

Let us denote the vacuum of the static theory by $|\,\Omega\rangle$, and the free vacuum as usual by $|\,0\rangle$. Then the statement that the free vacuum at $t = -\infty$ smoothly transforms into the physical vacuum at $t = 0$ may be written as

$$|\,\Omega\rangle = U_I(0,-\infty)|\,0\rangle \tag{1.27}$$

where $U_I(t_1,t_2) = T\exp\left(\int_{t_1}^{t_2} dt\, H_I(t)\right)$ is the usual time-evolution operator. Furthermore, since the system does not evolve past $t = 0$, we have $U_I(t,0) = 0$ for any positive time t. The 0 to n meson S-matrix elements may then be written

$$\begin{aligned} \langle\vec{k}_1,\ldots,\vec{k}_n\,|S|\,0\rangle &= \langle\vec{k}_1,\ldots,\vec{k}_n\,|U_I(\infty,-\infty)|\,0\rangle \\ &= \langle\vec{k}_1,\ldots,\vec{k}_n\,|U_I(0,-\infty)|\,0\rangle \\ &= \langle\vec{k}_1,\ldots,\vec{k}_n|\,\Omega\rangle \end{aligned} \tag{1.28}$$

which is exactly what we are looking for: the overlap of the true vacuum with the n-meson eigenstates of the free theory. Then, using the results for the theory with a time-dependent source, we recall that the resulting state $|\,\Omega\rangle$ is a coherent state of mesons, and that the probability for the state to contain n mesons is

$$P(n) = \frac{1}{n!}\alpha^n \exp(-\alpha) \tag{1.29}$$

where

$$\alpha = g^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left|\tilde{\rho}(k)\right|^2. \tag{1.30}$$

From the form of $\rho(x)$, we then find

$$\begin{aligned} \tilde{\rho}(k) &= \int d^4x e^{ik\cdot x}\rho(x) \\ &= \int d^3x e^{-i\vec{k}\cdot\vec{x}}\rho(\vec{x})\int_{-\infty}^{0} dt\, e^{ik_0 t}e^{\epsilon t} \end{aligned}$$

$$
\begin{aligned}
&= \quad \tilde{\rho}(\vec{k}) \frac{1}{ik^0 + \epsilon} \\
&\overset{\epsilon \to 0}{=} \quad -\frac{i}{k_0} \tilde{\rho}(\vec{k})
\end{aligned} \tag{1.31}
$$

and therefore

$$
\alpha = g^2 \int \frac{d^3k}{(2\pi)^3} \frac{|\tilde{\rho}(\vec{k})|^2}{2\omega_k^3}. \tag{1.32}
$$

Now, consider the simple case of a point charge at the origin, $\rho(\vec{x}) \to \delta^{(3)}(\vec{x})$. In this case, $\tilde{\rho}(\vec{k}) \to 1$ for all $\vec{k}$, and we find, for large $|\vec{k}|$,

$$
\alpha \sim \int \frac{d^3k}{\omega_k^3} \sim \int^{\infty} \frac{dk}{k} \tag{1.33}
$$

which is logarithmically divergent: if we only integrate up to $|\vec{k}| = \Lambda$, the upper limit of integration will contribute a piece proportional to $\ln \Lambda$. This is known as an *ultraviolet divergence*, and looks like bad news. Since the expectation value of the number of mesons in the ground state $\langle \Omega \, | N | \, \Omega \rangle = \alpha$ (from our previous results), we see that not only the energy of field becomes infinite in the limit of a point nucleon, but the ground state flees Fock Space. The problem is that a point source can excite mesons of arbitrarily short wavelength, and arbitrarily high energy. On the other hand, as we have already shown, physically observable quantities do not depend on this unpleasant fact. However, it does mean that the vacuum energy counterterm is formally divergent, and that we will have to be careful to make sure that the theory remains well-defined.

Even if we don't take the limit of a point source (so that our theory remains finite in the ultraviolet) we will also run into difficulties if we take the massless limit, $\mu \to 0$. In this case, we get, this time for small $|\vec{k}|$,

$$
\langle \Omega \, | N | \, \Omega \rangle \sim \int_0 \frac{d^3k}{k^3} \sim \int_0 \frac{dk}{k} \tag{1.34}
$$

which is also logarithmically divergent, this time at the lower limit of integration. This is known as an *infrared divergence*, and corresponds to an infinite number of mesons with very large wavelengths, and correspondingly very low energies.

While having an infinite number of arbitrarily low-energy particles sounds like trouble, it turns out that this divergence is also unmeasurable. The number of mesons with arbitrarily small energies is not observable. Any experiment has only some finite lower limit of energy which it can resolve. Even if there are 10^{20} photons with wavelengths between one light year and two light years hitting your detector, you'll never see them. It might be a problem if there were an infinite amount of energy stored by these photons, but there isn't:

$$
\langle \Omega \, | H | \, \Omega \rangle \sim \int d^3k \frac{\omega_k}{\omega_k^3} \sim \int_0 dk \tag{1.35}
$$

which is finite in the infrared. So, interestingly enough, the number of soft massless mesons (or photons, in QED) is not a physical observable. In general, infrared divergences signal that you have attempted to calculate something which is not observable.

To conclude, let me just restate the lessons we have learned from this section. While we have demonstrated them in a simple model, it shouldn't be hard to convince yourself that these issues will arise in any interacting field theory.

1. Properties of states of the free theory, such as the vacuum energy, may be dramatically modified by the interaction, and it in general it is necessary to introduce *counterterms* into the interaction Lagrangian to correct for this. This procedure is known as *renormalization*, and aspects of it will occupy us for much of this course.

2. When pointlike interacting particles are present, the energy and number of quanta of the states of the theory both suffer from *ultraviolet divergences*. Physical quantities are still finite, but this requires the introduction of formally divergent counterterms into the Lagrangian. The procedure of introducing an artificial prescription to make such terms finite (for example, by including a upper cutoff in momentum integrals and at the end taking the cutoff to infinity) is known as *regularization*.

3. When massless particles are present, the theory will have *infrared divergences*, resulting from the fact that there may be an arbitrarily large number of unobservably low-energy quanta in a state. Infrared divergences in a calculation indicate that an observable quantity (such as the number of soft photons in a state) is being calculated. Physical quantities (such as the energy difference between two states) are well-defined.

1.2 Counterterms in Scalar Field Theory

We now consider a more interesting theory with dynamics, and see how these considerations will affect it. While we could be bold and jump straight into QED at this point, QED suffers from all of the problems (particularly, infrared divergences) at once. Furthermore, it has special miracles due to gauge invariance. So for the next few lectures we will instead work with our scalar nucleon-meson theory,

$$\mathcal{L} = \mathcal{L}_\varphi + \mathcal{L}_\psi - g\psi^*\psi\varphi \tag{1.36}$$

keeping all the masses finite, to avoid infrared divergences.

First of all, we will clearly need a vacuum energy counterterm in this theory, since the ground state energy of the physical vacuum is not necessarily zero. Once again, the vacuum is not the simple vacuum state of the free theory, but something rather complicated. Vacuum-to-vacuum graphs (or "vacuum bubbles") will all give (generally divergent) contributions to the vacuum energy.

We are actually helped out by a very nice formula. Recall that in the previous section, we found for a theory with a source, that the sum of all vacuum-to-vacuum diagrams had a very simple form, shown in Fig. 1.5. In other words,

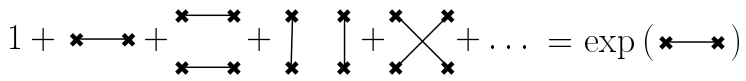


Figure 1.5: The sum of vacuum graphs is just the exponential of the simple connected vacuum-to-vacuum graph.

the sum of the vacuum-to-vacuum graphs (sometimes known, a bit confusingly, as *disconnected* graphs), is the exponential of the simple connected vacuum-to-vacuum graphs. In fact, this result holds in any theory. So in the theory we are considering, for example, the sum of all vacuum bubbles has the simple form shown in Fig. 1.6. Since this class of diagrams is present for *every* S matrix element, it factors out of any amplitude, and just corresponds to the overall phase in any amplitude due to the vacuum energy.

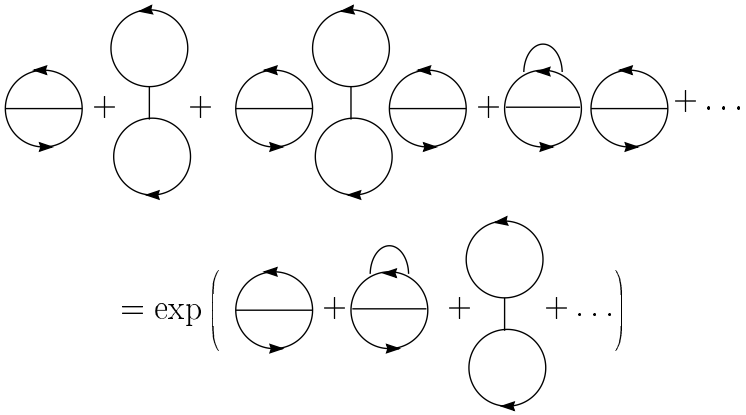


Figure 1.6: The complete sum of vacuum-to-vacuum diagrams is the exponential of the connected vacuum bubbles.

The argument for the exponentiation of the vacuum bubbles goes as follows: a graph with n_i copies of some connected vacuum-to-vacuum bubble gives a contribution to the S matrix of $V_i/n_i!$, where V_i is the value of the connected subgraph, and the $n_i!$ is a symmetry factor arising from the fact that there are $n_i!$ identical copies of the graph. (This is not obvious, but you can show this if you're careful). So any Feynman diagram which contains both a connected piece[1] and a number of vacuum bubbles may be written

$$(\text{graph}) = (\text{connected piece}) \times \prod_i \frac{1}{n_i!} V_i^{n_i} \tag{1.37}$$

[1]I am using the word "connected" here in two different ways: the connected piece of a graph refers to the subgraphs which are connected either to the incoming or outgoing mesons, or both. The connected vacuum-to-vacuum graphs are those vacuum-to-vacuum subgraphs which cannot be broken up into smaller subgraphs.

where the graph contains n_i copies of subgraph V_i. Therefore the sum of all Feynman graphs contributing to a given process may be factored into two pieces:

$$\begin{aligned}
\sum(\text{all graphs}) &= \sum_{\substack{\text{all possible}\\ \text{connected pieces}}} \sum_{\text{all } n_i} \begin{pmatrix}\text{value of}\\ \text{connected piece}\end{pmatrix} \times \prod_i \frac{1}{n_i!} V_i^{n_i} \\
&= \left(\sum \text{connected}\right) \times \sum_{\text{all } n_i} \left(\prod_i \frac{1}{n_i!} V_i^{n_i}\right) \\
&= \left(\sum \text{connected}\right) \times \left(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1}\right) \left(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2}\right) \ldots \\
&= \left(\sum \text{connected}\right) \times \prod_i \left(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i}\right) \\
&= \left(\sum \text{connected}\right) \times \prod_i \exp(V_i) \\
&= \left(\sum \text{connected}\right) \times \exp\left(\sum_i V_i\right). \qquad (1.38)
\end{aligned}$$

For the simple case of a vacuum to vacuum transition, we find

$$\langle 0\,|S|\,0\rangle = \exp\left(\sum_i V_i\right) = \exp(-iE_0T) \qquad (1.39)$$

we can identify the sum of all simple vacuum-to-vacuum graphs with the unrenormalized vacuum energy

$$\sum_i V_i = -iE_0T. \qquad (1.40)$$

Since this phase factors out of all graphs, we can simply renormalize the vacuum energy in any theory by *ignoring all vacuum bubbles*.

In the theory of a static source, fixing up the vacuum energy also fixed up the energies of the single particle states: the interactions don't modify the energy of the single-meson states relative to the vacuum. Thus, the only renormalization required in that theory was that of the vacuum energy. In a theory with nontrivial dynamics, life is more difficult. Even in the distant past and future, the mesons and nucleons are interacting, and this will shift their energies relative to the vacuum.

This happens even in classical physics. Imagine modelling the electron as a charged shell of mass m_0 (the "bare mass"), charge e and radius r. The measured mass of the electron (its rest energy) gets a contribution other than m_0: the energy in its electrostatic field. The measured, physical electron mass m is

$$m = m_0 + \frac{e^2}{2rc^2}. \qquad (1.41)$$

Note that as $r \to 0$, the measured mass differs from the bare mass by a divergent quantity. This is just the UV divergence associated with point particles again. Since the measured mass is some finite number fixed by experiment, this means the bare mass must also be divergent. This puts us again in the somewhat unnerving but nevertheless necessary position of having formally divergent terms in the Lagrangian.

Once again, this is going to be bad news for scattering theory. Just as the failure to match up the ground state energy for the noninteracting and full Hamiltonians in the previous section produced T dependent phases in $\langle 0 \,|S|\, 0\rangle$, the failure to match up one particle state energies in this theory will yield T dependent phases in $\langle \vec{k} \,|S|\, \vec{k}'\rangle$, when in fact we should have

$$\langle \vec{k} \,|S|\, \vec{k}'\rangle = \delta^{(3)}(\vec{k} - \vec{k}'). \tag{1.42}$$

Since there is nothing for a free particle to scatter off, it should just propagate freely (along with its meson field) from $t = -\infty$ to $t = +\infty$. To fix this up, we will require counterterms for the masses of both the nucleon and meson masses,

$$\mathcal{L}_{c.t.} = a + b\psi^*\psi + c\phi^2 \tag{1.43}$$

where b and c are determined by setting the energies of static mesons or nucleons to the physical values μ and m, respectively. But in fact there will be more counterterms to worry about, as you can see from the graph in Fig. 1.7(a), which contributes to $N\overline{N}$ scattering at $O(g^4)$ in perturbation theory. Imposing

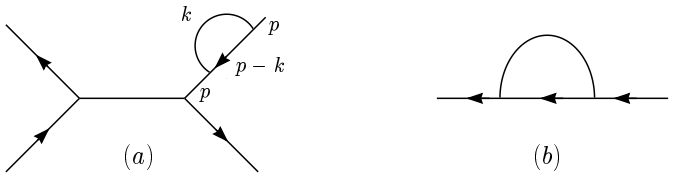


Figure 1.7: An external leg correction to $N\overline{N}$ scattering appears to give a divergent result.

energy-momentum conservation at each vertex, we may label the momenta as shown in the figure (all momenta are directed inward), and we find that one of the intermediate propagators gives a contribution to the graph of

$$\frac{i}{p^2 - m^2} = \frac{i}{0} \tag{1.44}$$

to the graph. This makes no sense.

The problem is that just as vacuum bubbles represent the evolution of the free vacuum $|\, 0\rangle$ to the physical vacuum $|\, \Omega\rangle$, so subgraphs such as Fig. 1.7(b) represent the evolution of the bare states into the "dressed" eigenstates of the full Hamiltonian. It will therefore require some care to correctly treat diagrams

with external leg corrections. The solution has to do with the normalization of the field operators themselves.

Recall that in free field theory fields produced particles from the vacuum with normalized probabilities: $\langle k\,|\varphi(0)|\,0\rangle = 1$ for single particle states $|\,k\rangle$. Furthermore, $\langle k_1, \ldots k_n\,|\varphi(0)|\,0\rangle = 0$ for multi-particle states $|\,k_1, \ldots k_n\rangle$, since there aren't enough annihilation operators in φ to convert a multiparticle state to the vacuum. In an interacting theory, though, this will be different. In the scalar nucleon-meson theory, If you act on the vacuum with a meson field $\varphi'(0)$, you will create a meson. But now that meson can propagate, and at first order in perturbation theory can turn into a nucleon-antinucleon pair; this gives a nonzero value for the matrix element $\langle N(k_1)\overline{N}(k_2)\,|\varphi(0)|\,0\rangle$. Similarly, at higher orders in perturbation theory all sorts of final states can be reached, as illustrated in Fig. 1.8. Furthermore, by conservation of probability, the amplitude to create a single meson from the vacuum must be reduced; thus, we expect that when interactions are included, $\langle k\,|\varphi(0)|\,0\rangle < 1$. Since to related Feynman diagrams to S-matrixes we would like our fields to have the right normalization to create particles, we will have to fix this problem up by rescaling the fields in the theory; this is known as "wavefunction renormalization."

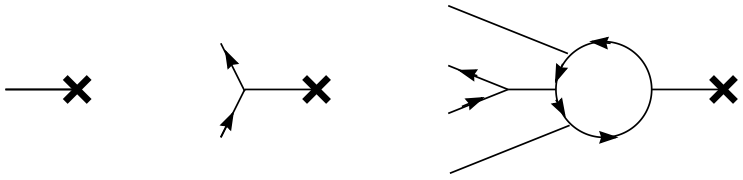


Figure 1.8: In an interacting theory, $\langle n\,|\varphi(0)|\,0\rangle$ may be nonzero for any state n with the same quantum numbers as φ. The figures above correspond to $\langle \phi(k_1)\,|\varphi(0)|\,0\rangle$, $\langle N(k_1), N(k_2)\,|\varphi(0)|\,0\rangle$ and $\langle N(k_1), N(k_2), \phi(k_3), \phi(k_4)\,|\varphi(0)|\,0\rangle$.

We can summarize all of this by rewriting the Lagrangian for this theory with a bunch of subscripts:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi_0)^2 - \frac{\mu_0^2}{2}\varphi_0^2 + \partial_\mu \psi_0^* \partial^\mu \psi_0 - m_0^2 \psi_0^* \psi_0 - g_0 \psi_0^* \psi_0 \varphi_0. \tag{1.45}$$

The subscripts indicate that the corresponding quantity is a *bare*, and so unphysical, quantity. The coefficient of $-1/2\varphi_0^2$ in the Lagrangian, μ_0^2, is not the measured meson mass squared - it's the bare mass, without the interactions turned on. Similarly, m_0^2 is not the real nucleon mass squared. Furthermore, as you can probably guess at this point, g_0 may not be what we want to call the coupling constant. In real electrodynamics there is a parameter e, defined by some experiment. It would be lucky, extremely lucky, if that were the coefficient of some term in the QED Lagrangian. It isn't. Higher order corrections will change the relation of the physically measured quantity to g_0 (or in QED,

e_0). Finally, the field φ in the Lagrangian is not the one we actually want to use to create and annihilate particles. The field φ_0 is normalized to satisfy the canonical commutation relations,

$$[\varphi_0(\vec{x},t),\dot{\varphi}_0(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y}) \tag{1.46}$$

and in general won't have the correct normalization to create a meson from the vacuum,

$$\langle k\,|\varphi_0(0)|\,0\rangle < 1. \tag{1.47}$$

Now, at this point we can proceed in one of two ways. The two approaches differ in which terms of $\mathcal{L}$ we treat exactly, and which as perturbations.

1. We could work directly with the Lagrangian (1.45), and calculate physical quantities from this. This is the approach used by Peskin & Schroeder in chapters 6 and 7. In this case, we would get expressions for the physical quantities μ, m and g in terms of the bare parameters μ_0, m_0 and g_0. All of our cross sections would also come out as functions of the bare parameters, but with a bit of work we could convert these to expressions in terms of the physical quantities μ, m and g. The disadvantage of this approach is that everything is expressed in terms of unphysical (generally divergent) quantities. Since we don't actually *care* what the bare masses or couplings are, this approach is rather unwieldy, particularly at higher orders in perturbation theory.

2. A better approach (used by Peskin & Schroeder in chapter 10 and beyond), known as "renormalized perturbation theory", is to express $\mathcal{L}$ in terms of the physical parameters right from the start. We therefore rewrite Eq. (1.45) as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\mu^2}{2}\varphi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\psi^*\psi\varphi\varphi + \mathcal{L}_{\text{c.t.}} \tag{1.48}$$

   where $\mathcal{L}_{\text{c.t.}}$ is the counterterm Lagrangian

$$\mathcal{L}_{\text{c.t.}} = \frac{A}{2}(\partial_\mu\varphi)^2 - \frac{B}{2}\varphi^2 + C\partial_\mu\psi^*\partial^\mu\psi - Dm^2\psi^*\psi - E\psi^*\psi\varphi + \text{constant}. \tag{1.49}$$

   It is a simple matter to compare Eqs. (1.45), (1.48) and (1.49) and read off the relation between the counterterms $A-E$ and the bare quantities: $\varphi_0 = \sqrt{1+B}\varphi$, $\mu_0^2 = (\mu^2+C)/(1+B)$, etc. The constant is just the vacuum energy counterterm. The two Lagrangians are identical; the only difference is that we have split up the free piece and the interacting piece differently. In the first approach, the meson propagator has a pole at the bare mass μ_0, as shown in Fig. 1.9(a). In the second, the meson propagator has a pole at the physical meson mass μ, and the counterterms give new interactions, as shown in Fig. 1.9(b). The counterterms $A-E$ are determined, order by order in perturbation theory, by requiring that they exactly cancel the contributions to the leading order masses and

couplings due to the interactions, just as the vacuum energy counterterm was adjusted to exactly cancel the corrections to the energy of the vacuum state. So for example, the meson mass shift produced at $O(g^2)$ by the diagram in Fig. 1.9(c) is precisely cancelled by C.[2]

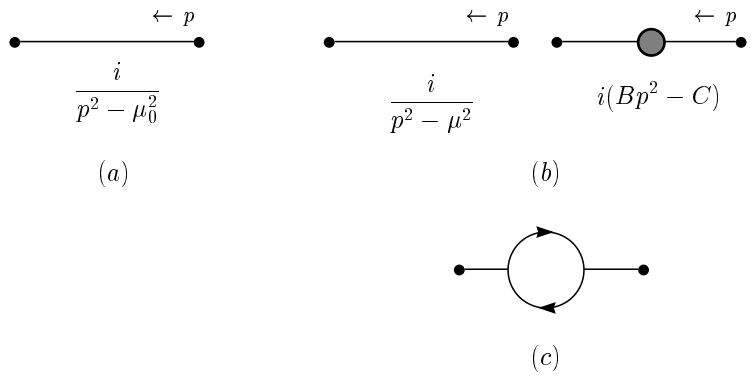


Figure 1.9: (a) In unrenormalized perturbation theory, the propagator has a pole at the bare mass μ_0. (b) In renormalized perturbation theory, the pole of the propagator is at the physical mass m; the counterterms B and C give additional interaction vertices. For example, at $O(g^2)$, the counterterm C precisely cancels the meson mass shift produced by the diagram (c).

In most instances the second approach is simpler. It allows us to avoid dealing with unphysical and uninteresting bare parameters, and deal instead directly with the physical quantities.

Unfortunately, we are still stuck with the clumsy turning on and off function $f(t)$ we have been using as the basis of scattering theory. It has proved sufficient for our limited purposes thus far, but it would be really nice to do away with it altogether. The real world doesn't have a turning on and off function. Is there a way to define scattering theory without it? In the next section we will discuss how this can be done, and reformulate scattering theory in a more elegant language. Then we can start calculating radiative corrections in renormalized perturbation theory.

[2]This does *not* mean that the complete diagram (c) is cancelled by the counterterm! This diagram is a function of p^2, and only for $p^2 = m^2$ is it cancelled by the counterterm; at other values of p^2 (relevant when the meson is off-shell) it contributes.

2 Reformulating Scattering Theory

With the last chapter as motivation, we now proceed to put scattering theory on a firmer foundation. To do that, it is useful first to think a bit about Feynman diagrams in a somewhat more general way than we are used to. Note that the next few subsections will be phrased in our old description of scattering theory, so we will not yet worry about the distinction between bare and full fields - we will have that forced on us later.

2.1 Feynman Diagrams with External Lines off the Mass Shell

Up to now, we've had a rather straightforward way to interpret Feynman diagrams: with all the external lines corresponding to physical particles, they correspond to S matrix elements. In order to reformulate scattering theory, we will have to generalize this notion somewhat, to include Feynman diagrams where the external legs are not necessarily on the mass shell; that is, the external momenta do not obey $p^2 = m^2$. Clearly, such quantities do not directly correspond to S matrix elements. Nevertheless, they will turn out to be extremely useful objects.

Let us denote the sum of all Feynman diagrams with n external lines carrying momenta $k_1, \ldots, k_n$ directed *inward* by

$$\tilde{G}^{(n)}(k_1, \ldots, k_n)$$

as denote in the figure for $n = 4$. (For simplicity, we will restrict ourselves

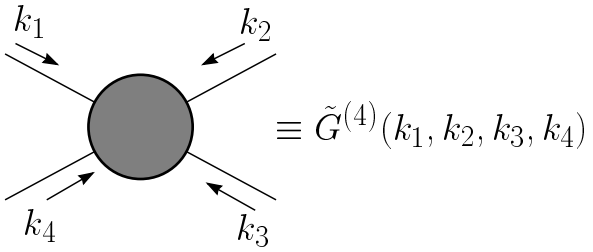


Figure 2.1: The blob represents the sum of all Feynman diagrams; the momenta flowing through the external lines is unrestricted.

to Feynman diagrams in which only one type of scalar meson appears on the external lines. The extension to higher-spin fields is straightforward; it just clutters up the formulas with indices). The question we will answer in this section is the following: *Can we assign any meaning to this blob if the momenta on the external lines are unrestricted, off the mass shell, and maybe not even satisfying* $k_1 + k_2 + k_3 + k_4 = 0$*?*

In fact, we will give three affirmative answers to this question, each one of which will give a bit more insight into Feynman diagrams.

2.1.1 Answer One: Part of a Larger Diagram

The most straightforward answer to this question is that the blob could be an internal part of a more complicated graph. Let's say we were interested in calculating the graphs in Fig. 2.2(a-d), which all have the form shown in Fig. 2.2(e), where the blob represents the sum of all graphs (at least up to some order in perturbation theory). Recalling our discussion of Feynman diagrams

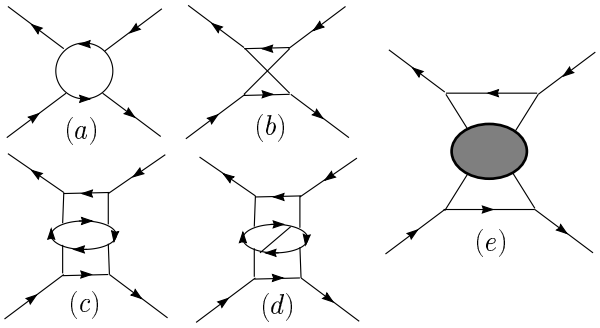


Figure 2.2: The graphs in (a-d) all have the form of (e).

with internal loops, we would label all internal lines with arbitrary momenta and integrate over them. So if we had a table of blobs, we could simply plug it into this graph, do the appropriate integrals, and have something which we do know how to interpret: an S-matrix element.

So this gives us a sensible, and possibly even useful, interpretation of the blob. Before we go any further, we should choose a couple of conventions. For example, we could include or not include the n propagators which hang off $\tilde{G}(k_1, \ldots, k_n)$. We could also include or not include the overall energy-momentum conserving δ-function. We'll include them both.

So, for example, here are a few contributions to $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$:

$$\tilde{G}^{(2)}(k_1,k_2,k_3,k_4) = \quad = \quad + \quad + \quad + O(g^2)$$

$$= (2\pi)^4\delta^{(4)}(k_1+k_4)\frac{i}{k_1^2-\mu^2+i\epsilon}(2\pi)^4\delta^{(4)}(k_2+k_3)\frac{i}{k_2^2-\mu^2+i\epsilon} + (2 \text{ permutations})$$

Figure 2.3: Lowest order contributions to $\tilde{G}^{(4)}(k_1, k_2, k_3, k_4)$.

One simple thing we can do with these blobs is to recover S-matrix elements. We cancel off the external propagators and put the momenta back on their mass

shells. Thus, we get

$$\langle k_3, k_4 \,|(S-1)|\, k_1, k_2\rangle = \prod_{r=1}^{4} \frac{k_r^2 - \mu^2}{i} \tilde{G}(-k_3, -k_4, k_1, k_2). \tag{2.1}$$

Because of the four factors of zero out front when the momenta are on their mass shell, the graphs that we wrote out above do not contribute to $S-1$, as expected (since they don't contribute to scattering away from the forward direction).

2.1.2 Answer Two: The Fourier Transform of a Green Function

We have found one meaning for our blob. We can use it to obtain another function, its Fourier transform, which we can then give another meaning to. Using the convention for Fourier transforms,

$$\begin{aligned} f(x) &= \int \frac{d^4k}{(2\pi)^4} \tilde{f}(k) e^{ik\cdot x} \\ \tilde{f}(k) &= \int d^4x f(x) e^{-ik\cdot x} \end{aligned} \tag{2.2}$$

(again keeping with our convention that each dk comes with a factor of $1/(2\pi)$), we have

$$G^{(n)}(x_1, \ldots, x_n) = \int \frac{d^4k_1}{(2\pi)^4} \cdots \int \frac{d^4k_n}{(2\pi)^4} \exp(ik_1\cdot x_1 + \ldots + ik_n \cdot x_n) \tilde{G}^{(n)}(k_1, \ldots, k_n) \tag{2.3}$$

(hence the tilde over G defined in momentum space).

Now, consider adding a source to any given theory,

$$\mathcal{L} \to \mathcal{L} + \rho(x)\varphi(x) \tag{2.4}$$

where $\rho(x)$ is a specified c-number source, not an operator. As you showed in a problem set back in the fall, this adds a new vertex to the theory, shown in Fig. 1.2.

Now, consider the vacuum-to-vacuum transition amplitude, $\langle 0\,|S|\,0\rangle$, in this modified theory. At n'th order in $\rho(x)$, all the contributions to $\langle 0\,|S|\,0\rangle$ come from diagrams of the form shown in the figure.

Thus, the n'th order (in $\rho(x)$) contribution to $\langle 0\,|S|\,0\rangle$ to all orders in g is

$$\frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \int \frac{d^4k_n}{(2\pi)^4} \tilde{\rho}(-k_1) \ldots \tilde{\rho}(-k_n)\, \tilde{G}^{(n)}(k_1, \ldots, k_n). \tag{2.5}$$

The reason for the factor of $1/n!$ arises because if I treat all sources as distinguishable, I overcount the number of diagrams by a factor of $n!$. Thus, to all orders, we have

$$\langle 0\,|S|\,0\rangle = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \int \frac{d^4k_n}{(2\pi)^4} \tilde{\rho}(-k_1) \ldots \tilde{\rho}(-k_n)\, \tilde{G}^{(n)}(k_1, \ldots, k_n)$$

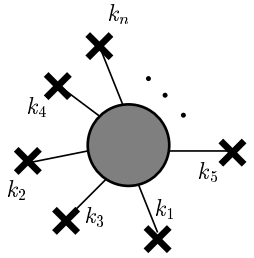


Figure 2.4: n'th order contribution to $\langle 0\,|S|\,0\rangle$ in the presence of a source.

$$= \quad 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \ldots d^4x_n \, \rho(x_1) \ldots \rho(x_n) \, G^{(n)}(x_1, \ldots, x_n). \quad (2.6)$$

This provides us with the second answer to our question. The Fourier transform of the sum of Feynman diagrams with n external lines off the mass shell is a Green function (that's what the G stands for). Recall we already introduced the $n = 2$ Green function (in *free* field theory) in connection with the exact solution to free field theory with a source.

Let's explicitly note that the vacuum-to-vacuum transition ampitude depends on $\rho(x)$ by writing it as

$$\langle 0\,|S|\,0\rangle_\rho.$$

$\langle 0\,|S|\,0\rangle_\rho$ is a *functional* of ρ, which is how mathematicians denote functions of functions. Really, it is just a function of an infinite number of variables, the value of the source at each spacetime point. It comes up often enough that it gets a name,

$$Z[\rho] = \langle 0\,|S|\,0\rangle_\rho. \quad (2.7)$$

(The square bracket reminds you that this is a function of the function $\rho(x)$.) $Z[\rho]$ is called the *generating functional* for the Green function because, in the infinite dimensional generalization of a Taylor series, we have

$$\left. \frac{\delta^n Z[\rho]}{\delta\rho(x_1) \ldots \delta\rho(x_n)} \right| = i^n G^{(n)}(x_1, \ldots, x_n) \quad (2.8)$$

where the δ instead of δ once again reminds you that we are dealing with functionals here: you are taking a partial derivatives of Z with respect to $\rho(x)$, holding a 4 dimensional continuum of other variable fixed. This is called a *functional derivative*. As discussed in Peskin & Schroeder, p. 298, the functional derivative obeys the basic axiom (in four dimensions)

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x - y), \quad \text{or} \quad \frac{\delta}{\delta J(x)} \int d^4y \, J(y)\varphi(y) = \varphi(x). \quad (2.9)$$

This is the natural generalization, to continuous functions, of the rule for discrete

vectors,

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij}, \quad \text{or} \quad \frac{\partial}{\partial x_i} \sum_j x_j k_j = k_i. \tag{2.10}$$

The generating functional $Z[J]$ will be particularly useful when we study the path integral formulation of QFT.

The term "generating functional" arises in analogy with the functions of two variables, which when you Taylor expand in one variable, the coefficients are a set of functions of the other. For example, the generating function for the Legendre polynomials is

$$f(x,z) = \frac{1}{\sqrt{z^2 - 2xz + 1}} \tag{2.11}$$

since when expanded in z, the coefficients of z^n is the Legendre polynomial $P_n(x)$:

$$\begin{aligned} f(x,z) &= 1 + xz + \frac{1}{2}(3x^2 - 1)z^2 + \ldots \\ &= P_0(x) + zP_1(x) + z^2 P_2(x) + \ldots . \end{aligned} \tag{2.12}$$

Similarly, when $Z[\rho]$ in expanded in powers of $\rho(x)$, the coefficient of ρ^n is proportional to the n-point Green function $G^{(n)}(x_1, \ldots, x_n)$. Thus, all Green functions, and hence all S matrix elements (and so all physical information about the system) are encoded in the vacuum persistance amplitude in the presence of an external source ρ.

2.1.3 Answer Three: The VEV of a String of Heisenberg Fields

But wait, there's more. Once again, let us consider adding a source term to the theory. Thus, the Hamiltonian may be written

$$\mathcal{H}_0 + \mathcal{H}_I \to \mathcal{H}_0 + \mathcal{H}_I - \rho(x)\varphi(x) \tag{2.13}$$

where $\mathcal{H}_0$ is the free-field piece of the Hamiltonian, and $\mathcal{H}_I$ contains the interactions. Now, as far as Dyson's formula is concerned, you can break the Hamiltonian up into a "free" and interacting part in any way you please. Let's take the "free" part to be $\mathcal{H}_0 + \mathcal{H}_I$ and the interaction to be $\rho\varphi$. I put quotes around "free", because in this new interaction picture, the fields evolve according to

$$\varphi(\vec{x}, t) = e^{iHt} \varphi(\vec{x}, 0) e^{-iHt} \tag{2.14}$$

where $H = \int d^3x \mathcal{H}_0 + \mathcal{H}_I$. These fields aren't free: they don't obey the free field equations of motion. You can't define a contraction for these fields, and thus you can't do Wick's theorem. They are what we would have called Heisenberg fields if there had been no source, and so we will subscript them with an H.

Now, just from Dyson's formula, we find

$$\begin{aligned} Z[\rho] &= \langle 0\,|S|\,0\rangle_\rho = \langle 0\,|T\exp\left(i\int d^4x\rho(x)\varphi_H(x)\right)|\,0\rangle \qquad (2.15)\\ &= 1+\sum_{n=1}^{\infty}\frac{i^n}{n!}\int d^4x_1\ldots d^4x_n\,\rho(x_1)\ldots\rho(x_n)\langle 0\,|T\left(\varphi_H(x_1)\ldots\varphi_H(x_n)\right)|\,0\rangle \end{aligned}$$

and so we clearly have

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0\,|T\left(\varphi_H(x_1)\ldots\varphi_H(x_n)\right)|\,0\rangle. \qquad (2.16)$$

For $n=2$, we have the two-point Green function $\langle 0\,|T\left(\varphi_H(x_1)\varphi_H(x_2)\right)|\,0\rangle$. This looks like our definition of the propagator, but it's not quite the same thing. The Feynman propagator was defined as a Green function for the *free* field theory; the two-point Green function is defined in the *interacting* theory. One of the tasks of the next few sections is to see how these are related.

2.2 Green Functions and Feynman Diagrams

From the discussion in the previous section, we have three logically distinct objects:

1. S-matrix elements (the physical observables we wish to measure),
2. the sum of Feynman diagrams (the things we know how to calculate), and
3. n-point Green functions, defined via by Eq. (2.8) or (2.16).

By introducing a turning on and off function, we could show that these were all related. Now we want to get rid of that crutch, and define perturbation theory in a more sensible way. The question we will then have to address is, what is the relation between these objects in our new formulation of scattering theory?

We set up the problem as follows. Imagine you have a well-defined theory, with a time independent Hamiltonian H (the turning on and off function is gone for good) whose spectrum is bounded below, whose lowest lying state is not part of a continuum (i.e. no massless particles yet), and the Hamiltonian has actually been adjusted so that this state, $|\,\Omega\rangle$, the physical vacuum, satisfies

$$H|\,\Omega\rangle = 0. \qquad (2.17)$$

The vacuum is translationally invariant and normalized to one

$$\vec{P}|\,\Omega\rangle = 0, \quad \langle\Omega\,|\Omega\rangle = 1. \qquad (2.18)$$

Now, let $\mathcal{H}\to\mathcal{H}-\rho(x)\varphi(x)$ and define

$$\begin{aligned} Z[\rho] &\equiv \langle\Omega\,|S|\,\Omega\rangle_\rho \\ &= \langle\Omega\,|U(\infty,-\infty)|\,\Omega\rangle_\rho \qquad (2.19) \end{aligned}$$

where the ρ subscript again means, in the presence of a source $\rho(x)$, and the evolution operator $U(t_1, t_2)$ is the Schorödinger picture evolution operator for the Hamiltonian $\int d^3x \, (\mathcal{H} - \rho(x)\varphi(x))$. We then define

$$G^{(n)}(x_1, \ldots, x_n) = \frac{1}{i^n} \frac{\delta^n Z[\rho]}{\delta\rho(x_1) \ldots \delta\rho(x_n)}. \tag{2.20}$$

Note that this is, at least in principle, *different* from our previous definitions of Z and G, which implicitly referred to S matrix elements taken between free vacuum states, with the interactions defined with the turning on and off function. Thus, we can ask two questions:

1. Is $G^{(n)}$ defined this way the Fourier transform of the sum of all Feynman graphs? Let's call the $G^{(n)}$ defined as the sum of all Feynman graphs $G_F^{(n)}$ and the Z which generates these Z_F. The question then is, is $G^{(n)} = G_F^{(n)}$? Or equivalently, is $Z = Z_F$?

   The answer, fortunately, will be "yes". In other words, we can compute Green functions just as we always did, as the sum of Feynman graphs. This is actually rather surprising, since we derived Feynman rules based on the action of interacting fields on the bare vacuum, not the full vacuum.

2. Are S matrix elements obtained from Green functions in the same way as before? For example, is

   $$\langle k_1', k_2' \, | S - 1 | \, k_1, k_2 \rangle = \prod_a \frac{k_a^2 - \mu^2}{i} \tilde{G}(-k_1', -k_2', k_1, k_2)? \tag{2.21}$$

   The answer will be, "almost." The problem will be, as we have already discussed, that in an interacting theory, the bare field $\varphi_0(x)$ no longer creates mesons with unit probability. The formula will hold, but only when the Green function $\tilde{G}$ is defined using renormalized fields φ instead of bare fields.

First we will answer the first question: Is $G^{(n)} = G_F^{(n)}$?[3] Our answer will be similar to the derivation of Wick's theorem on pages 82-87 of Peskin & Schroeder, which you should look at as well.

First of all, using Dyson's formula just as we did at the end of the last section, it is easy to show that

$$G^{(n)}(x_1, \ldots, x_n) = \langle \Omega \, | T \left(\varphi_H(x_1) \ldots \varphi_H(x_n) \right) | \, \Omega \rangle. \tag{2.22}$$

Now let's show that this is what we get by blindly summing Feynman diagrams.

The object which had a graphical expansion in terms of Feynman diagrams was

$$Z_F[\rho] = \lim_{t_\pm \to \pm\infty} \langle 0 \, | T \exp \left(-i \int_{t_-}^{t_+} \left[\mathcal{H}_I - \rho(x) \varphi_I(x) \right] \right) | \, 0 \rangle \tag{2.23}$$

[3] Since the answer to this question doesn't depend on using bare fields φ_0 or renormalized fields φ, we will neglect this distinction in the following discussion.

where I remind you that $|\,0\rangle$ refers to the bare vacuum, satisfying $H_0|\,0\rangle = 0$. Now, we know that we will have to adjust the constant part of $\mathcal{H}_I$ with a vacuum energy counterterm to eliminate the vacuum bubble graphs when $\rho = 0$. Equivalently, since as argued in the previous chapter the sum of vacuum bubbles is universal, an easier way to get rid of them is simply to divide by $\langle 0\,|S|\,0\rangle$; this gives

$$Z_F[\rho] = \lim_{t_\pm \to \pm\infty} \frac{\langle 0\,|T\exp\left(-i\int_{t_-}^{t_+}[\mathcal{H}_I - \rho(x)\varphi_I(x)]\right)|\,0\rangle}{\langle 0\,|T\exp\left(-i\int_{t_-}^{t_+}\mathcal{H}_I\right)|\,0\rangle}. \tag{2.24}$$

To get $G_F^{(n)}(x_1,\ldots,x_n)$, we do n functional derivatives with respect to ρ and then set $\rho = 0$:

$$G_F^{(n)}(x_1,\ldots,x_n) = \lim_{t_\pm \to \pm\infty} \frac{\langle 0\,|T\left[\varphi_I(x_1)\ldots\varphi_I(x_n)\exp\left(-i\int_{t_-}^{t_+}[\mathcal{H}_I - \rho\varphi_I]\right)\right]|\,0\rangle}{\langle 0\,|T\exp\left(-i\int_{t_-}^{t_+}\mathcal{H}_I\right)|\,0\rangle}. \tag{2.25}$$

Now, we have to show that this is equal to Eq. (2.22). This will take a bit of work.

First of all, since Eq. (2.25) is manifestly symmetric under permutations of the x_i's, we can simply prove the equality for a particularly convenient time ordering. So let's take

$$t_1 > t_2 > \ldots > t_n \tag{2.26}$$

In this case, we can drop the T-ordering symbol from $G^{(n)}(x_1,\ldots,x_n)$. Now, since

$$U_I(t_b,t_a) = T\exp\left(-i\int_{t_a}^{t_b} d^4x\,\mathcal{H}_I\right) \tag{2.27}$$

is the usual time evolution operator, we can express the time ordering in $G_F^{(n)}$ as

$$G_F^{(n)}(x_1,\ldots,x_n) = \lim_{t_\pm \to \pm\infty} \frac{\langle 0\,|U_I(t_+,t_1)\varphi_I(x_1)U_I(t_1,t_2)\varphi_I(x_2)\ldots\varphi_I(x_n)U_I(t_n,t_-)|\,0\rangle}{\langle 0\,|U_I(t_+,t_-)|\,0\rangle}. \tag{2.28}$$

Now, everywhere that $U_I(t_a,t_b)$ appears, rewrite it as $U_I(t_a,0)U_I(0,t_b)$, and then use the relation between Heisenberg and Interaction fields,

$$\begin{aligned} \varphi_H(x_i) &= U_I(t_i,0)^\dagger\varphi_I(x_i)U_I(t_i,0) \\ &= U_I(0,t_i)\varphi_I(x_i)U_I(t_i,0) \end{aligned} \tag{2.29}$$

to convert everything to Heisenberg fields, and get rid of those intermediate U's:

$$G_F^{(n)}(x_1,\ldots,x_n) = \lim_{t_\pm \to \pm\infty} \frac{\langle 0\,|U_I(t_+,0)\varphi_H(x_1)\varphi_H(x_2)\ldots\varphi_H(x_n)U_I(0,t_-)|\,0\rangle}{\langle 0\,|U_I(t_+,0)U_I(0,t_-)|\,0\rangle}. \tag{2.30}$$

Let's concentrate on the right hand end of the expression, $U_I(0,t_-)|\,0\rangle$ (in both the numerator and denominator), and refer to the mess to the left of it as some fixed state $\langle\Psi\,|$. First of all, since $H_0|\,0\rangle = 0$, we can trivially convert the evolution operator to the Schrödinger picture,

$$\lim_{t_-\to\infty}\langle\Psi\,|U_I(0,t_-)|\,0\rangle = \lim_{t_-\to\infty}\langle\Psi\,|U_I(0,t_-)\exp(iH_0t_-)|\,0\rangle = \lim_{t_-\to\infty}\langle\Psi\,|U(0,t_-)|\,0\rangle. \tag{2.31}$$

Next, insert a complete set of eigenstates of the *full* Hamiltonian, H,

$$\begin{aligned}\lim_{t_-\to\infty}\langle\Psi\,|U(0,t_-)|\,0\rangle &= \lim_{t_-\to\infty}\langle\Psi\,|U(0,t_-)\left[|\,\Omega\rangle\langle\Omega\,| + \sum_{n\neq 0}|\,n\rangle\langle n\,|\right]|\,0\rangle \\ &= \langle\Psi|\,\Omega\rangle\langle\Omega\,|0\rangle + \lim_{t_-\to-\infty}\sum_{n\neq 0}e^{iE_nt_-}\langle\Psi|\,n\rangle\langle n|\,0\rangle \end{aligned}\tag{2.32}$$

where the sum is over all eigenstates of the full Hamiltonian except the vacuum, and we have used the fact that $H|\,\Omega\rangle = 0$ and $H|\,n\rangle = E_n|\,n\rangle$, where the E_n's are the energies of the excited states.

We're almost there. This next part is the important one. The sum over eigenstates is actually a continuous integral, not a discrete sum. As $t_- \to -\infty$, the integrand oscillates more and more wildly, and in fact there is a theorem (or rather, a lemma - the Riemann-Lebesgue lemma) which states that as long as $\langle\Psi|\,n\rangle\langle n|\,0\rangle$ is a continuous function, the sum (integral) on the right is zero.

The Riemann-Lebesgue lemma may be stated as follows: for any "nice" function $f(x)$,

$$\lim_{\mu\to\infty}\int_a^b f(x)\left\{\begin{matrix}\sin\mu x\\ \cos\mu x\end{matrix}\right\} = 0. \tag{2.33}$$

It is quite easy to see the graphically, as shown in Fig. 2.5. Physically, what the lemma is telling you is that if you start out with any given state in some fixed region and wait long enough, the only trace of it that will remain is its (true) vacuum component. All the other one and multiparticle components will have gone away: as can be seen from the figure, the contributions from infinitesimally close states destructively interfere.

So we're essentially done. A similar argument shows that

$$\lim_{t_+\to\infty}\langle 0\,|U_I(t_+,0)|\,\Psi\rangle|\,0\rangle = \langle 0|\,\Omega\rangle\langle\Omega\,|\Psi\rangle \tag{2.34}$$

and applying this to the numerator and denominator of Eq. (2.30) we find

$$\begin{aligned}G_F^{(n)}(x_1,\ldots,x_n) &= \frac{\langle 0|\,\Omega\rangle\langle\Omega\,|\varphi_H(x_1)\ldots\varphi_H(x_n)|\,\Omega\rangle\langle\Omega\,|0\rangle}{\langle 0|\,\Omega\rangle\langle\Omega\,|\Omega\rangle\langle\Omega\,|0\rangle} \\ &= G^{(n)}(x_1,\ldots,x_n).\end{aligned}\tag{2.35}$$

So there is now no longer to distinguish between the sum of diagrams and the real Green functions.

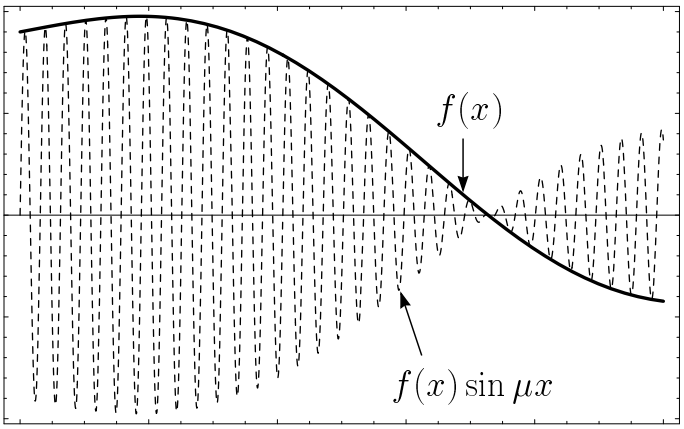


Figure 2.5: The Riemann-Lebesgue lemma: $f(x)$ multiplied by a rapidly oscillating function integrates to zero in the limit that the frequency of oscillation becomes infinite.

2.3 The LSZ Reduction Formula

We now turn to the second question: Are S matrix elements obtained from Green's functions in the same way as before?

By introducing a turning on and off function, we were able to show that

$$\langle l_1,\ldots,l_s\,|S-1|\,k_1,\ldots,k_r\rangle \;=\; \prod_{a=1}^{s}\frac{l_a^2-\mu^2}{i}\prod_{b=1}^{r}\frac{k_b^2-\mu^2}{i}\tilde{G}^{(r+s)}(-l_1,\ldots,-l_s,k_1,\ldots,k_r). \quad (2.36)$$

The real world does not have a turning on and off function. Is this formula correct? The answer is "almost."

The correct relation between S matrix elements (what we want) and Green functions (what, as we just showed, we get from Feynman diagrams) which we will derive is called the LSZ reduction formula. Since its derivation doesn't require resorting to perturbation theory, we no longer need to make any reference to free Hamiltonia, bare vacua, interaction picture fields, etc. So FROM NOW ON all fields will be in the Heisenberg representation (no more interaction picture), and states will refer to eigenstates of the *full* Hamiltonian (although for the rest of this section we will continue to denote the vacuum by $|\,\Omega\rangle$ to avoid confusion)

$$\varphi(x)\equiv\varphi_H(x),\quad |\,0\rangle\equiv|\,\Omega\rangle. \quad (2.37)$$

The physical one-meson states in the theory are now the complete one meson states, relativistically normalized

$$H|\,k\rangle=\sqrt{\vec{k}^2+\mu^2}|\,k\rangle\equiv\omega_k|\,k\rangle,\quad \langle k'\,|k\rangle=(2\pi)^3 2\omega_k\delta^{(3)}(\vec{k}-\vec{k}'). \quad (2.38)$$

We have actually been a bit cavalier with notation in this chapter; the fields we have been discussion have actually been the *bare* fields φ_0 which we discussed at the end of the last chapter. Thus, we have really been talking about *bare* Green functions, which we will now denote $G_0(n)$:

$$G_0^{(n)}(x_1, \ldots, x_n) = T\langle \Omega \left| \varphi_0(x_1) \ldots \varphi_0(x_n) \right| \Omega \rangle. \tag{2.39}$$

The reason the answer to our question is "almost" is because the bare field φ_0 which does not have quite the right properties to create and annihilate mesons. In particular, it is not normalized to create a one particle state from the vacuum with a standard amplitude - instead, it is normalized to obey the canonical commutation relations. For free field theory, these two properties were equivalent. For interacting fields, however, where the amplitude to create a meson from the vacuum has higher order perturbative corrections, these two properties are incompatible, as we have discussed. Furthermore, in an interacting theory φ_0 may also develop a vacuum expectation value, $\langle \Omega \left| \varphi(x) \right| \Omega \rangle \neq 0$.

We correct for these problems by defining a renormalized field, $\varphi(x)$, in terms of φ_0. By translational invariance,

$$\langle k \left| \varphi(x) \right| \Omega \rangle = \langle k \left| e^{iP\cdot x} \varphi(0) e^{-iP\cdot x} \right| \Omega \rangle = e^{ik\cdot x} \langle k \left| \varphi(0) \right| \Omega \rangle. \tag{2.40}$$

By Lorentz invariance, you can see that $\langle k \left| \varphi(0) \right| \Omega \rangle$ is independent of k. It is some number, which for historical reasons is denoted $Z^{1/2}$ (and traditionally called the "wave function renormalization"), and only in free field theory will it equal 1,

$$Z^{1/2} \equiv \langle k \left| \varphi(0) \right| \Omega \rangle. \tag{2.41}$$

We now can define a new field, φ, which is normalized to have a standard amplitude to create one meson, and a vanishing VEV (vacuum expectation value)

$$\begin{aligned} \varphi(x) &\equiv Z^{1/2} \left(\varphi_0(x) - \langle \Omega \left| \varphi_0(0) \right| \Omega \rangle \right) \\ \langle \Omega \left| \varphi(0) \right| \Omega \rangle &= 0, \quad \langle k \left| \varphi(x) \right| \Omega \rangle = e^{ik\cdot x}. \end{aligned} \tag{2.42}$$

We can now state the LSZ (Lehmann-Symanzik-Zimmermann) reduction formula: Define the renormalized Green functions $G^{(n)}$,

$$G^{(n)}(x_1, \ldots, x_n) \equiv \langle \Omega \left| T\left(\varphi(x_1) \ldots \varphi(x_n) \right) \right| \Omega \rangle \tag{2.43}$$

and their Fourier transforms, $\tilde{G}^{(n)}$. In terms of renormalized Green functions, S matrix elements are given by

$$\begin{aligned} &\langle l_1, \ldots, l_s \left| S - 1 \right| k_1, \ldots, k_r \rangle \\ &\qquad = \prod_{a=1}^{s} \frac{l_a^2 - \mu^2}{i} \prod_{b=1}^{r} \frac{k_b^2 - \mu^2}{i} \tilde{G}^{(r+s)}(-l_1, \ldots, -l_s, k_1, \ldots, k_r) \end{aligned} \tag{2.44}$$

That's it - almost, but not quite, what we had before. The only difference is that the S matrix is related to Green functions of the renormalized fields - in our new notation, the factors of $\tilde{G}$ in Eq. (2.1) should be $\tilde{G}_0$. Given that it is the φ which create normalized meson states from the vacuum, this is perhaps not so surprising. What is more surprising is that even the renormalized field $\varphi(x)$ creates a whole spectrum of *multiparticle* states from the vacuum as well, and that these do not pollute the relation between Green functions and S-matrix elements. Naıv̈ely, you might think that the Green function would be related to a sum of S-matrix elements, for all different incoming multiparticle states created by $\varphi(x)$. However, as we shall show, these additional states can all be arranged to oscillate away via the Riemann-Lebesgue lemma, much as in the last section.

2.3.1 Proof of the LSZ Reduction Formula

The proof can be broken up into three parts. In the first part, I will show you how to construct localized wave packets. The wave packet will have multiparticle as well as single particle components; however, the multiparticle components will be set up to oscillate away after a long time. In the second part of the proof, I will wave my hands vigorously and discuss the creation of multiparticle states in which the particles are well separated in the far past or future; these will be called *in* and *out* states, and we will find a simple expression for the S matrix in terms of the operators which create wave packets. In the third part of the proof, we massage this expression and take the limit in which the wave packets are plane waves, to derive the LSZ formula.

1. How to make a wave packet

Let us define a wave packet $|\, f\rangle$ as follows:

$$|\, f\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k}) |\, k\rangle \tag{2.45}$$

where $F(\vec{k}) = \langle k\,|f\rangle$ is the momentum space wave function of $|\, f\rangle$. Associate with each F a position space function, satisfying the Klein-Gordon equation with negative frequency,

$$f(x) \equiv \int \frac{d^3k}{(2\pi)^3 2\omega_k} F(\vec{k}) e^{-ik\cdot x}, \quad k_0 = \omega_k, \quad (\Box + \mu^2) f(x) = 0. \tag{2.46}$$

Note that as we approach plane wave states, $|\, v\rangle \to |\, k\rangle$, $f(x) \to e^{-ik\cdot x}$.

Now, define the following odd-looking operator which is only a function of the time, t (recall again that we are working in the Heisenberg representation, so the operators carry the time dependence)

$$\varphi^f(t) \equiv i \int d^3x \left(\varphi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\varphi(\vec{x},t)\right). \tag{2.47}$$

This is precisely the operator which makes single particle wave packets. First of all, it trivially satisfies

$$\langle \Omega \,|\varphi^f(t)|\, \Omega\rangle = 0 \tag{2.48}$$

and has the correct amplitude to produce a single particle state $|\, f\rangle$:

$$\begin{aligned}
&\langle k \,|\varphi^f(t)|\, \Omega\rangle \\
&\quad = i\int d^3x \int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')\, \langle k \,|\left(\varphi(x)\partial_0 e^{-ik'\cdot x} - e^{-ik'\cdot x}\partial_0\varphi(x)\right)|\, \Omega\rangle \\
&\quad = i\int d^3x \int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')\left(-i\omega_{k'} e^{-ik'\cdot x} - e^{-ik'\cdot x}\partial_0\right)\langle k \,|\varphi(x)|\, \Omega\rangle \\
&\quad = i\int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')(-i\omega_{k'} - i\omega_k)\int d^3x\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} e^{-i(\omega_{k'}-\omega_k)t} \\
&\quad = F(\vec{k})
\end{aligned} \tag{2.49}$$

where we have used

$$\int d^3x\, e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} = (2\pi)\delta^{(3)}(\vec{k}-\vec{k}') \tag{2.50}$$

and we note that the phase factor $e^{-i(\omega_{k'}-\omega_k)t}$ becomes one once the δ function constraint is imposed (this will change when we consider multiparticle states). Note that this result is independent of time.

A similar derivation, with one crucial minus sign difference (so that the factors of ω_k and $\omega_{k'}$ cancel instead of adding), yields

$$\langle \Omega \,|\varphi^f(t)k\rangle = 0. \tag{2.51}$$

Thus, as far as the zero and single particle states are concerned, $\varphi^f(t)$ behaves as a creation operator for wave packets. Now we will see that in the limit $t \to \pm\infty$ all the other states created by $\varphi^f(t)$ oscillate away.

Consider the multiparticle state $|\, n\rangle$, which is an eigenvalue of the momentum operator:

$$P^\mu|\, n\rangle = p_n^\mu|\, n\rangle. \tag{2.52}$$

Proceeding much as before, let us calculate the amplitude for $\varphi^f(t)$ to make this state from the vacuum:

$$\begin{aligned}
&\langle n \,|\varphi^f(t)|\, \Omega\rangle \\
&\quad = i\int d^3x \int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')\, \langle n \,|\left(\varphi(x)\partial_0 e^{-ik'\cdot x} - e^{-ik'\cdot x}\partial_0\varphi(x)\right)|\, \Omega\rangle \\
&\quad = i\int d^3x \int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')\left(-i\omega_{k'} e^{-ik'\cdot x} - e^{-ik'\cdot x}\partial_0\right)\langle n \,|\varphi(x)|\, \Omega\rangle \\
&\quad = i\int d^3x \int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')\left(-i\omega_{k'} e^{-ik'\cdot x} - e^{-ik'\cdot x}\partial_0\right) e^{ip_n\cdot x}\langle n \,|\varphi(0)|\, \Omega\rangle \\
&\quad = i\int \frac{d^3k'}{(2\pi)^2 2\omega_{k'}} F(\vec{k}')(-i\omega_{k'} - ip_n^0)\int d^3x\, e^{i(\vec{k}'-\vec{p}_n)\cdot\vec{x}} e^{-i(\omega_{k'}-p_n^0)t}\langle n \,|\varphi(0)|\, \Omega\rangle
\end{aligned}$$

$$= \frac{\omega_{p_n} + p_n^0}{2\omega_{p_n}} F(\vec{p}_n) e^{-i(\omega_{p_n} - p_n^0)t} \langle n \left| \varphi(0) \right| \Omega \rangle \tag{2.53}$$

where

$$\omega_{p_n} = \sqrt{\vec{p}_n^2 + \mu^2}. \tag{2.54}$$

Note that we haven't had to use any information about $\langle n \left| \varphi(x) \right| \Omega \rangle$ beyond that given by Lorentz invariance; thus, we haven't had to know anything about the amplitude to create multiparticle states from the vacuum. The crucial point is the existence of the phase factor $e^{-i(\omega_{p_n} - p_n^0)t}$ in Eq. (2.53), and the observation that for a multiparticle state with a finite mass,

$$\omega_{p_n} < p_n^0. \tag{2.55}$$

This should be easy to convince yourself of: a two particle state, for example, can have $\vec{p} = 0$ if the particles are moving back-to-back, while the energy of the state p_n^0 can vary between 2μ and ∞. In contrast, a single particle state with $\vec{p} = 0$ can *only* have $p^0 = \omega_p = \mu < p_n^0$. Thus, for the single particle state the oscillating phase vanishes, as already shown, whereas it can never vanish for multiparticle states.

So now we have the familiar Riemann-Lebesgue argument: take some fixed state $\left| \psi \right\rangle$, and consider $\langle \psi \left| \varphi^f(t) \right| \Omega \rangle$ as $t \to \pm\infty$. Inserting a complete set of states, we find

$$\begin{aligned} \lim_{t \to \pm\infty} \langle \psi \left| \varphi^f(t) \right| \Omega \rangle &= \lim_{t \to \pm\infty} \Bigg[\langle \psi | \, \Omega \rangle \langle \Omega \left| \varphi^f(t) \right| \Omega \rangle \\ &+ \int \frac{d^3k}{(2\pi)^3 2\omega_k} \langle \psi \left| k \right\rangle \langle k \left| \varphi^f(t) \right| \Omega \rangle + \sum_{n \neq 0,1} \langle \psi \left| n \right\rangle \langle n \left| \varphi^f(t) \right| \Omega \rangle \Bigg] \\ &= \langle \psi | \, f \rangle + 0 \end{aligned} \tag{2.56}$$

where we have used the fact that the multiparticle sum vanishes by the Riemann-Lebesgue lemma.

Similarly, one can show that

$$\lim_{t \to \pm\infty} \langle \Omega \left| \varphi^f(t) \right| \psi \rangle = 0 \tag{2.57}$$

for any fixed state $\left| \psi \right\rangle$.

Thus, our cunning choice of $\varphi^f(t)$ was arranged so that the oscillating phases cancelled only for the single particle state, so that only these states survived after infinite time. We have a similar interpretation as before: in the $t \to -\infty$ limit, $\varphi^f(t)$ acts on the true vacuum and creates states with one, two, ... n particles. Taking the inner product of this state with any fixed state, we find that at $t = 0$ the only surviving components are the single-particle states, which make up a localized wave packet.

2. How to make widely separated wave packets

The results of the last section were rigorous (or at least, could be made so without a lot of work). By contrast, in this section we will wave our hands violently and rely on physical arguments.

We now wish to construct multiparticle states of interest to scattering problems, that is, states which in the far past or far future look like well separated wavepackets. The physical picture we will rely upon is that if $F_1(\vec{k})$ and $F_2(\vec{k})$ do not have common support, in the distant past and future they correspond to widely separated wave packets. Then when $\varphi^{f_2}(t)$ acts on a state in the far past or future, it shouldn't matter if this state is the vacuum state or the state $|\, f_1\rangle$, since the first wavepacket is arbitrarily far away. Let us denote states created by the action of $\varphi^{f_2}(t)$ on $|\, f_1\rangle$ in the distant past as "in" states, and states created by the action of $\varphi^{f_2}(t)$ on $|\, f_1\rangle$ in the far future as "out" states. Then we have

$$\lim_{t\to\infty\ (-\infty)} \langle\psi\,|\varphi^{f_2}(t_2)|\, f_1\rangle = |\, f_1, f_2\rangle^{\text{out (in)}}. \tag{2.58}$$

Now by definition, the S matrix is just the inner product of a given "in" state with another given "out" state,

$${}^{\text{out}}\langle f_3, f_4\,|f_1, f_2\rangle^{\text{in}} = \langle f_3, f_4\,|S|\, f_1, f_2\rangle. \tag{2.59}$$

Thus, we have shown that

$$\begin{aligned}\langle f_3, f_4\,|S|\, f_1, f_2\rangle &= \lim_{t_4\to\infty}\lim_{t_3\to\infty}\lim_{t_2\to-\infty}\lim_{t_1\to-\infty} \\ &\langle\Omega\,|\varphi^{f_4\dagger}(t_4)\varphi^{f_3\dagger}(t_3)\varphi^{f_2}(t_2)\varphi^{f_1}(t_1)|\,\Omega\rangle.\end{aligned} \tag{2.60}$$

3. Massaging the resulting expression

In principle, we have achieved our goal in Eq. (2.60): we have written an expression for S matrix elements in terms of a weighted integral over vacuum expectation values of Heisenberg fields. However, it doesn't look much like the LSZ reduction formula yet, but we can do that with a bit of massaging.

What we will show is the following

$$\begin{aligned}\langle f_3, f_4\,|S-1|\, f_1, f_2\rangle &= \int d^4x_1\ldots d^4x_n\, f_4^*(x_4)f_3^*(x_3)f_2(x_2)f_1(x_1) \\ &\times\ \ i^4\prod_r(\Box_r+\mu^2)\langle\Omega\,|T\varphi(x_1)\ldots\varphi(x_4)|\,\Omega\rangle.\end{aligned} \tag{2.61}$$

This looks messy and unfamiliar, but it's not. If we take the limit in which the wave packets become plane wave states, $|\, f_i\rangle \to |\, k_i\rangle$, $f_i(x) \to e^{ik_i\cdot x_i}$, we have

$$\begin{aligned}\langle k_3, k_4\,|S-1|\, k_1, k2\rangle &= \int d^4x_1\ldots d^4x_n\, e^{ik_3\cdot x_3+ik_4\cdot x_4-ik_2\cdot x_2-ik_1\cdot x_1} \\ &\times i^4\prod_r(\Box_r+\mu^2)G^{(4)}(x_1,\ldots,x_4)\end{aligned}$$

$$= \prod_r \frac{k_r^2 - \mu^2}{i} \tilde{G}^{(4)}(k_1, k_2, -k_3, -k_4), \tag{2.62}$$

which is precisely the LSZ formula. Note that we have taken the plane wave limit *after* taking the limit in which the limit $t \to \pm\infty$ required to define the in and out states; thus, in this order of limits even the plane wave in and out states are widely separated.

Before showing this, let us prove a useful result: for an arbitrary interacting field A and function $f(x)$ satisfying the Klein-Gordon equation and vanishing as $|x| \to \infty$,

$$\begin{aligned} i\int d^4x\, f(x)(\Box + \mu^2)A(x) &= i\int d^4x\, f(x)\partial_0^2 A(x) + A(x)(-\nabla^2 + \mu^2)f(x) \\ &= i\int d^4x\, f(x)\partial_0^2 A(x) - A(x)\partial_0^2 f(x) \\ &= \int dt\, \partial_0 \int d^3x\, i\left(f(x)\partial_0 A(x) - A(x)\partial_0 f(x)\right) \\ &= -\int dt\, \partial_0 A^f(t) \\ &= \left(\lim_{t\to-\infty} - \lim_{t\to+\infty}\right) A^f(t) \end{aligned} \tag{2.63}$$

where we have integrated once by parts, and $A^f(t)$ is defined as in Eq. (2.47). Similarly, we can show

$$i\int d^4x\, f^*(x)(\Box + \mu^2)A(x) = \left(\lim_{t\to+\infty} - \lim_{t\to-\infty}\right) A^{f\dagger}(t). \tag{2.64}$$

Note the difference in the signs of the limits. We can now use these relations to convert factors of $(\Box + \mu^2)\varphi(x)$ to $\varphi^f(t)$ on the RHS of Eq. (2.61). Doing this for each of the x_i's, we obtain

$$\begin{aligned} \text{RHS} &= \left(\lim_{t_4\to+\infty} - \lim_{t_4\to-\infty}\right)\left(\lim_{t_3\to+\infty} - \lim_{t_3\to-\infty}\right)\left(\lim_{t_2\to-\infty} - \lim_{t_2\to+\infty}\right) \\ &\quad \times \left(\lim_{t_1\to-\infty} - \lim_{t_1\to+\infty}\right) \langle\Omega\,|T\varphi^{f_1}(x_1)\varphi^{f_2}(x_2)\varphi^{f_3\dagger}(x_3)\varphi^{f_4\dagger}(x_4)|\,\Omega\rangle. \end{aligned} \tag{2.65}$$

Now we can evaluate the limits one by one.

First of all, when $t_4 \to -\infty$, it is the earliest (in the order of limits which we have taken), and so acts on the vacuum. Thus, according to the complex conjugate of Eq. (2.57), we get zero. When $t_4 \to +\infty$, it is the latest, and acts on the vacuum state on the left, to give $\langle f_4\,|$. Similarly, only the $t_3 \to \infty$ limit contributes, creating the state ${}^{\text{out}}\langle f_3, f_4\,|$. Next, taking the two limits of t_2, we find

$$\text{RHS} = \left(\lim_{t_1\to-\infty} - \lim_{t_1\to+\infty}\right)\left({}^{\text{out}}\langle f_3, f_4\,|\varphi^{f_1}(t_1)|\,f_2\rangle\right.$$

$$- \lim_{t_2 \to +\infty} {}^{\text{out}}\langle f_3, f_4 \,|\varphi^{f_2}(t_2)\varphi^{f_1}(t_1)|\, \Omega\rangle \Big) . \qquad (2.66)$$

Unfortunately, we don't know how $\varphi^{f_2}(t_2 \to \infty)$ acts on a multi-meson out state, and so it's not clear what the second term is. Let's parameterize our ignorance and define

$$\langle \psi \,| \equiv \lim_{t_2 \to \infty} {}^{\text{out}}\langle f_3, f_4 \,|\varphi^{f_2}(t_2). \qquad (2.67)$$

Then taking the t_1 limits, we find

$$\begin{aligned} RHS &= {}^{\text{out}}\langle f_3, f_4 \,| f_1, f_2\rangle^{\text{in}} - {}^{\text{out}}\langle f_3, f_4 \,| f_1, f_2\rangle^{\text{out}} - \langle \psi \,| f_1\rangle + \langle \psi \,| f_1\rangle \\ &= \langle f_3, f_4 \,|S - 1|\, f_1, f_2\rangle \end{aligned} \qquad (2.68)$$

as required. Note that we have used the fact (from Eq. (2.56)) that

$$\lim_{t_1 \to \infty} \langle \psi \,|\varphi^{f_1}(t_1)|\, \Omega\rangle = \lim_{t_1 \to -\infty} \langle \psi \,|\varphi^{f_1}(t_1)|\, \Omega\rangle = \langle \psi \,| f_1\rangle. \qquad (2.69)$$

So that's it - we've proved the reduction formula. It was a bit involved, but there are a few important things to remember:

1. The proof relied *only* on the properties

$$\langle \Omega \,|\varphi(0)|\, \Omega\rangle = 0, \quad \langle k \,|\varphi(0)|\, \Omega\rangle = 1. \qquad (2.70)$$

   No other properties of φ were assumed. In particular, φ was not assumed to have any particular relation to the bare field φ_0 which appears in the Lagrangian - the simplest relation is Eq. (2.42), but the Green functions of any field φ which satisfies the requirements (2.70) will give the correct S-matrix elements. For example,

$$\tilde{\varphi}(x) = \varphi(x) + \tfrac{1}{2} g \varphi(x)^2 \qquad (2.71)$$

   is a perfectly good field to use in the reduction formula.

   Physically, this is again because of the Riemann-Lebesgue destructive interference. One appropriately renormalized, $\varphi(0)$ and $\tilde{\varphi}(0)$ only differ in their vacuum to multiparticle state matrix elements. But this difference just oscillates away - the multiparticle states created by the field are irrelevant.

   Practically, this has a very useful consequence: you can always make a nonlinear field redefinition for any field in a Lagrangian, and it doesn't change the value of S-matrix elements (although it will change the Green functions off shell, but that is irrelevant to the physics). In some cases this is quite convenient, since some complicated nonrenormalizable Lagrangians may take particularly simple forms after an appropriate field redefinition. A simple example of this is given on the first problem set.

This is a good result to remember, if only to save a few trees. A lot of papers have been written (even in the past few years) which claim that some particular field is the "correct" one to use in a given problem. Most of these papers are idiotic - the authors' pet form of the Lagrangian has been obtained by a simple nonlinear field redefinition from the standard form, and so is guaranteed to give the same physics.

2. We can also use the same methods as above to derive expressions for matrix elements of fields between in and out states (remembering that the S matrix is just the matrix element of the unit operator between in and out states). For example,

$$\begin{aligned} {}^{\text{out}}\langle k_1, \ldots, k_n \left| A(x) \right| \Omega \rangle &= \int d^4x_1 \ldots d^4x_n \, e^{ik_1 \cdot x_1 + \ldots + ik_n \cdot x_n} \\ &\times i^n \prod_r \left(\Box_r + \mu^2 \right) T \langle \Omega \left| \varphi(x_1) \ldots \varphi(x_n) A(x) \right| \Omega \rangle \end{aligned} \tag{2.72}$$

for any field $A(x)$. Substituting the expression for the Fourier transformed Green function, the matrix element can be calculated in terms of Feynman diagrams:

$${}^{\text{out}}\langle k_1, \ldots, k_n \left| A(x) \right| \Omega \rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \prod_{r=1}^{n} \frac{k_r^2 - \mu^2}{i} \tilde{G}^{(2,1)}(k_1, \ldots, k_n; k) \tag{2.73}$$

where $G^{(2,1)}(x_1, \ldots, x_n; x)$ is the Green function with n φ fields and one A field.

3. In principle, there is no problem in obtaining matrix elements for scattering bound states - you just need a field with some overlap with the bound state, which can then be renormalized to satisfy Eq. (2.70). For example, in QCD mesons have the quantum numbers of quark-antiquark pairs. So if $q(x)$ is a quark field and $\overline{q}(x)$ an antiquark field, $\overline{q}(x)q(x)$ should have a nonvanishing matrix elements to make a meson. So "all" you need to calculate for meson-meson scattering from QCD is

$$T \langle \Omega \left| (\overline{q}q)_R(x_1) (\overline{q}q)_R(x_2) (\overline{q}q)_R(x_3) (\overline{q}q)_R(x_4) \right| \Omega \rangle \tag{2.74}$$

where the renormalized product of fields satisfies $\langle \text{meson} \left| (\overline{q}q)_R(0) \right| \Omega \rangle = 1$. Of course, nobody can calculate this T-product since perturbation theory fails for the strong interactions, but that's not a problem of the formalism (as opposed to the turning on and off function, which had no way of dealing with bound states). One could use perturbation theory to calculate the scattering amplitudes of an e^+e^- pair to the various states of positronium.

3 Renormalizing Scalar Field Theory

The practical upshot of the last section is that we can just calculate Feynman diagrams as we always have, as long as we use renormalized fields. As we shall soon see, this will fix up our problem with loops on external legs - these will just get cancelled by the rescaling of φ_0 to φ. As a point of notation, from this point on we will dispense with the notation $|\,\Omega\rangle$ to denote the true vacuum: since we no longer need to consider the free vacuum, we will return to $|\,0\rangle$ to denote the *true* vacuum of the theory.

In this section we will look at the renormalization of our meson-"nucleon" theory. The Lagrangian for the theory is

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\varphi_0)^2 - \frac{\mu_0^2}{2}\varphi_0^2 + \partial_\mu\psi_0^*\partial^\mu\psi_0 - m_0^2\psi_0^*\psi_0 - g_0\psi_0^*\psi_0\varphi_0 + \text{constant} \\ &= \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\mu^2}{2}\varphi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi - g\psi^*\psi\varphi + \mathcal{L}_{\text{c.t.}} \qquad (3.1)\end{aligned}$$

where

$$\mathcal{L}_{\text{c.t.}} = A\varphi + \frac{B}{2}(\partial_\mu\varphi)^2 - \frac{C}{2}\varphi^2 + D\partial_\mu\psi^*\partial^\mu\psi - Em^2\psi^*\psi - F\psi^*\psi\varphi + \text{constant}. \qquad (3.2)$$

Note that we have added one more counterterm, corresponding to a term linear in φ. This will be required to cancel out any vacuum expectation value of the bare field φ_0 induced by the interaction. We can then proceed to calculate Green functions in this theory, with the counterterms $A - F$ being determined by the conditions

1. $\langle 0\,|\varphi(0)|\,0\rangle = 0$
2. $\langle k\,|\varphi(0)|\,0\rangle = 1$ (where $|\,k\rangle$ is a meson state)
3. $\langle N(k)\,|\psi^*(0)|\,0\rangle = 1$ (where $N(k)$ is a nucleon state)
4. the meson mass is μ
5. the nucleon mass is m
6. g agrees with a conventionally defined coupling.

Six conditions, six unknowns. Note that we haven't included a counterterm $A'\psi$ to cancel a possible VEV of ψ, since such a term would break the $U(1)$ symmetry. (In fact, as we will see later on in the course, if a field acquires a symmetry breaking VEV this is a real physical effect, which shouldn't be cancelled by a counterterm.) Note that these relations aren't expressed in terms of renormalized Green functions, which is what we know how to calculate. However, in the next few sections we will see how to do this.

I will actually not say much about the counterterm A, because it is simple to show that we never need to think about it. Just as we could ignore the vacuum energy counterterm if you never calculate disconnected graphs, it turns out that

you can ignore A if you consistently neglect "tadpole" graphs, of the form shown in Fig. 3.1: the counterterm A serves to exactly cancel graphs of this form at each order in perturbation theory. This is easy to see, because by momentum conservation the line connecting the tadpole to the rest of the graph must carry zero momentum. If that line were instead attached to a source instead of the rest of the n-point Green function, it would give a VEV to $\varphi(0)$, and thus the counterterm A would cancel it. Hence, all such subgraphs must vanish.

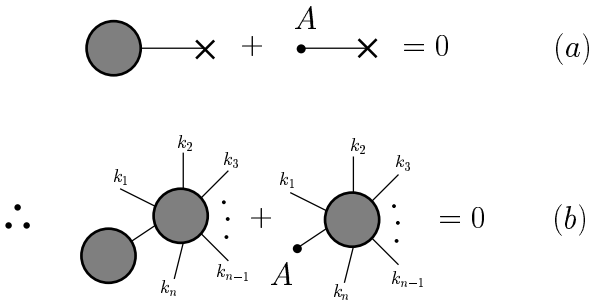


Figure 3.1: (a) A "tadpole" graph is cancelled by the counterterm A. By energy-momentum conservation, $k = 0$ on the external line (the cross denotes an insertion of the field operator φ). (b) Since $k = 0$ on the line connecting the tadpole to the rest of the Green function, all graphs containing tadpoles are also cancelled by A.

3.1 The Two-Point Function: Wavefunction Renormalization

To see how to implement the renormalization conditions, we have to study the renormalized two point function, $G^{(2)}(k_1, k_2)$,

$$\tilde{G}^{(2)}(k_1, k_2) = \int d^4x\, d^4y\, e^{-ik_1\cdot x - ik_2\cdot y} \langle 0\,|T\left(\varphi(x)\varphi(y)\right)|\,0\rangle. \tag{3.3}$$

First of all, since

$$T\left(\varphi(x)\varphi(y)\right) = \theta(x^0 - y^0)\varphi(x)\varphi(y) + \theta(y^0 - x^0)\varphi(y)\varphi(x) \tag{3.4}$$

it is sufficient to study $\langle 0\,|\varphi(x)\varphi(y)|\,0\rangle$ and then take this combination at the end. Inserting a complete set of states between the fields (again, the sum is continuous rather than discrete), we get

$$\langle 0\,|\varphi(x)\varphi(y)|\,0\rangle = \sum_n \langle 0\,|\varphi(x)|\,n\rangle\langle n\,|\varphi(y)|\,0\rangle. \tag{3.5}$$

Now,

$$\langle 0\,|\varphi(x)|\,n\rangle = \langle 0\,|e^{iP\cdot x}\varphi(x)e^{-iP\cdot x}|\,n\rangle = e^{-ip_n\cdot x}\langle 0\,|\varphi(0)|\,n\rangle \tag{3.6}$$

where P is the momentum operator, $P^\mu|\,n\rangle \equiv p_n^\mu|\,n\rangle$, and so

$$\begin{aligned}
\langle 0\,|\varphi(x)\varphi(y)|\,0\rangle &= \sum_n e^{-ip_n\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \qquad (3.7)\\
&= |\langle 0\,|\varphi(0)|\,0\rangle|^2 + \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,k\rangle|^2 \\
&\qquad + \sum_{n\neq|\,0\rangle,|\,k\rangle} e^{-ip_n\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot(x-y)} + \sum_{n\neq|\,0\rangle,|\,k\rangle} e^{-ip_n\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \\
&= i\Delta_+(x-y,\mu^2) + \sum_{n\neq|\,0\rangle,|\,k\rangle} e^{-ip_n\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,n\rangle|^2
\end{aligned}$$

where we have used the fact that $\langle 0\,|\varphi(0)|\,0\rangle = 0$, $\langle 0\,|\varphi(0)|\,k\rangle = 1$, and have explicitly removed the vacuum and single-particle states from the summation over $|\,n\rangle$. Thus, the states $|\,n\rangle$ only refer to multi-particle eigenstates of H. Also, our old friend the Δ_+ function has returned. To make explicit the fact that the Δ_+ function depends on the (physical) mass of the meson, μ^2, we will include it as an argument.

Now let's massage the sum over all momentum eigenstates. Inserting an integral over p and a δ function is just a fancy way of writing 1, but we will do something slick with it:

$$\begin{aligned}
&\sum_{n\neq|\,0\rangle,|\,k\rangle} e^{-ip_n\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \\
&\qquad = \sum_{n\neq|\,0\rangle,|\,k\rangle} e^{-ip_n\cdot(x-y)} \int d^4p\,\delta^{(4)}(p-p_n)\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \\
&\qquad = \int d^4p\,e^{-ip\cdot(x-y)} \sum_{n\neq|\,0\rangle,|\,k\rangle} \delta^{(4)}(p-p_n)\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \qquad (3.8)
\end{aligned}$$

Now, the expression in the summation is a manifestly Lorentz invariant function of p, which vanishes when $p_0 < 0$. To agree with the longstanding convention, we will abandon the convention that every p integration gets a factor of $1/2\pi$, and use this expression to define the function $\sigma(p^2)$:

$$\frac{1}{(2\pi)^3}\sigma(p^2)\theta(p^0) \equiv \sum_{n\neq|\,0\rangle,|\,k\rangle} \delta^{(4)}(p-p_n)\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 \qquad (3.9)$$

and thus we have

$$\sum_{n\neq|\,0\rangle,|\,k\rangle} e^{-ip_n\cdot(x-y)}\,|\langle 0\,|\varphi(0)|\,n\rangle|^2 = \frac{d^4p}{(2\pi)^3} e^{-ip\cdot(x-y)}\sigma(p^2)\theta(p^0). \qquad (3.10)$$

Of course, the last few lines were just definition - we traded the summation over states for an integral over the unknown function $\sigma(p^2)$. But this function has

a nice physical interpretation: it is a *spectral density* function, which tells you something about the density of states at a given value of p^2. It is always positive, and in perturbation theory it is zero if p^2 is less than the the momentum-squared of the lightest bound state. If there is only one kind of particle, of mass m, in the theory, it is zero in perturbation theory if $p^2 < 4m^2$. Beyond perturbatoin theory, there may be bound states with masses less than $4m^2$, due to the binding energy of the state, in which case $\sigma(p^2)$ will be nonzero at the appropriate energy. (Note also that Peskin & Schroeder use the notation $\rho(p^2)$ instead of $\sigma(p^2)$, and also include in ρ the contribution from the single meson state.) The form of $\sigma(M^2)$ is shown in Fig. 3.2.

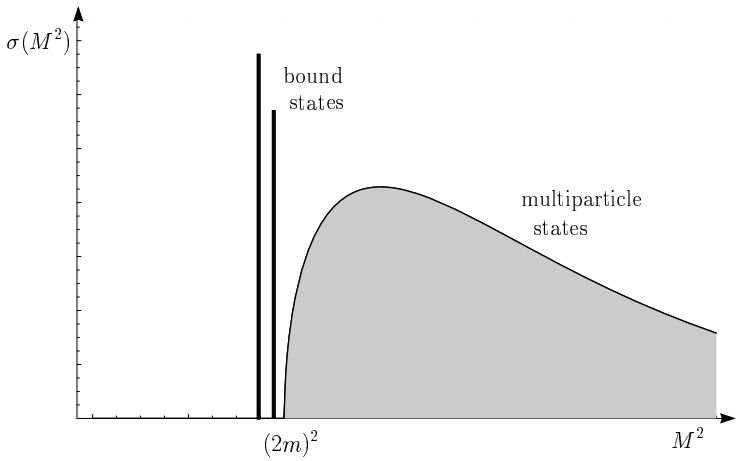


Figure 3.2: The form of $\sigma(M^2)$, the spectral density function. It vanishes below the multiparticle continuum, with the exception of isolated poles at the location of bound states.

So, what we have found so far is

$$\begin{aligned}\langle 0\,|\varphi(x)\varphi(y)|\,0\rangle &= i\Delta_+(x-y,\mu^2) + \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot(x-y)}\sigma(p^2)\theta(p^0) \\ &= i\Delta_+(x-y,\mu^2) + \int \frac{d^4p}{(2\pi)^3} e^{-ip\cdot(x-y)} \int_0^\infty dM^2\,\delta(M^2-p^2)\sigma(M^2)\theta(p^0) \\ &= i\Delta_+(x-y,\mu^2) + \int_0^\infty dM^2\,\sigma(M^2)\,i\Delta_+(x-y,M^2). \qquad (3.11)\end{aligned}$$

This is known as the *Lehmann-Källen spectral decomposition.* Note that it allows us to make a statement about Z, using $\varphi = Z^{-1/2}\varphi_0$, and the fact that φ_0 obeys the canonical commutation relations:

$$\langle 0\,|[\varphi(\vec{x},t),\dot{\varphi}(\vec{y},t)]|\,0\rangle = Z^{-1}i\delta^{(3)}(\vec{x}-\vec{y}). \qquad (3.12)$$

On the other hand, using the spectral representation and using the fact that

$$\frac{\partial}{\partial y^0} i\Delta_+(\vec{x}-\vec{y}) = i\delta^{(3)}(\vec{x}-\vec{y}), \tag{3.13}$$

we find

$$\langle 0\,|[\varphi(\vec{x},t),\dot{\varphi}(\vec{y},t)]|\,0\rangle = i\delta^{(3)}(\vec{x}-\vec{y}) + \int dM^2\,\sigma(M^2)\,i\delta^{(3)}(\vec{x}-\vec{y}). \tag{3.14}$$

Thus, we find

$$Z^{-1} = 1 + \int_0^\infty dM^2\,\sigma(M^2) \geq 1 \tag{3.15}$$

and so $Z \leq 1$, as expected by conservation of probability.

Now, combining the other time ordering, we at last get an expression for the renormalized two-point Green function,

$$\begin{aligned} \tilde{G}^{(2)}(k,k') &= (2\pi)^4\delta^{(4)}(k+k')\left(\frac{i}{k^2-\mu^2+i\epsilon} + \int_0^\infty dM^2\,\sigma(M^2)\frac{i}{k^2-M^2+i\epsilon}\right) \\ &= (2\pi)^4\delta^{(4)}(k+k')D(k^2) \end{aligned} \tag{3.16}$$

where we defined the full renormalized propagator, $D(k^2)$, by

$$D(k^2) = \frac{i}{k^2-\mu^2+i\epsilon} + \int_0^\infty dM^2\,\sigma(M^2)\frac{i}{k^2-M^2+i\epsilon}. \tag{3.17}$$

Since there was a lot of algebra in getting here, perhaps we should pause for a moment and interpret this result. Multiplying through by Z, we find the expression for the unrenormalized 2-point function,

$$\tilde{G}_0^{(2)}(k,k') = (2\pi^4)\delta^{(4)}(k+k')\left(\frac{iZ}{k^2-\mu^2+i\epsilon} + \int_0^\infty dM^2\,Z\sigma(M^2)\frac{i}{k^2-M^2+i\epsilon}\right). \tag{3.18}$$

The $(2\pi^4)\delta^{(4)}(k+k')$ is just the usual factor we include by definition in Green functions. For free field theory, the remainder of the expression would just be the usual free propagator, $i/(k^2-\mu^2+i\epsilon)$. By introducing interactions, all that has changed is that the field can now produce multiparticle states, so there are now additional contributions for the amplitude for the meson to propagate, and the amplitude for the field to produce a single meson has been reduced; hence the factor of Z.

3.2 The Analytic Structure of $G^{(2)}$, and 1PI Green Functions

The expression, Eq. (3.17), for $D(k^2)$ is actually a highly nontrivial expression. It defines a function *everywhere* in the complex k^2 plane, even though the propagator was not originally defined there. Let us denote the p^2 (called the

"invariant mass") of the lightest multiparticle state by $\mu^2 + \Delta$. In our nucleon-meson theory, for example, $\mu^2 + \Delta = 4m^2$, the invariant mass of a 2-nucleon state. The function $D(k^2)$ is analytic, except at $k^2 = \mu^2$, where it has a simple pole with residue i, and along the positive real axis beginning at $k^2 = \mu^2 + \Delta$, where it has a branch cut. The value of the function on the positive real axis is given by the $+i\epsilon$ prescription, which says that you take the value just above the cut. The analytic structure of $D(k^2)$ is shown in Fig. 3.3.

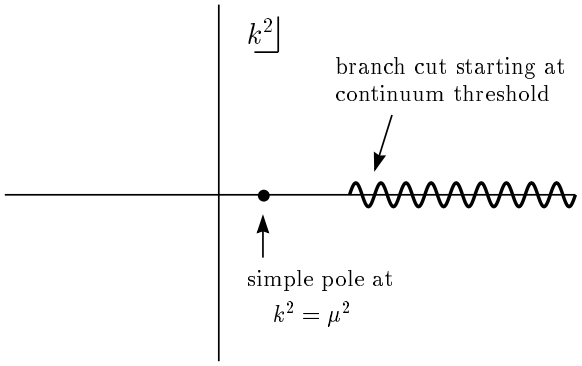


Figure 3.3: The analytic structure of $D(k^2)$. It has a simple pole at the invariant mass of the single-meson state, and a branch cut starting at the continuum for multiparticle states. (If there are bound states, there will be additional poles below the continuum threshold).

This is useful - it allows us to rephrase renormalization conditions 2 and 4 (and consequently 3 and 5 when we consider the nucleon two-point function) in terms of D. First of all, the fact that the meson mass is μ means that D has a simple pole and $k^2 = \mu^2$. Secondly, the requirement that

$$\langle 0 \,|\varphi(0)|\, k \rangle = 1 \tag{3.19}$$

corresponds to the statement that the residue of the pole at $k^2 = \mu^2$ is i.

We can actually massage all of this a bit more. Define another new kind of Green function, called a *one particle irreducible* (1PI) Green function as the sum of all connected graphs that cannot be disconnected by cutting a single internal line. Just to be confusing, we will stick by the standard convention that this Green function does *not* include the overall energy-momentum conserving δ function or the external propagators. The nice thing about 1PI Green functions for $n = 2$ is that we can express the full Green function simply in terms of the 1PI functions, as shown in Fig. 3.4.

By definition, the left-hand-side of Fig. 3.4 is $(2\pi)^4\delta^{(4)}(k + k')D(k^2)$. The 2-point, 1PI Green function has a name, the (renormalized) "self-energy" of the particle, and is denoted $-i\Pi(k^2)$.

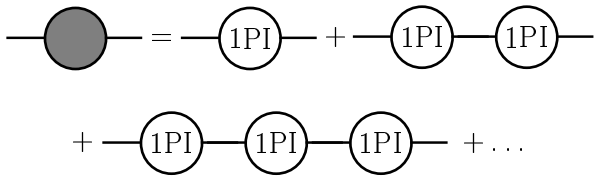


Figure 3.4: The 2-point Green function $\tilde{G}^{(2)}(k_1, k_2)$ can be expressed as a geometric series in terms if 1PI functions.

We can now sum the series for $\tilde{G}^{(2)}$ in terms of $\Pi(k^2)$:

$$\begin{aligned}
\tilde{G}^{(2)}(k,k') &= (2\pi)^4\delta^{(4)}(k+k')\frac{i}{k^2-\mu^2+i\epsilon} \\
&\qquad \times\left(1+\frac{\Pi(k^2)}{k^2-\mu^2+i\epsilon}+\left(\frac{\Pi(k^2)}{k^2-\mu^2+i\epsilon}\right)^2+\ldots\right) \\
&= (2\pi)^4\delta^{(4)}(k+k')\frac{i}{k^2-\mu^2+i\epsilon}\frac{1}{1-\frac{\Pi(k^2)}{k^2-\mu^2+i\epsilon}} \\
&= \frac{i}{k^2-\mu^2-\Pi(k^2)+i\epsilon}(2\pi)^4\delta^{(4)}(k+k'). \qquad (3.20)
\end{aligned}$$

This shows you why $\Pi(k^2)$ is called the "self-energy": it is like a momentum dependent mass. In terms of $\Pi(k^2)$, we find

$$D(k^2) = \frac{i}{k^2-\mu^2-\Pi(k^2)+i\epsilon} \qquad (3.21)$$

Now we can rephrase our previous renormalization conditions even more succinctly:

$$\begin{aligned}
D(k^2) \text{ has a pole at } k^2=\mu^2 &\iff \Pi(\mu^2)=0 \\
\text{The residue of this pole is } i &\iff \left.\frac{d\Pi}{dk^2}\right|_{k^2=\mu^2}=0
\end{aligned}$$

This is easy to see if you expand $\Pi(k^2)$ around $k^2=\mu^2$ in a power series,

$$\Pi(k^2) = \Pi(\mu^2) + (k^2-\mu^2)\left.\frac{d\Pi}{dk^2}\right|_{k^2=\mu^2} + \ldots. \qquad (3.22)$$

The first two terms in the series must vanish, or it screws up the location and residue of the pole.

So, having expressed our renormalization conditions in terms of 1PI Green functions, we can determine the counterterms B and C order by order in perturbation theory.

A single insertion of B and C gives the Feynman rule $i(Bk^2-C)$, where k is the momentum flowing through the vertex (recall that in passing from

the Lagrangian to Feynman rules, a derivative acting on an incoming line with momentum k brings down a factor of $-ik$). We can write B and C as power series expansions,

$$B = \sum_r B_r, \quad C = \sum_r C_r \tag{3.23}$$

where B_r and C_r are of order g^r (and $B_0 = C_0 = 0$, since there are no interactions at $O(g^0)$). Now, suppose we have calculated everything up to $O(g^n)$ in perturbation theory, including all counterterms. Then, at $O(g^{n+1})$, the contribution to $\Pi(k^2)$ is given by known stuff (that is, loops we can calculate, including insertions of lower order counterterms) plus single insertions of B_{n+1} and C_{n+1}, as shown in Fig. 3.5. Our renormalization conditions (3.22) therefore give us

$$\begin{aligned} iB_{n+1}\mu^2 - iC_{n+1} &= -(\text{known stuff})_{k^2=\mu^2} \\ iB_{n+1} &= -\left.\frac{d\text{known stuff}}{dk^2}\right|_{k^2=\mu^2}, \end{aligned} \tag{3.24}$$

determining B_{n+1} and C_{n+1}.

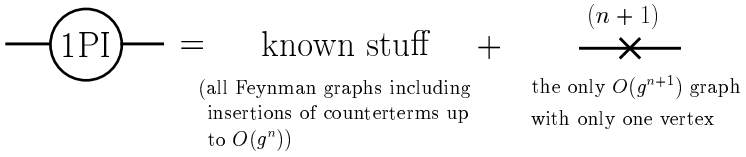


Figure 3.5: Determining counterterms at $O(g^{n+1})$.

Similarly, we can calculate the nucleon self-energy, $\Sigma(p^2)$, and impose the corresponding renormalization conditions

$$\Sigma(m^2) = 0, \quad \left.\frac{d\Sigma}{dp^2}\right|_{p^2=m^2} = 0. \tag{3.25}$$

Note that these subtractions do not allow you to ignore corrections to the 1PI two point function: $\Pi(k^2)$ has a complicated momentum dependence, and this is not eliminated by just subtracting the constant and first derivative terms in the vicinity of the pole. Since internal lines are not on shell, this momentum dependence is essential. However, it does solve our problem of loop graphs on external legs. Since the LSZ formula comes along with a factor of $k^2 - \mu^2$ for each external leg, only the location and residue of the pole are relevant for S matrix elements.

To see this explicitly, consider the sum of all graphs renormalizing an external leg of a diagram, including counterterms, as shown in Fig. 3.6. The contribution of this sum of graphs to the amplitude is therefore

$$\lim_{k^2\to\mu^2} \frac{k^2-\mu^2}{i} \times D(k^2) \times (\text{rest of graph})$$

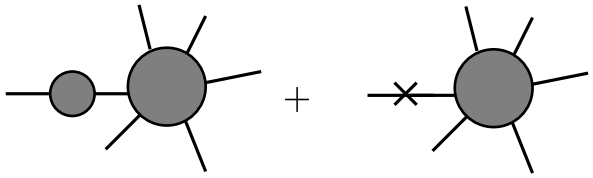


Figure 3.6: A generic Green function, including graphs renormalizing the external leg. The counterterm contribution has been denoted separately.

$$\begin{aligned} &= \lim_{k^2\to\mu^2} \frac{k^2-\mu^2}{i}\left(\frac{i}{k^2-\mu^2} + (\text{regular at } k^2=\mu^2)\right) \times (\text{rest of graph}) \\ &= (\text{rest of graph}) \end{aligned} \tag{3.26}$$

where we have used the fact that the counterterms have been chosen such that $D(k^2)$ has a simple pole $i/(k^2-\mu^2)$ and no other singularity at $k^2=\mu^2$. Thus, the counterterm precisely cancels the external leg corrections, and so we can ignore all diagrams which renormalize external legs.

3.3 Calculation of $\Pi(k^2)$ to order g^2

Let us denote the contribution to $\Pi(k^2)$ from terms other than the counterterms B_2 and C_2 by $\Pi_f(k^2)$, as shown in Fig. 3.7 where $B = B_2g^2 + B_3g^3 + \ldots$, $C = C_2g^2 + C_3g^3 + \ldots$, and the renormalization conditions (3.22) are

$$\begin{aligned} \Pi_f(\mu^2) - B_2\mu^2 + C_2 &= 0 \\ \left.\frac{d\Pi_f(k^2)}{dk^2}\right|_{k^2=\mu^2} - B_2 &= 0. \end{aligned} \tag{3.27}$$

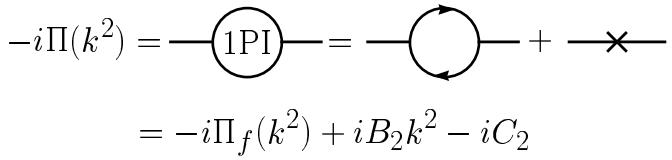


Figure 3.7: Feynman diagrams contribution to $\Pi(k^2)$ at $O(g^2)$.

Equivalently, if you don't care what B_2 and C_2 are, this may be written in terms of the Feynman diagram $\Pi_f(k^2)$:

$$\Pi(k^2) = \Pi_f(k^2) - \Pi_f(\mu^2) - (k^2-\mu^2)\left.\frac{d\Pi_f(k^2)}{dk^2}\right|_{k^2=\mu^2}. \tag{3.28}$$

The single graph contributing to Π_f at this order is given in Fig. 3.8, and applying the Feynman rules (including the integration over unconstrained loop

momenta) gives

$$-i\Pi_f(k^2) = (-ig)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \frac{i}{(q+k)^2 - m^2 + i\epsilon}. \tag{3.29}$$

This look pretty bad - a quadruple integral. Actually, it wouldn't be so bad,

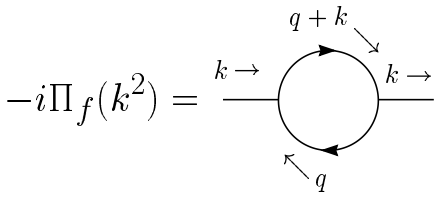


Figure 3.8: One loop graph contributing to Π_f at $O(g^2)$.

except for the facts that

1. It's not spherically symmetric (in 4D), so we can't just write it as an angular factor times an integral over the magnitude of q.
2. Actually, even if it were just a function of q^2, it wouldn't be (4D) spherically symmetric, since we're in Minkowski space. So we still couldn't write it as an angular factor times the integral over the magnitude of q.
3. It's divergent. At large q, it looks like

$$\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4} \tag{3.30}$$

which, if it were 4D spherically symmetric, would be

$$\sim \int \frac{q^3 dq}{q^4} \tag{3.31}$$

which, if we cut it off at some large momentum Λ, would have a piece proportional to $\log \Lambda$. Thus, the integral is logarithmically UV divergent.

This last problem we have discussed, and now we see how renormalized perturbation theory has saved us from worrying about infinities in our results: the quantity we are interested in, $\Pi(k^2)$, depends on $\Pi(k^2) - \Pi(\mu^2)$, and in the difference the divergent term cancels. Only the counterterms are divergent, but as we discussed several weeks ago, we expected this.

So now we will attack the other two problems. First of all, to make the integral "spherically" symmetric in Minkowski space (i.e. Lorentz invariant), we use Feynman's trick for combining denominators:

$$\begin{aligned} \int_0^1 \frac{1}{(ax + b(1-x))^2} &= \frac{1}{b-a} \left. \frac{1}{ax + b(1-x)} \right|_0^1 \\ &= \frac{1}{b-a} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{ab}. \end{aligned} \tag{3.32}$$

So applying this to the two propagators in Π_f, with

$$a = (q+k)^2 - m^2 + i\epsilon, \quad b = q^2 - m^2 + i\epsilon \tag{3.33}$$

we find

$$\begin{aligned} -i\Pi_f(k^2) &= g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \, \frac{1}{\left[\left((q+k)^2 - m^2 + i\epsilon\right) x + \left((q^2 - m^2 + i\epsilon\right)(1-x)\right]^2} \\ &= g^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \, \frac{1}{\left[q^2 + 2xk\cdot q + xk^2 - m^2 + i\epsilon\right]^2}. \end{aligned} \tag{3.34}$$

Now we can shift the integration variable and complete the square

$$q' \equiv q + xk \Rightarrow q'^2 = q^2 + 2xk\cdot q + k^2x^2 \tag{3.35}$$

to get

$$-i\Pi_f(k^2) = g^2 \int_0^1 dx \int \frac{d^4q'}{(2\pi)^4} \frac{1}{\left[q'^2 + k^2x(1-x) - m^2 + i\epsilon\right]^2}. \tag{3.36}$$

This would be easy, except that we're still in Minkowskian space, so we can't use the usual formulas for n dimensional volume elements. So now we deal with the second problem, and relate this to an integral which really is 4D spherically symmetric. We'll do this for the general case.

In general, when doing one-loop graphs we will get integrals of the form

$$I_n(a) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 + a + i\epsilon)^n} = \int \frac{d^3q}{(2\pi)^3} \frac{dq_0}{2\pi} \frac{1}{(q_0^2 - \vec{q}^2 + a + i\epsilon)^n} \tag{3.37}$$

(in the case at hand, $a = k^2x(1-x) - m^2$). Now, the locations of the poles of the integrand in the q_0 plane are shown in Fig. 3.9 for the two cases $\vec{q}^2 - a > 0$ and $\vec{q}^2 - a < 0$. In either case, the contour of integration may be *rotated* to the imaginary axis as shown, where q_0 runs from $-i\infty$ to $i\infty$, since no poles are crossed and the integrand vanishes at ∞. This is known as the "Wick rotation."

Let us define

$$q_4 = -iq_0 \tag{3.38}$$

and so we have

$$dq^0 = idq_4, \; d^4q = idq_4\, d^3q = id^4q_E \tag{3.39}$$

where q_E is a *Euclidean* 4 vector, satisfying

$$q_E^2 = q_4^2 + \vec{q}^2. \tag{3.40}$$

In terms of q_E, we can now write the integral as

$$\begin{aligned} I_n(a) &= i \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{(-q_4^2 - \vec{q}^2 + a + i\epsilon)^n} \\ &= i \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{(-q_E^2 + a + i\epsilon)^n}. \end{aligned} \tag{3.41}$$

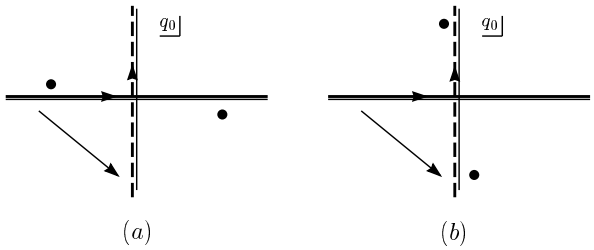


Figure 3.9: The Wick rotation: In both the cases $\vec{q}^2 - a > 0$ (a) and $\vec{q}^2 - a < 0$ (b), the path of the q_0 integration may be rotated as shown in the q_0 plane, without crossing any poles.

Now we're jamming - we have transformed our integral into something which is (four-dimensionally) spherically symmetric. The volume element in 4D space is

$$\int d^4 q_E = \int_0^\infty q_E^3 \, dq_E \int d\Omega_4 = 2\pi^2 \int_0^\infty q_E^3 \, dq_E \tag{3.42}$$

and setting $z = q_E^2$, $q_E^3 \, dq_E = \frac{1}{2} z \, dz$, we get

$$I_n(a) = \frac{i}{16\pi^2} \int_0^\infty z \, dz \frac{1}{(-z + a + i\epsilon)^n} \tag{3.43}$$

which is straightforward to evaluate.

Returning to the case at hand, we need

$$I_2(a) = \frac{i}{16\pi^2} \int_0^\infty \frac{z \, dz}{(-z + a)^2}. \tag{3.44}$$

This is divergent, although the in the combination we are interested the divergence cancels. So to make sense of the intermediate steps, let's *regulate* the integral by putting an artifical upper limit Λ^2 on the integration, which we will then take to ∞ at the end:

$$\begin{aligned} I_2^\Lambda(a) &= \frac{i}{16\pi^2} \int_0^{\Lambda^2} \frac{z \, dz}{(-z + a)^2} \\ &= \frac{i}{16\pi^2} \left(-\ln(-a) + \frac{a}{a - \Lambda^2} + \ln(\Lambda^2 - a) - 1 \right) \\ &= \frac{i}{16\pi^2} \left(\ln \Lambda^2 - 1 - \ln(-a) \right) + \ldots \end{aligned} \tag{3.45}$$

where the dots denote terms which vanish as $\Lambda \to \infty$. We therefore have

$$\Pi_f(k^2) = \frac{g^2}{16\pi^2} \int_0^1 dx \, \ln(-k^2 x(1-x) + m^2 - i\epsilon) + \frac{g^2}{16\pi^2} \left(\ln \Lambda^2 - 1 \right) \tag{3.46}$$

where the second term will vanish in the convergent combination $\Pi_f(k^2) - \Pi_f(\mu^2)$. Thus, we find, at long last, the desired result:

$$\begin{aligned}\Pi(k^2) &= \Pi_f(k^2) - \Pi_f(\mu^2) - (k^2-\mu^2)\left.\frac{d\Pi_f(k^2)}{dk^2}\right|_{k^2=\mu^2} \qquad (3.47)\\ &= \frac{g^2}{16\pi^2}\int_0^1 \left[\ln\frac{-k^2x(1-x)+m^2-i\epsilon}{-\mu^2x(1-x)+m^2-i\epsilon} + \frac{(k^2-\mu^2)x(1-x)}{-\mu^2x(1-x)+m^2}\right]dx\end{aligned}$$

We could actually do the remaining integral without too much trouble (it can be expressed in terms of arctangents), but it's not necessary for the discussion which follows.

The physical result (3.47) is finite - the only divergence in the calculation has been cancelled by the divergent counterterm C_2.[4] Thus, as expected, the bare mass of the interacting meson field is divergent. One thing we should check, however, is that the bare mass should still be real; otherwise, the bare Lagrangian will not be Hermitian. The only way $C_2 = -\Pi_f(\mu^2) + (\text{convergent})$ can have a complex piece is if the argument of the logarithm in Eq. (3.46) is less than zero for $k^2 = \mu^2$; that is, if

$$\mu^2 x(1-x) > m^2 \qquad (3.48)$$

for x in the range $(0,1)$. Since the maximum value of $x(1-x)$ in this interval is $1/4$, this means that C_2 will be real as long as

$$\mu^2 < 4m^2. \qquad (3.49)$$

In other words, as long as the meson is a stable particle, kinematically forbidden to decay to two nucleons, C_2 will be real. On the other hand, if it can decay to two nucleons it looks like its mass counterterm picks up an imaginary piece. We will discuss this case shortly, but for the moment we note that this is perhaps not unexpected, since if the meson can decay we really had no business treating it as a stable particle anyway. Furthermore, the analytic structure of $D(k^2)$ will be more complicated than we have discussed, since the single particle pole will be contained within the multiparticle continuum, and we will have to modify our renormalization conditions accordingly. With these modified conditions, C_2 will once again be purely real. As a bonus, we will discover a nice interpretation of the imaginary piece of Π.

Recall that we already determined, on general principles, the analytic structure of

$$D(k^2) = \frac{i}{k^2 - \mu^2 - \Pi(k^2) + i\epsilon}. \qquad (3.50)$$

Let's compare this with the one-loop result (3.47). For $\mu^2 < 4m^2$, $\int_0^1 \ln(-k^2x(1-x)+m^2-i\epsilon)\,dx$ has a branch cut starting at $k^2 = 4m^2$, continuing up to $k = \infty$,

[4] We would normally expect B_2 to be divergent as well. However, this theory is actually a special type called a *super-renormalizable* theory (due to the fact that the coupling constant g has positive mass dimension) and it has particularly nice UV behaviour. More on this shortly.

but it is analytic everywhere else, and the $+i\epsilon$ prescription says that the value of the function along the cut is obtained by approaching it from above. This agrees with what we found before, from the Lehmann-Källén spectral representation. It is always reassuring to see that our perturbative result agrees with results we obtained independent of perturbation theory.

Comments:

1. From Eq. (3.46), we also find

$$B_2 = -\frac{g^2}{16\pi^2}\int \frac{x(1-x)}{m^2-\mu^2 x(1-x)}\,dx < 0 \text{ (for } 4m^2 > \mu^2) \tag{3.51}$$

   and since $Z = 1 + B$, we find $Z < 1$, as required.

2. Feynman's trick may be used to combined any number of denominators. Using a similar technique, we find

$$\begin{aligned}\prod_{r=1}^{n}\frac{1}{a_r+i\epsilon} &= (n-1)!\int_0^1 d\alpha_1\ldots d\alpha_n\,\delta(1-\sum_i\alpha_i)\frac{1}{\left[\sum_r \alpha_r(a_r+i\epsilon)\right]^n}\\ &= (n-1)!\int_0^1 d\alpha_1\int_0^{1-\alpha_1} d\alpha_2\ldots\int_0^{1-\alpha_1\ldots-\alpha_{n-2}} d\alpha_{n-1}\\ &\qquad \frac{1}{\left[\sum_{r=1}^{n-1}\alpha_r a_r+\left(1-\sum_r^{n-1}\alpha_r\right)a_n+i\epsilon\right]^n}.\end{aligned} \tag{3.52}$$

3. For $n \geq 3$, where the integral is convergent, we have

$$I_n(a) = \frac{i}{16\pi^2(n-1)(n-2)a^{n-2}}. \tag{3.53}$$

   The integral is divergent for $n = 1,\ 2$; introducing a cutoff Λ^2 we find

$$\begin{aligned} I_1 &= \frac{i}{16\pi^2}a\ln(-a)+\ldots\\ I_2 &= -\frac{i}{16\pi^2}\ln(-a)+\ldots\end{aligned} \tag{3.54}$$

   where the dots denote additional terms which cancel in a convergent sum of such terms (i.e. a sum of such terms such that the total integrand vanishes faster than $1/q^4$ for high q).

4. Should the fact that the bare mass is divergent be a worry, even philosophically? No - the ultraviolet divergence comes from the region of loop integration where k is very large, and so the distances are correspondingly short. Experimentally, we don't know much about what happens at momenta greater than about 100 GeV (about $5\times 10^(-17)$ m), and so we really can't expect the integral to be correct above this region. If all

particles were actually composite, for example, their interactions would be cut off by some form factor at about the scale of their sizes, which would make all the loop integrals finite. However, the beauty of renormalized perturbation theory is that this is totally irrelevant: any effects of the cutoff Λ which grow with Λ cancel out of physical quantities. If we were to leave a finite cutoff in the calculation, rather than taking it to ∞, we would only be left with terms which were suppressed by $p_{\rm typ}/Lambda$, where $p_{\rm typ}$ is the typical momentum in the scattering process of interest. Thus, as long as we are doing physics well below the scale at which the physics changes, these effects are irrelevant.

5. Finally, note that if we insist that the renormalized mass $\mu = 0$, C_2 is still logarithmically divergent. In other words, a massless meson can only arise due to a very precise cancellation between the bare mass and the interactions - the infinite energy stored in the field surrounding the particle exactly cancels its divergent bare mass to leave a massless particle. Thus, this is an extremely unnatural situation - even if you started with a particle with vanishing bare mass, the interactions would produce a mass for it. Thus, massless scalar particles are highly unnatural. (This will not be the case for fermions, as we will see).

3.4 The definition of g

We would like to relate the coupling g in the Lagrangian to some definition which has been approved by a committee of eminent persons. Let us define $\Gamma(p^2, p'^2, q^2)$ to be the 1PI three-point function, as shown in Fig. 3.10. Γ is defined with an i so that it is always g at leading order in perturbation theory; however, at higher orders in perturbation theory it will be momentum-dependent. (Note that since Γ is Lorentz invariant, it can only be a function of scalar products. Since there are only two indepenent momenta, it can only depend on p^2, p'^2 and $p \cdot p'$, but $p \cdot p'$ can be traded in for q^2.)

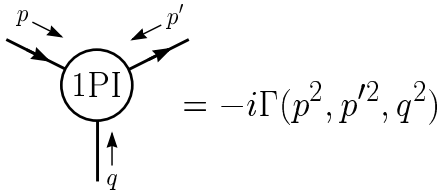


Figure 3.10: The 1PI three-point function.

A sensible committee definition would be to set $\Gamma(\overline{p}^2, \overline{p}'^2, \overline{q}^2) = g$ for some fixed point $(\overline{p}^2, \overline{p}'^2, \overline{q}^2)$ in momentum space. Here's a particularly sensible committee definition:

$$g = \Gamma(m^2, m^2, \mu^2). \tag{3.55}$$

This point is actually not kinematically accessible (an on-shell nucleon cannot scatter off an on-shell meson and remain on-shell) but nevertheless we will show that it has a nice experimental significance.

Consider meson-nucleon scattering. We can split all graphs contributing to this process into 1PI graphs and non-1PI graphs. Now, consider the behaviour of the amplitude as a function of $s = (p_N + p_\varphi)^2$, the invariant mass of the scattering particles. Although $s = m^2$ is kinematically forbidden, we can certainly define our scattering amplitudes there by analytic continuation, and we can extrapolate our experimental results to this point.

Now, since the 1PI contribution to the amplitude never has a single nucleon carrying all the incident momentum, it should not have a pole at $s = m^2$, since that would indicate the presence of a single-particle resonance. Furthermore, it shouldn't be hard to convince yourself that all non-1PI contributions to the rate have the form shown in Fig. 3.11(b), (c) and (d). Of these, only (b) has a pole at $s = m^2$, since at this point the internal nucleon propagator $D(s)$ has pole. Thus, in the vicinity of $s = m^2$, this class of graphs dominates the amplitude, and we have

$$\begin{aligned} i\mathcal{A} &= -i\Gamma(s, m^2, \mu^2)D(s)(-i\Gamma(m^2, s, \mu^2)) \\ &= -i\frac{g^2}{s - m^2} + (\text{regular at } s = m^2) \end{aligned} \tag{3.56}$$

where the second line follows as a consequence of our renormalization condition, Eq. (3.4). Thus, measuring the scattering amplitude and extrapolating to $s = m^2$ (that is, vanishingly small incident meson momentum) gives a direct measurement of g with this renormalization condition.

As an aside, if you do this experiment with real nucleons and mesons, you find $g \sim 13.5$. This put the last nail in the coffin in the attempt to describe the nucleon-meson strong interactions via a renormalizable, perturbative field theory such as this model. (Actually, the original experiment was $\gamma N \to \pi N$ inelastic scattering which gives a measurement of eg, since it's not easy to make pion beams. The general idea was the same, though.)

3.5 Unstable Particles and the Optical Theorem

We now return to the situation where $\mu > 2m$. In this case, $\Pi_f(k^2)$ is not real at $k^2 = \mu^2$, and so our previous renormalization condition for C yields a non-Hermitian bare Lagrangian. In this section we will modify the renormalization prescription, and also discover the physical interpretation of the imaginary piece of Π.

From the spectral representation, it is straightforward to show that

$$\mathrm{Im}\Pi(k^2) = -\pi\frac{\sigma(k^2)}{|D(k^2)|^2} \tag{3.57}$$

and so Π picks up an imaginary piece when k is large enough to make multi-particle states - this is a real physical effect and should not be subtracted away

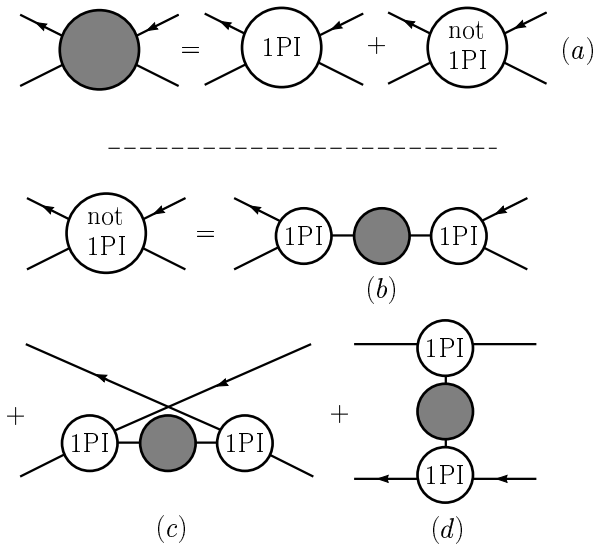


Figure 3.11: (a) Contributions to nucleon-meson elastic scattering may be divided into one-particle irreducible and non-1PI Green functions. The 1PI contribution does not have a pole at $s = m^2$. The non-1PI contributions all have the form (b), (c) or (d), and only (b) has a pole at $s = m^2$.

by our renormalization condition. In this theory, $\sigma(k^2) \neq 0$ for $k^2 > 4m^2$, since this is the start of the two-nucleon continuum. Thus, $\mathrm{Im}\Pi(\mu)^2 \neq 0$ for $\mu^2 > 4m^2$, and so we shouldn't be subtracting this imaginary piece away with a counterterm. What has happened is that the multiparticle continuum has moved down so that it overlaps with the single particle pole.

So let's modify our renormalization prescription, so that for $\mu > 2m$,

$$\mathrm{Re}\Pi(\mu^2) = 0, \quad \mathrm{Re}\left.\frac{d\Pi(k^2)}{dk^2}\right|_{k^2=\mu^2} = 0. \tag{3.58}$$

Now, let's interpret the imaginary piece of $\Pi(k^2)$. From the defininition of σ, we have

$$\begin{aligned} \mathrm{Im}\Pi(k^2) &= -\pi\frac{\sigma(k^2)}{|D(k^2)|^2} \\ &= -\frac{1}{2}|D(k^2)|^{-2}\sum_{n\neq|\,0\rangle,|\,k\rangle}|\langle n\,|\varphi(0)|\,0\rangle|^2(2\pi)^4\delta^{(4)}(k-p_n) \end{aligned} \tag{3.59}$$

where the sum is over all multiparticle states $|\,n\rangle$, and $p_n \equiv \sum_i k_i$. Now, consider the contribution of some n particle state $|\,k_1,\ldots,k_n\rangle$ in the sum (the state could include particles of different types; it doesn't matter for this discussion). From

the LSZ formula and its generalizations, we know that

$$\langle k_1, \ldots, k_n \,|\varphi(0)|\, 0\rangle = \int \frac{d^4k}{(2\pi)^4} \prod_{r=1}^{n} \frac{k_r^2 - m_r^2}{i} \tilde{G}(k_1, \ldots, k_n; k) \tag{3.60}$$

where the Green function $\tilde{G}$ is illustrated in Fig. 3.12.

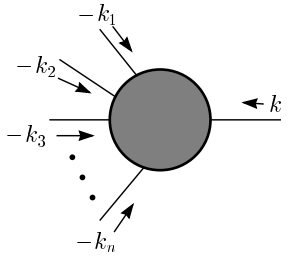


Figure 3.12: The Green function which determines $\langle k_1, \ldots, k_n \,|\varphi(0)|\, 0\rangle$.

Thus, we have

$$\begin{aligned} \langle k_1, \ldots, k_n \,|\varphi(0)|\, 0\rangle &= [\text{sum of Feynman diagrams}] \times D(p_n^2) \\ &= i\mathcal{A}_{fi} D(p_n^2) \end{aligned} \tag{3.61}$$

where $\mathcal{A}_{fi}$ is the usual invariant Feynman amplitude for the process $\varphi \to n$.

Thus, at $k^2 = \mu^2$, summing over all states $|\, n\rangle$ the factors of the external meson propagators $|D(\mu^2)|$ cancel, and we find

$$\begin{aligned} \mathrm{Im}\Pi(k^2 = \mu^2) &= -\frac{1}{2} \sum_f |\mathcal{A}_{fi}|^2 (2\pi)^4 \delta^{(4)}(k - p_n) \\ &= -\frac{1}{2} \sum_n \frac{1}{n!} \int \frac{d^3p_1}{(2\pi)^2 2E_1} \cdots \frac{d^3p_n}{(2\pi)^2 2E_n} |\mathcal{A}_{fi}|^2 (2\pi)^4 \delta^{(4)}(k - p_n) \\ &= -\mu\Gamma \end{aligned} \tag{3.62}$$

where Γ is the decay width of the φ, which we calculated last semester. (The factors of $1/n!$ arise from the overcounting of identical particles.)

Now let's use this result to look at the full meson propagator $D(k^2)$ in the vicinity of $k^2 = \mu^2$. For convenience, let's look at the region where $k^2 - \mu^2 \leq O(g^2)$. Then we have

$$\begin{aligned} [-iD(k^2)]^{-1} &= k^2 - \mu^2 - \Pi(\mu^2) - (k^2 - \mu^2) \left.\frac{d\Pi}{dk^2}\right|_{k^2=\mu^2} + O(g^4) \\ &= k^2 - \mu^2 - \mathrm{Re}\Pi(\mu^2) - i\mathrm{Im}\Pi(\mu^2) + O(g^4) \\ &= k^2 - \mu^2 + i\mu\Gamma + O(g^4) \\ &= k^2 - \left(\mu - \frac{i\Gamma}{2}\right)^2 + O(g^4) \end{aligned} \tag{3.63}$$

where we have used the renormalization condition (3.58), and the factor that both $k^2 - \mu^2$ and $\frac{d\Pi(\mu^2)}{dk^2}$ are both $O(g^2)$. Thus, the renormalized propagator, in the vicinity of the pole, is

$$D(k^2) = \frac{i}{k^2 - \left(\mu - \frac{i\Gamma}{2}\right)^2} + O(g^4). \tag{3.64}$$

Thus, the pole in the propagator has moved off the real axis, as shown in Fig. 3.13.

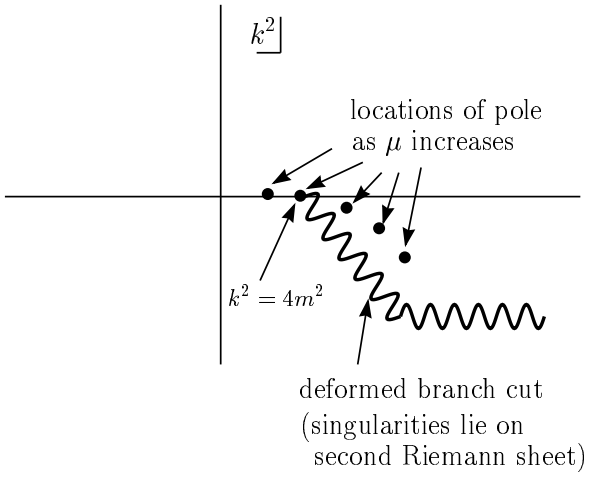


Figure 3.13: Location of pole in the full propagator $D(k^2)$ as μ increases past $4m^2$. The $+i\epsilon$ prescription dictates that the singularity be approached from above; hence the singularities lie in the second Riemann sheet, seen here by deforming the branch cut.

We have found the physical significance of the imaginary part of the full propagator. Let me make a few (somewhat lengthy) comments.

1. Experimentally, Γ really is a width. When an experimentalist says she has discovered an unstable particle, what she means is that she has found a peak in the momentum distribution of the decay products hitting her detector. So let's analyze this situation.

An experimentalist blasts the vacuum by crashing two protons together in the middle of her detector. A theorist blasts the vacuum by turning on a δ function source:

$$\mathcal{L} \to \mathcal{L} + \rho(x)\varphi(x), \quad \rho(x) = \lambda\delta^{(4)}(x). \tag{3.65}$$

The amplitude to find any momentum eigenstate $|\, n\rangle$ is then proportional to

$$\lambda\langle n\,|\varphi(0)|\,0\rangle + O(\lambda^3).$$

Thus, the probability of finding a state with momentum k is

$$\begin{aligned}\lambda^2 \sum_n |\langle n\,|\varphi(0)|\,0\rangle|^2 (2\pi)^4 \delta^{(4)}(p_n - k) + O(\lambda^3) \\ = 2\pi^2 \lambda^2 \sigma(k^2)\theta(k^0) + O(\lambda^3)\end{aligned} \tag{3.66}$$

where we have used the definition of σ. From Eq. (3.57) we therefore find, to leading order in the strength λ of the source, the probability to find a final state with momentum k is proportional to

$$-\lambda^2 \operatorname{Im}\Pi(k^2)\,|D(k^2)|2.$$

Using our previous result that near $k^2 = \mu^2$, $\operatorname{Im}\Pi(k^2) = -\mu\Gamma$, this becomes

$$\frac{\lambda^2 \mu\Gamma}{(k^2-\mu^2)^2 + \mu^2\Gamma^2}$$

and thus the probability to find a state with energy E in the centre of mass frame is proprtional to

$$\begin{aligned}\frac{\mu\Gamma}{(E^2-\mu^2)^2+\mu^2\Gamma^2} &= \frac{\mu\Gamma}{(E-\mu)^2(E+\mu)^2+\mu^2\Gamma^2} \\ \sim \frac{\mu\Gamma}{(E-\mu)^2(2\mu)^2+\mu^2\Gamma^2} &= \frac{\mu\Gamma}{(4\mu^2)\left[(E-\mu)^2+\frac{\Gamma^2}{4}\right]}\end{aligned} \tag{3.67}$$

where the second line follows if $\Gamma \ll \mu$, the particle's width is much smaller than its mass.

The probability distribution is plotted in Fig. 3.14; this is called a *Breit-Wigner*, or *Lorentzian* lineshape. The pole in the propagator has been smeared out by the width; as Γ decreases, the peak gets narrower and higher, approaching the pole shape of a stable particle.

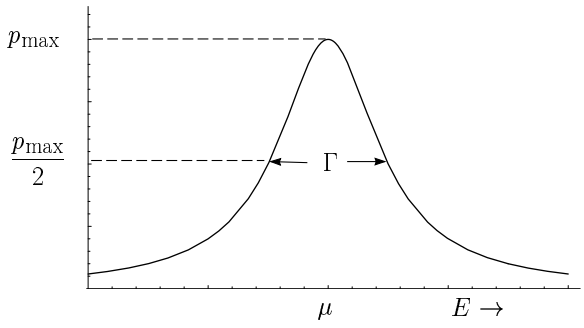


Figure 3.14: A Lorentzian lineshape. The full-width at half maximum is Γ.

Thus, μ and Γ are what an experimentalist means when she says she has found an unstable particle with mass μ and width Γ.

2. The previous comment actually justifies our previous expression for Γ, which we obtained in the theory with a turning on and off function. The problem is, you can't use the LSZ formula in the usual way to calculate the width, since you can't make an "in" state of an unstable particle - it decays away in the time from $t = -\infty$ to $t = 0$. So instead, we get the expression for the partial decay width $d\Gamma$ to some final state $|\, n\rangle$ by taking the matrix element

$$^{\text{out}}\langle n\,|\varphi(0)|\,0\rangle$$

which we *can* evaluate via the extended form of the LSZ formula discussed in the last chapter. That is, we blast the vacuum at $t = 0$, and look at the out states this produces in the region of the single-particle pole.

3. This is an example of a more general result, known as the Optical Theorem. This actually goes back to scattering theory in quantum mechanics. There it was based on a simple idea: there is an incoming wave incident on a target, and an outgoing wave. The outgoing wave is the superpositioni of the incoming wave that passes right through the target and the scattered wave. By conservation of probability, if the wave is scattered there must be a decrease in the intensity of the beam in the forward direction.

The total probability to scatter in any direction *but* directly forward is proportional to the cross section σ. This therefore is equal to the decrease in the probability of going exactly forward, which is the interference between the unscattered and scattered waves.

The field theory version of the optical theorem is trivial to derive. By conservation of probability,

$$S^\dagger S = 1$$

and thus we have

$$(S-1)(S-1)^\dagger = SS^\dagger - S - S^\dagger + 1 = -(S-1) - (S-1)^\dagger. \tag{3.68}$$

Now consider the case of forward scattering. Taking the matrix element of this result between the in and out states $|\, f\rangle$ and $|\, i\rangle$, and inserting a complete set of states $|\, n\rangle$ we have

$$\sum_n \langle f\,|(S-1)|\,n\rangle\langle n\,|(S-1)^\dagger|\,i\rangle - \langle f\,|(S-1)|\,i\rangle - \langle f\,|(S-1)^\dagger|\,i\rangle. \tag{3.69}$$

Working on the LHS, we find[5]

$$\begin{aligned} \text{LHS} \;\; &= \;\; \sum_n \int \frac{d^3k_1 \ldots d^3k_n}{(2\pi)^3 2E_1 \ldots (2\pi)^3 2E_n} \mathcal{A}_{fm}\mathcal{A}^*_{im} \\ &\qquad \times (2\pi)^4\delta^{(4)}(p_f - p_m)(2\pi)^4\delta^{(4)}(p_i - p_m). \end{aligned} \tag{3.70}$$

[5] In the case in which some of the n_m particles are identical, we will have additional factors of $1/n_m!$ in these formulas.

Now consider the case of forward scattering, $i = f$. In this case, we find, factoring out a single δ function from both sides,

$$-i\mathcal{A}_{ii} + i\mathcal{A}^*_{ii} = \sum_n \int \frac{d^3k_1 \dots d^3k_n}{(2\pi)^3 2E_1 \dots (2\pi)^3 2E_n} |\mathcal{A}_{im}|^2 (2\pi)^4 \; delta^{(4)}(p_i - p_m) \tag{3.71}$$

and using the fact that the integral over final states times the δ function is what we used to call the invariant density of states D_m, we obtain the result:

$$2\mathrm{Im}\mathcal{A}_{ii} = \sum_n \int D_m |\mathcal{A}_{im}|^2. \tag{3.72}$$

Thus, the total transiiton probability is proportional to the imaginary part of the forward scattering amplitude. For example, for $2 \to 2$ scattering, the RHS of Eq. (3.72) is proportional to the cross section σ, and we have from the definition of σ,

$$\mathrm{Im}\mathcal{A}_{ii} = 2E_T p_i \sigma. \tag{3.73}$$

The optical theorem is shown in diagrams in Fig. 3.15.

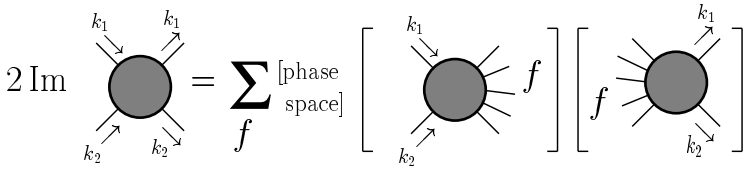


Figure 3.15: The optical theorem: the imaginary piece of the forward scattering amplitude is related to the total transition probability to some intermediate state $|f\rangle$, summed over all possible states.

In our previous case, if we take $-i\Pi(k^2)$ as the amplitude for $1 \to 1$ "scattering", we get

$$\mathrm{Im}\Pi(\mu^2) = -\frac{1}{2}\sum_n \int D_m |\mathcal{A}_{im}|^2 = -\mu\Gamma \tag{3.74}$$

as before.

3.6 Renormalizability

The $O(g^3)$ correction to Γ in this simple model is finite - the counterterm F is only needed to make g agree with our committee definition of g. This can be seen by simple power counting: at high q, the one-loop graph goes like

$$\int \frac{d^4q}{q^6}$$

which is convergent. This is peculiar to theories with positive mass dimension in the couplings (so-called superrenormalizable theories). For spin 1/2 nucleons coupled to mesons (where the coupling g is dimensionless), the graph goes like

$$\int \frac{d^4q}{q^4}$$

which is logarithmically divergent, and the counterterm is required to cancel the divergence.non

Consider, for simplicity, a theory of a single meson with a $\lambda\varphi^4$ interaction. Then the wave function renormalization comes from the two loop diagram in Fig. 3.16 (a) which is proportional to

$$\int \frac{d^8q}{q^4}$$

and is quartically divergent, but this is cancelled by the appropriate counterterm. Similarly, the graph in Fig. 3.16 (b) is logarithmically divergent, but this is cancelled by the φ^4 counterterm.

Now consider the graph in Fig. 3.16 (c). If this term were divergent, it would be trouble - we would have to introduce at φ^6 counterterm into our theory, even though there was no interaction in the bare Lagrangian that looked like this. Thus, our committee would have to meet again, and comeup with a renormalization condition for $3 \to 3$ scattering, requiring us to introduce another coupling constant. Our theory would then have no predictive power for $3 \to 3$ scattering. Fortunately, this graph is proportional (at high q) to

$$\int \frac{d^4q}{q^6}$$

and so is convergent, and no counterterm is required.

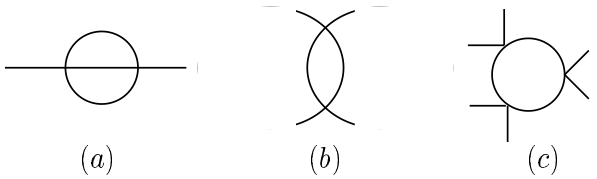


Figure 3.16: Loops graphs in φ^4 theory.

With this in mind, we make the following definition:

A Lagrangian is said to be renormalizable (in the strict sense) only if all of the counterterms required to remove infinities from Green functions are of the same type as those present in the original Lagrangian.

(This is actually slightly more stringent than some other definitions of this term; hence the "strict sense".)

For example, suppose we had a φ^5 interaction. Then the graph in Fig. 3.17 (a) is logarithmically divergent, so we would need a φ^6 counterterm. Thus, φ^5 theory is not renormalizable. Of course, we could add a φ^5 term to the Lagrangian, but then you'd get graphs like those in Fig. 3.17 (b) and (c), requiring φ^7 and φ^8 terms, and so on. The problem with this, as we have already mentioned, is that the theory would no longer have predictive power. Our committee would have to come up with an infinite number of renormalization conditions for $2 \to 3$, $2 \to 4, \ldots, 2 \to n$ scattering, and so our theory would have an infinite number of adjustable parameters.[6]

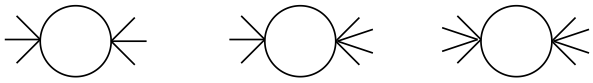


Figure 3.17: Divergent loops graphs (a) in φ^5 theory, (b, c) with a φ^6 counterterm added.

Thus, theories with polynomial interactions of scalars with more than four fields are nonrenormalizable. Although we haven't proved it, it seems plausible that scalar theories with poynomial interactions with four or fewer fields are renormalizable. In the next section we will study this problem in more detail, and determine the conditions for a general field theory to be renormalizable.

[6] Actually, this statement is a bit misleading. As we may discuss later in the course, nonrenormalizable theories can be very useful. Even though they cannot make *exact* predictions without tuning an arbitrary number of parameters, the effects of operators of higher and higher dimension are suppressed by powers of the momentum relevant to a given process over some large momentum scale where new physics comes into play. It turns out that nonrenormalizable Lagrangians are very useful for calculating in theories with widely separated mass scales.

4 Renormalizability

At the end of the last section, we considered the renormalizability of a simple scalar field theory. We would now like to extend this discussion to more general theories, such as QED. To do this, let us first approach scalar field theory a bit more systematically and look at all the possible divergences one could encounter.

4.1 Degrees of Divergence

Consider scalar field theory with a φ^n interaction,

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu \varphi\right)^2 - \frac{\mu^2}{2}\varphi^2 - \frac{\lambda}{n!}\varphi^n \tag{4.1}$$

and consider an arbitrary diagram in this theory. We define

$$\begin{aligned} N &= \#\text{ of external scalars} \\ P &= \#\text{ of propagators} \\ V &= \#\text{ of vertices} \\ L &= \#\text{ of loops.} \end{aligned} \tag{4.2}$$

At large loop momentum, where m is irrelevant, the graph will thus behave like

$$\int \frac{d^4k_1 \ldots d^4k_L}{k_1^2 \ldots k_P^2}. \tag{4.3}$$

The graph will be divergent is the number of powers of k in the numerator (including the (d^4k's) is greater than or equal to the number of powers of k in the denominator. Thus, we define the "superficial degree of divergence" D of a graph to be

$$\begin{aligned} D &= (\text{powers of } k \text{ in numerator})\text{-}(\text{powers of } k \text{ in denominator}) \\ &= 4L - 2P. \end{aligned} \tag{4.4}$$

Then, if the integral is regulated by cutting it off at some large scale Λ, we would expect that graph to scale like Λ^D (if $D = 0$, we expect the graph to be logarithmically divergent, growing like $\ln \Lambda$.) Thus, if $D \geq 0$ the graph is said to be superficially divergent, while if $D < 0$ it is superficially convergent.

The reason for the pejorative adjective "superficial" is that this is not necessarily the correct degree of divergence of the graph. This is simply because all the propagators don't necessarily carry loop momenta, as tacitly assumed in the above argument. The examples in Fig. 4.1 illustrate the point. Fig. 4.1(a) has $D = 4 - 2 = 0$, and is logarithmically divergent, as expected. Similarly, Fig. 4.1(b) has $D = 8 - 6 = 2$ and is in fact quadratically divergent. However, Fig. 4.1(c) is superficially convergent, $D = 8 - 10 = -2$, but nevertheless contains a divergent subgraph. Our previous arguments fail for this graph because two of the propagators are not in the loop, and so don't contribute to making the loop graphs any more convergent.

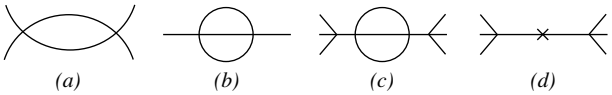


Figure 4.1: (a) and (b) are superficially divergent; (c) is superficially convergent, but the divergence is cancelled by the counterterm for (b).

Despite this shortcoming, D is in fact a very useful quantity, because of a theorem due to Hepp: *If all superficially divergent 1PI diagrams are cancelled by counterterms, then all divergences are removed from scattering amplitudes.* The reason for this can be determined by the graph in Fig. 4.1(c): superficially convergent graphs which are still divergent contain divergent subgraphs, which on their own are also superficially divergent. Once these divergences are cancelled by the appropriate counterterms, any graph which contains them as subgraphs is also rendered finite. (This is illustrated in Fig. 4.1(d)).

This is a very useful result: once you have identified all the superficially divergent diagrams in a theory, you can determine all the required counterterms. The further constraint that you only have to worry about 1PI graphs arises because any non-1PI graph can be written as a product of 1PI graphs without introducing any additional loops.

So, now that we have demonstrated the usefulness of D, let us go a bit farther with our formula Eq. (4.4) in φ^N theory. Recall that every propagator gives us a $\int d^4k$, and each vertex a δ functions. There will be one overall energy-momentum conservation δ function left over, and any momentum which is unconstrained will give us a loop integral. Therefore, the number of leftover loop integrals is

$$L = P - V + 1 \tag{4.5}$$

and since n lines meet at each vertex and each propagator lands on two vertices,

$$nV = N + 2P \tag{4.6}$$

and so we find

$$D = 4L - 2P = (n-4)V + 4 - N. \tag{4.7}$$

So now we see there are three cases:

- If $n > 4$, adding vertices to a diagram increases D. This is BAD, because if means at higher orders in perturbation theory (graphs with more vertices) you can always continue to get new superficially divergent graphs for any value of N. Thus, at some order in perturbation theory we will need a $\phi^9 9$ counterterm. Thus, an infinite number of counterterms are required, since there are an infinite number of superficially divergent amplitudes, and so the theory is *nonrenormalizable*, in agreement with our arguments at the end of the last section.

- If $n = 4$, only a finite number of Green's functions will superficially diverge, those with $N < 4$, and so only a finite number of counterterms will

be required to make all amplitudes finite. This class of theory is called *renormalizable*. Note that there will be divergent graphs at all orders in perturbation theory, but only for Green functions with less than or equal to N external legs.

- If $n < 4$, not only is the theory renormalizable, but each time you add a vertex you make any given graph more convergent. In this case, not only is the number of superficially divergent Green functions finite, but the number of superficially divergent *graphs* is finite. Divergences are restricted to low orders in perturbation theory (so that V isn't too big), in contrast with renormalizable theories, where divergences occur at all orders in perturbation theory. This class of theory is called *super-renormalizable*; the $\psi^*\psi\varphi$ theory we have been playing with is in this class.

Now that we have formulated our criteria for renormalizability in a more general way, we can determine all necessary counterterms just be writing down all superficially divergent amplitudes. For example, in φ^4 theory, we have $D = 4 - N$, and so the only superficially divergent amplitudes have $N \leq 4$:

1. $N = 0$: This is a zero-point Green function, just correponding to the vacuum energy, and is cancelled by the appropriate constant energy shift.

2. $N = 1,\ 3$: No such counterterms can exist, because $\mathcal{L}$ is invariant under the symmetry $\varphi \to -\varphi$, and such terms break this symmetry.

3. $N = 2$: The self-energy graph has $D = 2$. We can Taylor expand this amplitude in powers of p, the external momentum, to a series of the form $a_0 + a_1 p^2 + a_2 p^4 + \ldots$. Now, each term in the expansion will have a smaller D than the previous one, since each time we differentiate the amplitude with respect to p it reduces the degree of divergence by 1 (adding one power of momentum to the denominator of the loop integral). Thus, a_0 is quadratically divergent ($D = 2$), but a_0 is only logarithmically divergent ($D = 0$) while all other a_i's are convergent. Thus, this Green function has two independent divergences, requiring two counterterms of the form $Ap^2 + B$. These are just mass and wavefunction renormalization.

4. $N = 4$: The four-point function has $D = 0$, so diverges like $\ln \Lambda$. It therefore requires a single counterterm.

That's it! As expected, there are no superficially divergent amplitudes with more than 4 external legs.

We can also compare this result with the superrenormalizable $\psi^*\psi\varphi$ theory. In this case, we have $D = 4 - N - V$. For the two-point function, $D = 2 - V$ and so the only graph with $D > 0$ is the one-loop 1PI graph we calculated in the last chapter! Since $D = 0$, it is log divergent, so expanding the amplitude as $a_0 + a_1 p^2 + \ldots$, only a_0 is divergent, requiring a mass counterterm. As we found in the last chapter, the wavefunction renormalization is finite. Furthermore, graphs at higher orders in perturbation theory are all superficially convergent, since they

will have $V > 2$. Similarly, the one-loop renormalization of g already has $D = 4 - 3 - 3 = -2$, and so there is no divergent coupling constant renormalization in this theory, as we found.

A few comments:

1. The renormalizability of a theory depends on the number of dimensions you are working in. More generally, for φ^n theory in d dimensions, similar manipulations yield

$$D = d + \left[n\left(\frac{d-2}{2}\right) - d \right] V - \left(\frac{d-2}{2}\right) N. \tag{4.8}$$

   Ths, φ^4 theory is renormalizable in four dimensions, while in three dimensions it is superrenormalizable, and φ^6 theory is renormalizable. In two dimensions, φ^n theory is renormalizable for *any* n. We can also summarize this in terms of dimensional analysis. In d dimensions, $[\mathcal{L}] = d$ (so that the action is dimensionless), and since $[m^2\varphi^2] = d$, we have $[\varphi] = (d-2)/2$. Thus, $[\lambda\varphi^n] = d$, and the units of the coupling constant λ in φ^n theory in d dimensions are $[\lambda] = d - n[\varphi] = d - n(d-2)/2$. Comparing this with Eq. (4.8), we find that

   (a) $[\lambda] > 0$ (the coupling has positive mass dimension $\rightarrow$ the theory is superrenormalizable

   (b) $[\lambda] = 0$ (the coupling is dimensionless) $\rightarrow$ the theory is renormalizable

   (c) $[\lambda] < 0$ (the coupling has negative mass dimension) $\rightarrow$ the theory is nonrenormalizable.

   As we shall see, this requirement is generally true, even in theories with fermions and (with additional restrictions) gauge bosons. Thus, the class of renormalizable theories in four dimensions is very small - only terms with dimension ≤ 4 are allowed.

2. It is straightforward to add spin 1/2 fermions to this discussion, as well as derivative interactions such as $\varphi^2\partial_\mu\varphi\partial^\mu\varphi$. Fermions contribute differently to D than bosons since their propagators don't fall off as fast for large k - like $1/\not{k}$ instead of $1/k^2$. Thus, in this case we have

$$D = 4L - 2P_B - P_F \tag{4.9}$$

   where P_B is the number of scalar boson propagators and P_F is the number of spin 1/2 fermion propagators. Now suppose that the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \lambda_i \mathcal{L}_i \tag{4.10}$$

   where the i'th interaction vertex contains

b_i	scalar bosons
f_i	spin 1/2 fermiosn
d_i	derivatives

and has coupling constant λ_i. Then, going through the same rigamarole as before, we find that a graph with n_i vertices of type i has (in four dimensions)

$$D = 4 - N_B - 3/2N_F + \sum_i n_i \delta_i \tag{4.11}$$

where

$$\delta_i \equiv d_i + b_i + 3/2 f_i - 4 \tag{4.12}$$

is the "index of divergence" of the interaction $\mathcal{L}_i$. (Note that because of their slower UV falloff, fermions give a larger contribution to D than scalars). THus, if any interaction has positive index of divergence, the theory is nonrenormalizable, since by adding vertices a positive D may be obtained for any value of N_B and N_F. But again, even in this case, in four dimensions we have $[\varphi] = 1$, $[\psi] = 3/2$, and so $\delta_i = [\mathcal{L}_i] - 4 = -[\lambda_i]$. Thus, as before, we find that as long as all λ_i's are ≥ 0 (have positive or zero mass dimension) the theory is renormalizable.

3. Generally, *all* renormalizable interactions are required to absorb divergences in superficially divergent subgraphs. For example, consider a pseudoscalar-fermion-antifermion coupling

$$\mathcal{L}_I = -ig\overline{\psi}\gamma_5\psi\varphi. \tag{4.13}$$

The coupling g has 0 mass dimension. However, this theory is not, strictly speaking, renormalizable. The box graph in Fig. 4.2 is logarithmically divergent ($D = 0$), and so it will need a φ^4 counterterm, which wasn't originally present in $\mathcal{L}_i$. However, the interaction

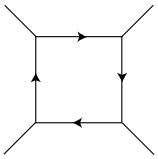


Figure 4.2: Logarithmically divergent "box graph".

$$\mathcal{L}_I = -ig\overline{\psi}\gamma_5\psi\varphi + \frac{\lambda}{4}\varphi^4 \tag{4.14}$$

is, strictly speaking, renormalizable. This is also in accord with our earlier assertion that massless scalars are unnatural - even if the renormalized mass μ is set to zero, a mass counterterm is nevertheless required.

4. There is one exception to the above rule: a theory which is invariant under some internal symmetry will not need all terms with $d \leq 4$ to be renormalizable, but only those consistent with the symmetry. In the above example, for instance, the Lagrangian is invariant under parity,

with φ transforming as a pseudoscalar, $\varphi(t,\vec{x}) \to -\varphi(t,-\vec{x})$. However, the scalar interaction $\overline{\psi}\psi\varphi$ violates this symmetry, and so will never be required as a counterterm.[7] Indeed, if this were not the case, parity would be inconsistent with quantum field theory, since renormalization would induce parity violating counterterms.

With all of this in mind, we now turn to ...

4.2 Renormalization of QED

4.2.1 Troubles with Vector Fields

We haven't yet discussed vector fields. Offhand, they look like trouble. Consider adding a generic massive spin one field to our previous discussion. As we discussed in the first semester, the propagator for a massive spin one field is

$$\frac{i}{k^2-\mu^2+i\epsilon}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{\mu^2}\right). \tag{4.15}$$

The second term looks bad - it doesn't fall off at all for large momenta! So it will give a large contribution to D. If we work through as before we find the index of divergence of an interaction with V_i massive vector bosons is

$$\delta_i = d_i - 4 + b_i + 3/2 f_i + 2v_i. \tag{4.16}$$

Note that since $[A^\mu] = 1$, this is *not* $[\mathcal{L}_i] - 4$. The vector bosons behave much worse than their dimension suggests. In fact, there are *no* possible renormalizable interaction terms containing v_i! For example, $-g\overline{\psi}\gamma^\mu\psi A_\mu$ has $\delta_i = 2\times 3/2 + 2 - 4 = 1$, while $-g\varphi\partial_\mu\varphi A^\mu$ has $\delta_i = 3+2-4=1$. Thus, any theory with interacting massive vector bosons in not renormalizable, because of this second term in the popagator $k^\mu k^\nu/\mu^2$.

However, recall that this term arises in the spin summation from the 3-D longitudinal mode of the vector meson. There is, however, one class of theories which does not have such a propagating mode - gauge theories with massless vector bosons. Recall from last semester that if a massive vector boson couples to a conserved current, the second term in the propagator doesn't contribute to scattering amplitudes, and so (at least at tree level; we will worry about loops later), we can set

$$\frac{i}{k^2-\mu^2+i\epsilon}\left(-g^{\mu\nu}+\frac{k^\mu k^\nu}{\mu^2}\right) \to -\frac{ig^{\mu\nu}}{k^2-\mu^2+i\epsilon}. \tag{4.17}$$

Thus, in this case, A^μ behaves just like a scalar field as far as D is concerned, and so in the case that A_μ couples to a conserved current,

$$\delta_i = b_i + \frac{3}{2}f_i + d + i - 4 \tag{4.18}$$

[7] The exception to this rule are *anomalous symmetries*, classical symmetries that are broken in the quantum theory. We won't discuss these in this course, but you can read the discussion in chapter 19 of Peskin and Schroeder).

where now b_i is the number of scalar *plus* vector bosons. In this case, $\overline{\psi}\gamma^\mu\psi A_\mu$ is a renormalizable interaction, $\delta_i = 0$, and so for QED the superficial degree of divergence of a graph with N_γ external photons and N_e external electrons is

$$D = 4 - N_\gamma - \frac{3}{2}N_e. \tag{4.19}$$

So it looks like we have a fighting chance.

4.2.2 Counterterms

The Lagrangian for QED is

$$\mathcal{L}_{\rm QED} = -\frac{1}{4}\left(F_0^{\mu\nu}\right)^2 + \overline{\psi}_0\left(i\partial\!\!\!/ + m_0\right)\psi_0 - e_0\overline{\psi}_0\gamma_\mu\psi_0 A_0^\mu \tag{4.20}$$

where as usual the subscript 0 denotes bare quantites. We defined the renormalized fields

$$\psi \equiv Z_2^{-1/2}\psi_0, \quad A^\mu \equiv Z_3^{-1/2}A_0^\mu \tag{4.21}$$

in terms of which the interaction term is

$$-e_0 Z_2 Z_3^{1/2}\overline{\psi}\gamma^\mu\psi A_\mu \equiv -eZ_1\overline{\psi}\gamma^\mu\psi A_\mu \tag{4.22}$$

where we have defined the renormalized coupling

$$e = \frac{e_0 Z_2 Z_3^{1/2}}{Z_1}. \tag{4.23}$$

As usual, the precise definition of e (or of Z_1) will be set by some convention which we don't yet specify. In terms of the renormalized quantites, we split the Lagrangian into the renormalized piece and the counterterms:

$$\begin{aligned}\mathcal{L}_{\rm QED} &= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - e\overline{\psi}\gamma^\mu\psi A_\mu \\ &\quad -\frac{1}{4}\delta_3 F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\left(i\delta_2\partial\!\!\!/ - \delta m\right)\psi - e\delta_1\overline{\psi}\gamma^\mu\psi A_\mu\end{aligned} \tag{4.24}$$

where the second line is the counterterm Lagrangian, and we have defined

$$\delta_3 \equiv Z_3 - 1,\ \delta_2 \equiv Z_2 - 1,\ \delta_m \equiv Z_2 m_0 - m,\ \delta_1 \equiv Z_1 - 1. \tag{4.25}$$

Thus, we have four counterterms, with Feynman rules given in Fig. 4.3.

Now, the statement that QED is renormalizable is the statement that these four counterterms will absorb *all* the divergences of QED. This isn't obvious! Although the theory looks fine by power counting, Eq. (4.24) *doesn't* include all possible interactions with $\delta_i \leq 0$, and there is no guarantee that these won't be required as counterterms. For example, the one loop graph in Fig. 4.4 (a) has $D = 2$, so we would expect to have a quadratically divergent mass counterterm for the photon, as shown in the figure. Furthermore, the four point vertex in

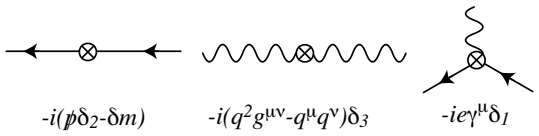


Figure 4.3: Counterterms for QED.

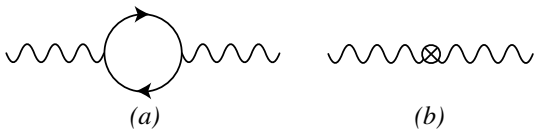


Figure 4.4: (a) has $D = 2$, so we expect a divergence $\sim \Lambda^2 g^{\mu\nu}$, requiring the photon mass counterterm (b).

Fig. 4.4 (b) is logarithmically divergent ($D = 0$), so we would expect an $(A^\mu A_\mu)^2$ counterterm.

This would be a disaster.

First of all, it would make the masslessness of the photon highly unnatural, just as a massless scalar is unnatural. The bare photon mass would have to be precisely tuned to exactly cancel the effects of the interactions. Second, an $(A_\mu A^\mu)^2$ interaction term would mean that the photon no longer couples to a conserved current; we could no longer write the interaction as $A_\mu J^\mu$, where $\partial_\mu J^\mu = 0$. In this case we could no longer argue away the longitudinal piece of the photon propagator. This would spoil the renormalizability of the theory (as well as the $\mu \to 0$ limit!).

So what saves us? The only reason a counterterm wouldn't arise is if it were forbidden by a symmetry of the theory. Of course, in this case, the symmetry is *gauge invariance*. Since gauge invariance is a somewhat peculiar internal symmetry, let me remind you of how it goes.

A $U(1)$ gauge transformation has the form

$$\psi \to e^{-ieQ\lambda(x)}\psi, \; A_\mu \to A_\mu - \partial_\mu \lambda(x). \tag{4.26}$$

Since $\lambda(x)$ is not a constant, but instead a function of space and time, the electron kinetic term on its own is not invariant under this transformation - the derivative also acts on $\lambda(x)$. In order for a theory to be gauge invariant, derivatives acting on ψ and gauge fields A_μ must occur in the combination

$$D^\mu \equiv \partial^\mu - ieA^\mu. \tag{4.27}$$

(Note that there is a subtlety hidden here - is this e the same as our committee definition of the renormalized coupling e? Really, the second term is $-ie_0 A_0^\mu$ - we shall see later on that $e_0 A_0^\mu = eA^\mu$, so the two are equivalent. This is one of the miracles of gauge invariance.) The two problematic terms, A^2 and A^4, are not invariant under the gauge transformation (4.26). Thus, as long as

gauge symmetry is preserved by renormalization, they shouldn't arise. However, since gauge symmetries are not as straightforward as global symmetries, this again isn't obvious. Let's see how it works. First, however, recall one of the consequences of coupling a photon to a conserved current that we discovered last semester. An amplitude with an external photon with polarization ϵ^μ will have the form $\epsilon^\mu M_\mu$, where M_μ is the rest of the diagram. We found that as long as the current is conserved,

$$k^\mu M_\mu = 0. \tag{4.28}$$

This just follows from $\partial_\mu J^\mu = 0$, and is known as the *Ward Identity*, and it tells us that the helicity zero mode of the photon decouples, since the helicity zero mode has $\epsilon_\mu \propto k^\mu$.

We will prove the Ward Identity (and its generalization) soon, but for the moment let us assume it remains true beyond tree level.

Now let us look at all the 1PI graphs with $D > 0$ in QED. There are six, as illustrated in Fig. 4.5, where the starred Green functions denote possible trouble, since they have no gauge invariant counterterms.

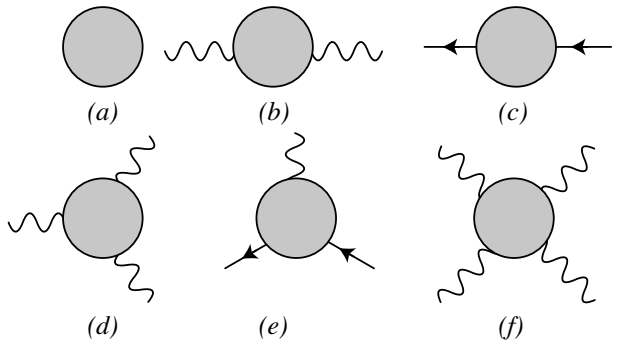


Figure 4.5: Superficially divergent Green's function in QED.

We will dispatch with the easy ones first. (a) is just the vacuum energy counterterm, which is trivial. (c) corresponds to electron mass and wavefunction renormalization. Expanding in powers of $\not{p}$,

$$\text{(c)} \sim A_0 + A_1 \not{p} + A_2 p^2 + \ldots \tag{4.29}$$

the usual arguments would lead us to expect that A_0 is linearly divergent, $\sim \Lambda$, $A_1 \sim \ln \Lambda$, and all the other A's are finite. Thus, there are two infinities, A_0 and A_1, and two counterterms, δ_2 and δ_m, so all is well. Actually, there is a subtlety here. A theory with massless fermions has an extra symmetry, *chiral* symmetry, since the left- and right-handed helicity modes of a massless fermion decouple. Thus, we can perform *independent* $U(1)$ transformations on the two helicities,

$$\psi_L \to e^{i\alpha}\psi_L, \ \psi_R \to e^{i\beta}\psi_R. \tag{4.30}$$

A mass counterterm $\delta m\overline{\psi}\psi = \delta m\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right)$ breaks this symmetry down to the diagonal $U(1)$ of fermion number, and so is forbidden. (Thus, in contrast to scalars, massless fermions are quite natural, since they are protected from getting a mass by a symmetry). As far as our counterterms go, it means that A_0 vanishes when $m = 0$, so by dimensional analysis it must be proportional to $m\ln\Lambda$, and not Λ, so is only logarithmically divergent.

Diagram (e) is straightforward - it has $D = 0$ and so has a single logarithmic divergence proportional to $-ie\gamma^\mu \ln\Lambda$, which is cancelled by δ_1.

Diagram (d) would required a counterterm proportional to A^3. However, this amplitude is actually zero - it is forbidden by charge conjugation symmetry. (This result is known as "Furry's Theorem.") Recall that the charge conjugation operator C changes particles to antiparticles. You can easily show that this implies that the electromagnetic current operator must flip sign under C:

$$CJ^\mu C^\dagger = -J^\mu \tag{4.31}$$

where $J^\mu = \overline{\psi}\gamma^\mu\psi$ is the electromagnetic current. Since the three-point function is related to the Fourier transform of

$$\langle 0\,|T\left(J^\mu(x_1)J^\nu(x_2)J^\alpha(x_3)\right)|\,0\rangle \tag{4.32}$$

we can act with the C operator to show that

$$\begin{aligned}\langle 0\,|J^\mu(x_1)J^\nu(x_2)J^\alpha(x_3)|\,0\rangle &= \langle 0\,|CC^\dagger J^\mu(x_1)CC^\dagger J^\nu(x_2)CC^\dagger J^\alpha(x_3)CC^\dagger|\,0\rangle \\ &= -\langle 0\,|J^\mu(x_1)J^\nu(x_2)J^\alpha(x_3)|\,0\rangle \end{aligned}\tag{4.33}$$

since $CC^\dagger = 1$ and the vacuum is invariant under charge conjugation, $C|\,0\rangle = |\,0\rangle$. Thus, the matrix element of an odd number of currents vanishes, so the photon three-point function in QED is identically zero.

This leaves us with diagram (b). This one looks like trouble. Defining the photon self-energy $\Pi^{\mu\nu}$ in analogy with the scalar self energy, we can split $\Pi^{\mu\nu}$ up into two Lorentz structures,

$$\Pi^{\mu\nu}(q) = ag^{\mu\nu} + b\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right). \tag{4.34}$$

A divergence in b is fine - this is precisely the correct form to be absorbed by the counterterm δ_3. The a term is problematic - it corresponds to $D = 2$, or a quadratic divergence, requiring a counterterm of the form $A^\mu A_\mu$. This is a photon mass counterterm, precisely the disaster we must avoid. But now the Ward Identity saves us - it states in this case that

$$q^\mu\Pi_{\mu\nu}(q) = 0 \tag{4.35}$$

and so we will find $a = 0$ identically. Thus, gauge invariance guarantees that no divergences in loop graphs will have the form of a photon mass.

Thus, we may write

$$\Pi^{\mu\nu}(q) = \Pi(q^2)\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right). \tag{4.36}$$

(Note that we expect $\Pi(q^2)$ to be regular about $q^2 = 0$, otherwise the self-energy would have a pole, corresponding to a massless state. Since the graph is 1PI, no such pole should exist.) Since we have taken two powers of q out of the self-energy, $\Pi(q^2)$ is at most logarithmically divergent, and this divergence is cancelled by δ_3, leaving a finite expression for $\Pi^{\mu\nu}(q^2)$.

Finally, we have diagram (f). Once again, by the Ward Identity, we know that this vanishes when contracted with any of the external momenta $k_1^\mu,\ k_2^\nu, k_3^\alpha, k_4^\beta$. By exhaustion, you can show that this requires it to have the form

$$\left(g^{\mu\nu}k_1^\beta - g^{\mu\beta}k_1^\alpha\right) \times (\text{permutations})\,. \tag{4.37}$$

Thus, the graph is order k^4, and so the first nonvanishing term in the taylor series expansion has $D = 0 - 4 = -4$. Thus, the four-point photon vertex function introduces no new divergences requiring an A^4 counterterm.

Hence, QED is renormalizable. We have shown that all the divergences in the theory may be absorbed by four gauge invariant counterterms. Defining the electron self-energy $\Sigma(\not{p})$, the photon self-energy $\Pi(q^2)$ and the vertex function $\Gamma(p, p')$ as shown in Fig. 4.6 and proceeding in the same way as we proceeded

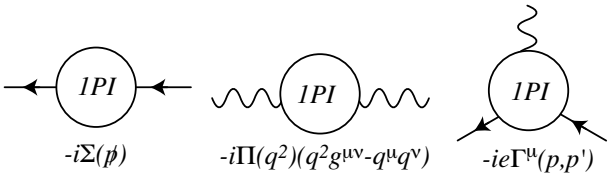


Figure 4.6: 1PI Green's functions in QED which require renormalization.

in the scalar case, we define the following renormalization conditions for QED:

$$\begin{aligned} &\Sigma(\not{p} = m) = 0, \quad \left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m} = 0, \\ &\Pi(q^2 = 0) = 0, \quad \Gamma^\mu(p' - p = 0) = \gamma^\mu. \end{aligned} \tag{4.38}$$

The first three just guarantee that the mass of the electron is m and that the photon and electron fields are properly normalized. The last ensures that the charge measured at very low momentum transfer (or at very large distances) corresponds to the coupling constant e.