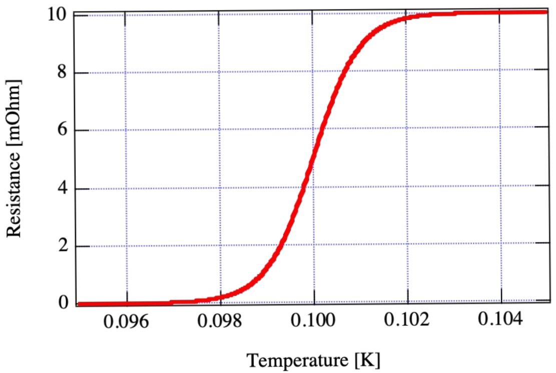
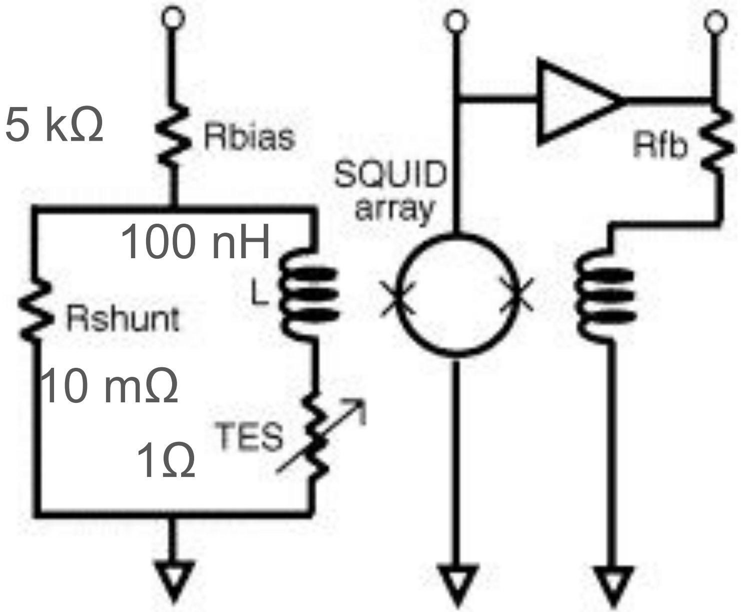
Transition Edge Sensor Modeling

Following “Transition-Edge Sensors, K.D. Irwin and G. C. Hilton”

TES operation

- Set a **voltage** across the TES (V_{TES})
- Put TES in series with an inductor
- Current in TES → magnetic flux in inductor
- → SQUID measures magnetic flux
- → SQUID measures current I_{TES}
- Calculate TES resistance with V_{TES}/I_{TES}
- R → T

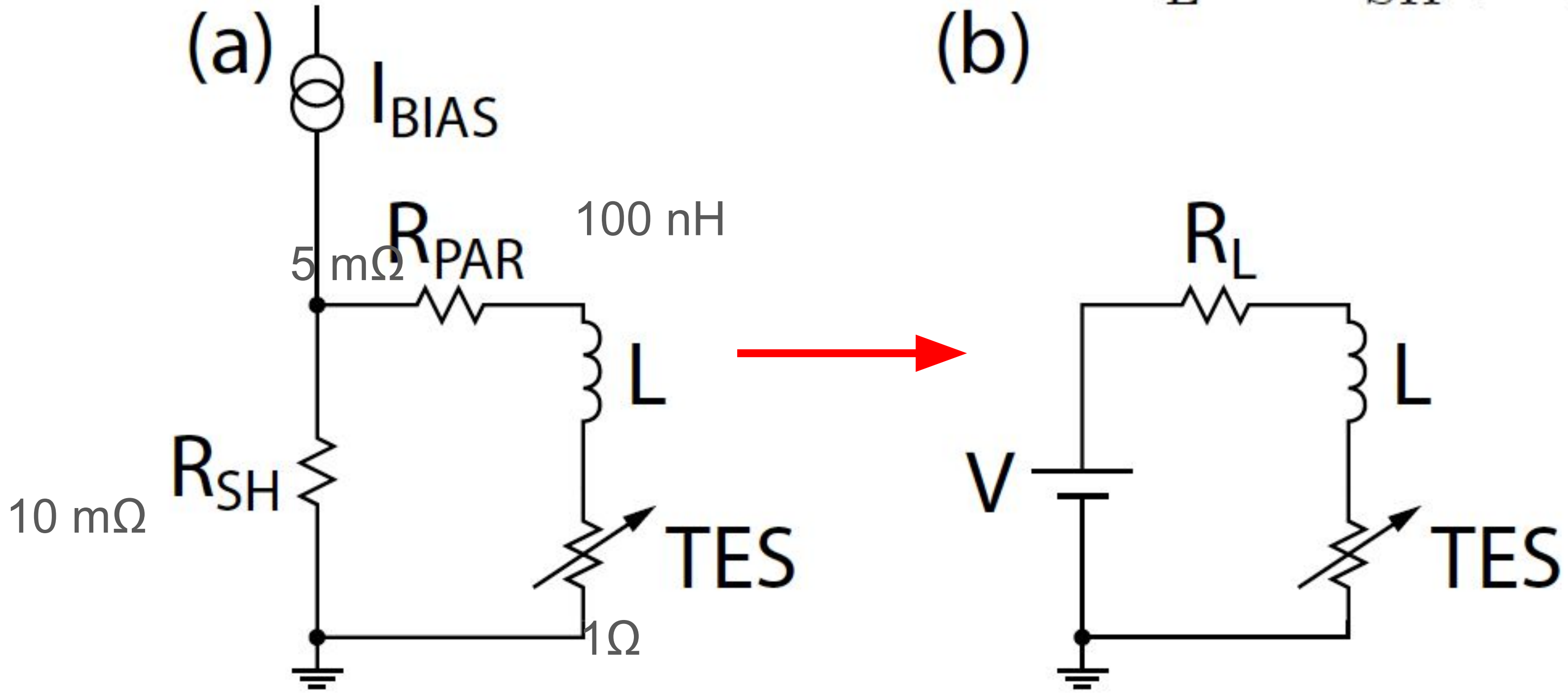




Thevenin Equivalent Circuit

$$V = I_{\mathrm{BIAS}} R_{\mathrm{SH}}$$

$$R_{\mathrm{L}} = R_{\mathrm{SH}} + R_{\mathrm{PAR}}$$

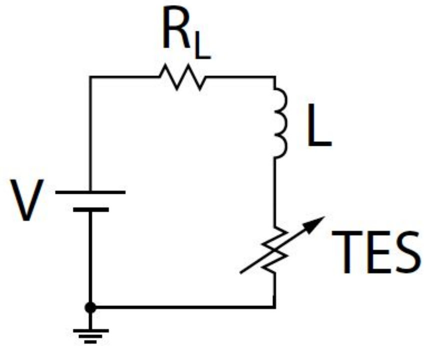


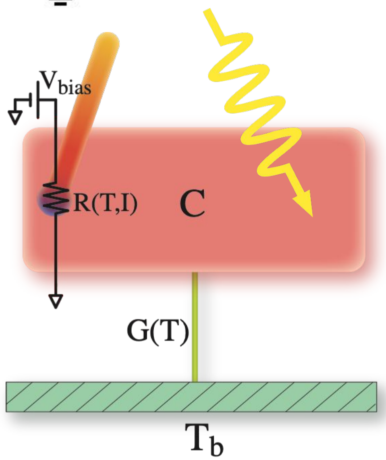
Differential equations

$$L\frac{\mathrm{d}I}{\mathrm{d}t} = V - IR_L - IR(T,I)$$

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = -P_{\mathrm{bath}} + P_{\mathrm{J}} + P$$

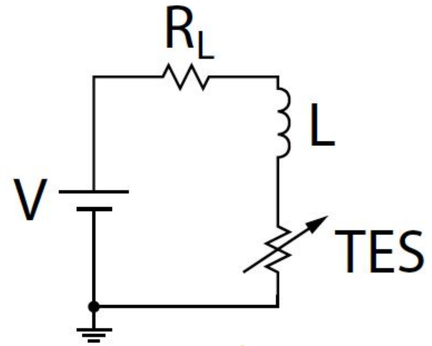
$$P_{\mathrm{bath}} = K\left(T^n - T^n_{\mathrm{bath}}\right)$$

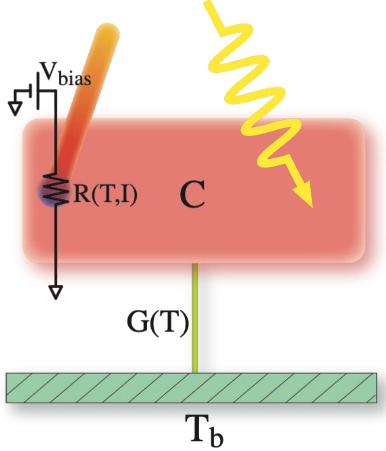




Quiescent solution and Electro-thermal stability

- Typical scenario:
  - Tb ~ 10 mK
  - T_{TES} (critical temperature Tc) ~ 50 mK
- Joule heating keeps TES at elevated temperature
  - $P = V^2/R$
- In Quiescence, TES operates at
  - T_0 : Operating temperature, close to Tc
  - R_0: TES operating resistance
    - ~30% of Rn (normal Resistance)
  - P_0: "Bias power", power to heat TES from Tb to T0
- When particle hit → TES heats up → R goes up → P decreases → TES cools back down





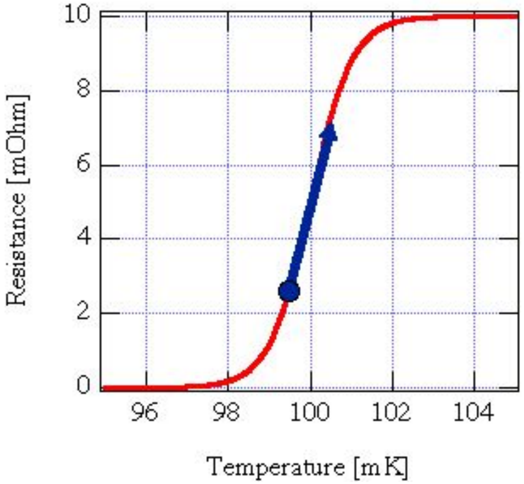
TES model

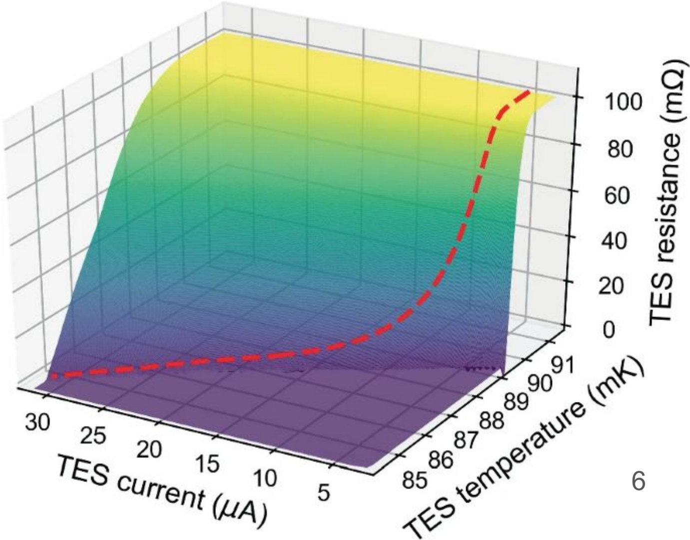
$$R(T,I) \approx R_0 + \left.\frac{\partial R}{\partial T}\right|_{I_0} \delta T + \left.\frac{\partial R}{\partial I}\right|_{T_0} \delta I$$

$$\alpha_I \equiv \left.\frac{\partial \log R}{\partial \log T}\right|_{I_0} = \frac{T_0}{R_0} \left.\frac{\partial R}{\partial T}\right|_{I_0}$$

$$\beta_I \equiv \left.\frac{\partial \log R}{\partial \log I}\right|_{T_0} = \frac{I_0}{R_0} \left.\frac{\partial R}{\partial I}\right|_{T_0}$$

$$\boxed{R(T,I) \approx R_0 + \alpha_I \frac{R_0}{T_0} \delta T + \beta_I \frac{R_0}{I_0} \delta I}$$





More “random” definitions (aka jargons)

- Constant temperature dynamic resistance $$R_{\text{dyn}} \equiv \left.\frac{\partial V}{\partial I}\right|_{T_0} = R_0\left(1+\beta_I\right)$$
- Natural thermal time constant $$\tau \equiv \frac{C}{G}$$
- Low frequency loop gain with constant current $$\mathscr{L}_I \equiv \frac{P_{J_0}\alpha_I}{GT_0}$$
- Electrical time constant $$\tau_{\text{el}} = \frac{L}{R_L + R_0\left(1+\beta_I\right)} = \frac{L}{R_L + R_{\text{dyn}}}$$
- Current biased thermal time constant $$\tau_I = \frac{\tau}{1-\mathscr{L}_I}$$

Linearized differential equation

- $$L\frac{\mathrm{d}I}{\mathrm{d}t} = V - IR_L - IR(T, I)$$
$$C\frac{\mathrm{d}T}{\mathrm{d}t} = -P_{\mathrm{bath}} + P_{\mathrm{J}} + P$$
- Small signal limit, Taylor expansion, neglecting all higher order terms

$$\frac{\mathrm{d}\delta I}{\mathrm{d}t} = -\frac{R_L + R_0\left(1+\beta_I\right)}{L}\delta I - \frac{\mathscr{L}_I G}{I_0 L}\delta T + \frac{\delta V}{L}$$

$$\frac{\mathrm{d}\delta T}{\mathrm{d}t} = \frac{I_0 R_0\left(2+\beta_I\right)}{C}\delta I - \frac{\left(1-\mathscr{L}_I\right)}{\tau}\delta T + \frac{\delta P}{C}.$$

δV and δP are small perturbations around bias voltage V and input power P

  - Put in for later analyses, eg. particle power injection, noise

Matrix differential equation

$$\frac{\mathrm{d}}{dt}\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} = -\begin{pmatrix}\dfrac{1}{\tau_{\mathrm{el}}} & \dfrac{\mathscr{L}_I G}{I_0 L}\\ -\dfrac{I_0 R_0 (2+\beta_I)}{C} & \dfrac{1}{\tau_I}\end{pmatrix}\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} + \begin{pmatrix}\dfrac{\delta V}{L}\\ \dfrac{\delta P}{C}\end{pmatrix}$$

- Solve for homogeneous equations first → setting δV and δP to 0
- Diagonalize the matrix → two eigenvalues with two eigenvectors

$$\frac{d}{dt}f_{\pm}(t) = -\lambda_{\pm} f_{\pm}(t)$$

Differential equation solution

$$\frac{\mathrm{d}}{dt}\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} = -\begin{pmatrix}\frac{1}{\tau_{\mathrm{el}}} & \frac{\mathscr{L}_I G}{I_0 L}\\ -\frac{I_0 R_0\,(2+\beta_I)}{C} & \frac{1}{\tau_I}\end{pmatrix}\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} + \begin{pmatrix}\frac{\delta V}{L}\\ \frac{\delta P}{C}\end{pmatrix}$$

$$\frac{d}{dt} f_\pm(t) = -\lambda_\pm f_\pm(t)$$

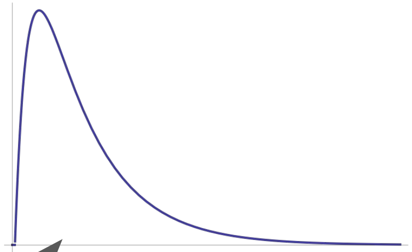
- Solve for homogeneous equations first → setting δV and δP to 0
- Diagonalize the matrix → two eigenvalues with two eigenvectors

$$\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} = A_+ e^{-\lambda_+ t}\,\vec{v_+} + A_- e^{-\lambda_- t}\,\vec{v_-}$$

$$\frac{1}{\tau_\pm} \equiv \lambda_\pm = \frac{1}{2\tau_{\mathrm{el}}} + \frac{1}{2\tau_I} \pm \frac{1}{2}\sqrt{\left(\frac{1}{\tau_{\mathrm{el}}} - \frac{1}{\tau_I}\right)^2 - 4\frac{R_0}{L}\frac{\mathscr{L}_I(2+\beta_I)}{\tau}}$$

$$\vec{v_\pm} = \begin{pmatrix}\frac{1-\mathscr{L}_I-\lambda_\pm\tau}{2+\beta_I}\frac{G}{I_0 R_0}\\ 1\end{pmatrix}$$

Specific solution 1 -- Impulse energy input



- Particle incident and instantaneously thermalization

$$\delta T(0) = \delta T = E/C \qquad \delta I(0) = 0$$

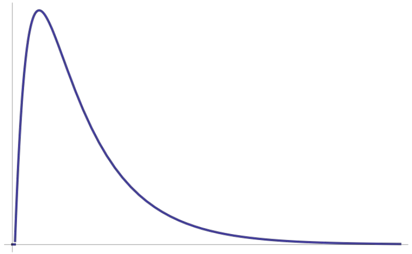
- Solve for A±, then…

$$\delta I(t) = \left(\frac{\tau_I}{\tau_+} - 1\right)\left(\frac{\tau_I}{\tau_-} - 1\right)\frac{1}{(2+\beta_I)}\frac{C\delta T}{I_0 R_0 \tau_I^2}\frac{\boxed{\left(e^{-t/\tau_+} - e^{-t/\tau_-}\right)}}{(1/\tau_+ - 1/\tau_-)}$$

$$\delta T(t) = \left(\left(\frac{1}{\tau_I} - \frac{1}{\tau_+}\right)e^{-t/\tau_-} + \left(\frac{1}{\tau_I} - \frac{1}{\tau_-}\right)e^{-t/\tau_+}\right)\frac{\delta T}{(1/\tau_+ - 1/\tau_-)}$$

- **τ+**, **τ-** : "rise time" and "fall time" of a pulse with impulse energy deposition

Specific solution 1 -- Impulse energy input



- When L (inductance) is small, **τ+<< τ-**

$$\tau_+ \to \tau_{\rm el},$$

$$\tau_- \to \tau \frac{1+\beta_I + R_L/R_0}{1+\beta_I + R_L/R_0 + (1-R_L/R_0)\mathscr{L}_I} = \tau_{\rm eff}$$

$$\mathscr{L}_I \equiv \frac{P_{J_0}\alpha_I}{GT_0}$$

- Recall from last lecture, when R_L/R_0 << 1,"Stiff voltage bias limit"
- $\beta_I \rightarrow 0$ (not modelled in last lecture)

$$\tau_{\rm eff} = \frac{\tau_0}{1+\frac{\alpha P_0}{T_0 G}} = \frac{\tau_0}{1+\frac{\alpha}{n}(1-\left(\frac{T_b}{T_0}\right)^n)}$$

Specific solution 2 -- small sinusoidal power input

$$\delta P = \mathrm{Re}(\delta P_0 e^{i\omega t})$$

- Useful to derive **responsivity**

$$\frac{d}{dt}\begin{pmatrix}\delta I \\ \delta T\end{pmatrix} = -\begin{pmatrix}\frac{1}{\tau_{\mathrm{el}}} & \frac{\mathscr{L}_I G}{I_0 L} \\ \frac{-I_0 R_0 (2+\beta_I)}{C} & \frac{1}{\tau_I}\end{pmatrix}\begin{pmatrix}\delta I \\ \delta T\end{pmatrix} + \begin{pmatrix}0 \\ \frac{\delta P_0}{C}\end{pmatrix} e^{i\omega t}$$

- After some math……

$$\frac{dI}{dP}(\omega) = s_I(\omega) = -\frac{1}{I_0 R_0}\frac{1}{(2+\beta_I)}\frac{(1-\tau_+/\tau_I)}{(1+i\omega\tau_+)}\frac{(1-\tau_-/\tau_I)}{(1+i\omega\tau_-)}$$

$$\frac{dT}{dP}(\omega) = s_T(\omega) = \frac{1}{G}\frac{\tau_+\tau_-}{\tau^2}\frac{(\tau/\tau_+ + \tau/\tau_- + \mathscr{L}_I - 1 + i\omega\tau)}{(1+i\omega\tau_+)(1+i\omega\tau_-)}.$$

- Two low-pass filters
  - For a review of one-pole filter and its response in Laplace domain, see eg. https://www.embeddedrelated.com/showarticle/590.php

Complex impedance

- Impedance of the system in a complex plane, as a function of frequency
  - Given Voltage excitation of ω angular frequency, what's current and phase?

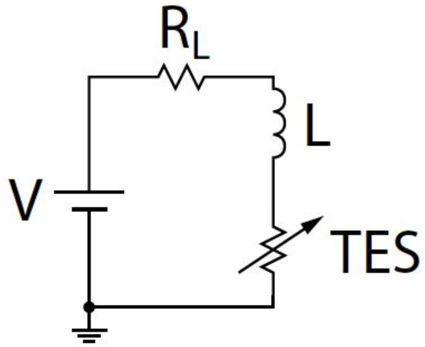
$$\frac{\mathrm{d}}{dt}\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} = -\begin{pmatrix}\frac{1}{\tau_{\mathrm{el}}} & \frac{\mathscr{L}_I G}{I_0 L}\\ -\frac{I_0 R_0\,(2+\beta_I)}{C} & \frac{1}{\tau_I}\end{pmatrix}\begin{pmatrix}\delta I\\ \delta T\end{pmatrix} + \begin{pmatrix}\frac{\delta V}{L}\\ \frac{\delta P}{C}\end{pmatrix}$$

Fourier Transform

$$\begin{pmatrix}\frac{1}{\tau_{\mathrm{el}}} + i\omega & \frac{\mathscr{L}_I G}{I_0 L}\\ \frac{-I_0 R_0(2+\beta_I)}{C} & \frac{1}{\tau_I} + i\omega\end{pmatrix}\begin{pmatrix}I_\omega\\ T_\omega\end{pmatrix} = \begin{pmatrix}\frac{V_\omega}{L}\\ 0\end{pmatrix}$$

Voltage excitation of Vω

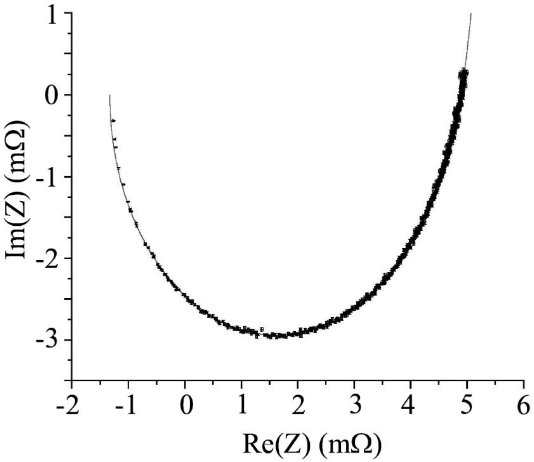
Complex impedance



$$\begin{pmatrix} \frac{1}{\tau_{\mathrm{el}}} + i\omega & \frac{\mathscr{L}_I G}{I_0 L} \\ \frac{-I_0 R_0 (2+\beta_I)}{C} & \frac{1}{\tau_I} + i\omega \end{pmatrix} \begin{pmatrix} I_\omega \\ T_\omega \end{pmatrix} = \begin{pmatrix} \frac{V_\omega}{L} \\ 0 \end{pmatrix}$$

$$Z_\omega = V_\omega / I_\omega = R_L + i\omega L + Z_{\mathrm{TES}}$$

$$Z_{\mathrm{TES}} = R_0(1+\beta_I) + \frac{R_0 \mathscr{L}_I}{1-\mathscr{L}_I}\frac{2+\beta_I}{1+i\omega\tau_I}$$

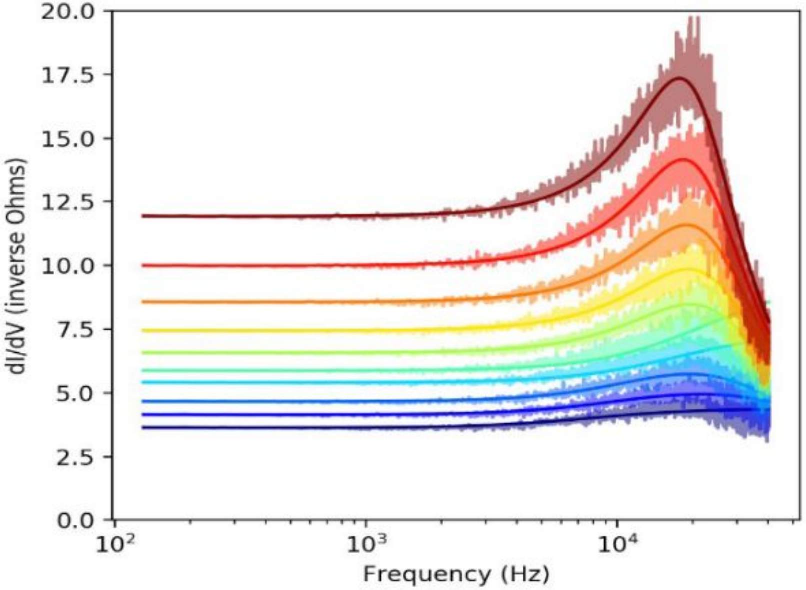


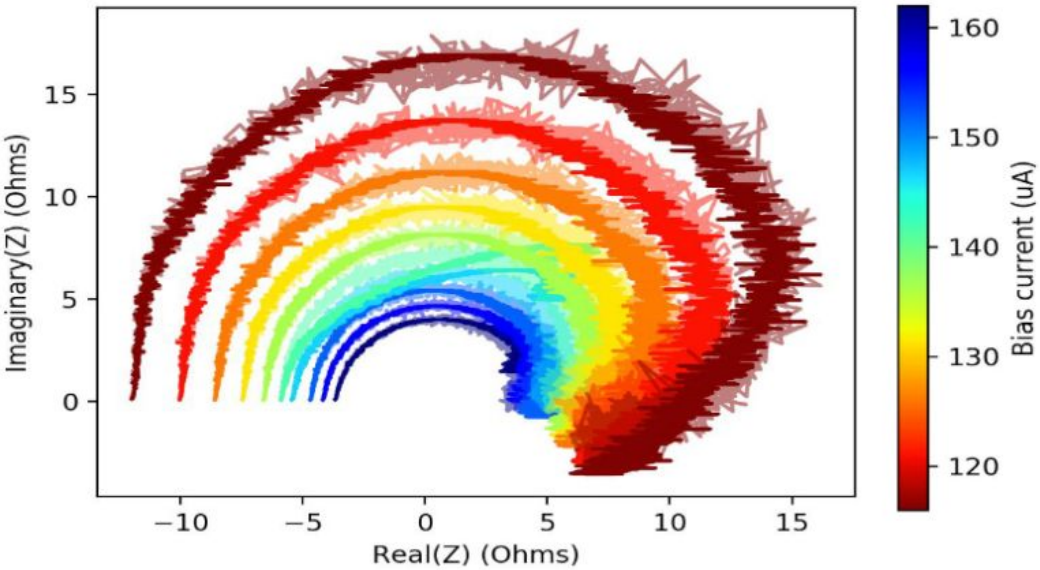
Or.. complex admittance Y=1/Z

$$Y(\omega) = s_I(\omega) I_0 \frac{\mathscr{L}_I - 1}{\mathscr{L}_I}(1 + i\omega\tau_I)$$

Complex impedance used to extract TES parameters

- Often takes data at different Tb and different R0





TES Stability

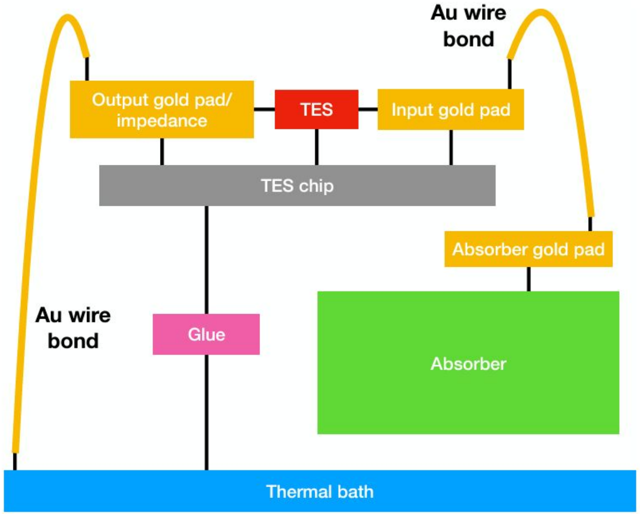
- For a TES to be stable:
  - **τ**± need to be **real (not oscillating)**
  - Re(**τ**±) need to be **positive (converging back to quiescence)**

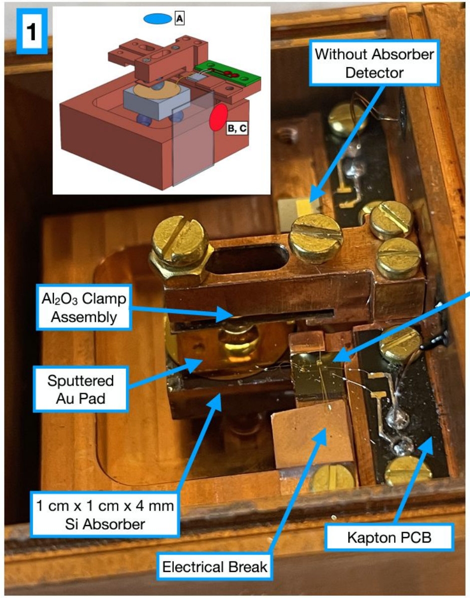
$$\frac{1}{\tau_\pm} \equiv \lambda_\pm = \frac{1}{2\tau_{\mathrm{el}}} + \frac{1}{2\tau_I} \pm \frac{1}{2}\sqrt{\left(\frac{1}{\tau_{\mathrm{el}}} - \frac{1}{\tau_I}\right)^2 - 4\frac{R_0}{L}\frac{\mathscr{L}_I(2+\beta_I)}{\tau}}$$

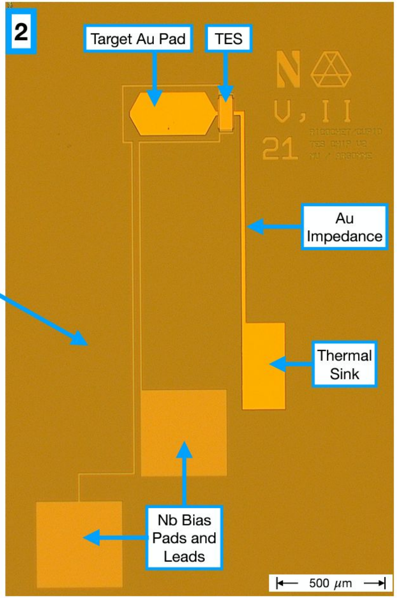
$$R_0 > \frac{(\mathscr{L}_I - 1)}{(\mathscr{L}_I + 1 + \beta_I)} R_L$$

- In large L_I limit (large amplification), $R_0 > R_L$
- If Re(**τ**±) is positive, but **τ**± isn't real → TES is underdamped (oscillate) but will converge → **Electrothermal oscillation**
- If Re(**τ**±) is negative → thermal runaway

Complex detector modeling







Doug Pinckney,
undergrad honor thesis

$$
\begin{aligned}
L\frac{dI}{dt} &= I_0 * R_0 - I * R_{tes} \\
C_a\frac{dT_a}{dt} &= \frac{E}{c\sqrt{2\pi}}e^{\frac{-(t-t_0)^2}{2c^2}} - K_{a,b}(T_a^{n_{a,b}} - T_b^{n_{a,b}}) - K_{a,auq}(T_a^{n_{a,au1}} - T_{au1}^{n_{a,au1}}) - K_{a,g}(T_a^{n_{a,g}} - T_g^{n_{a,g}}) \\
C_{au1}\frac{dT_{au1}}{dt} &= K_{a,au1}(T_{au1}^{n_{a,au1}} - T_a^{n_{a,au1}}) - K_{au1,wb1}(T_{au1}^{n_{au1,wb1}} - T_{wb1}^{n_{au1,wb1}}) \\
C_g\frac{dT_g}{dt} &= K_{a,g}(T_g^{n_{a,g}} - T_a^{n_{a,g}}) - K_{g,si}(T_g^{n_{g,si}} - T_{si}^{n_{g,si}}) \\
C_{wb1}\frac{dT_{wb1}}{dt} &= K_{au1,wb1}(T_{wb1}^{n_{au1,wb1}} - T_{au1}^{n_{au1,wb1}}) - K_{wb1,au2}(T_{wb1}^{n_{wb1,au2}} - T_{au2}^{n_{wb1,au2}}) \\
C_{au2}\frac{dT_{au2}}{dt} &= K_{si,au2}(T_{au2}^{n_{si,au2}} - T_{si}^{n_{si,au2}}) - K_{wb1,au2}(T_{au2}^{n_{wb1,au2}} - T_{wb1}^{n_{wb1,au2}}) - K_{au2,tes}(T_{au2}^{n_{au2,tes}} - T_{tes}^{n_{au2,tes}}) \\
C_{si}\frac{dT_{si}}{dt} &= K_{g,si}(T_{si}^{n_{g,si}} - T_g^{n_{g,si}}) - K_{si,au2}(T_{si}^{n_{si,au2}} - T_{au2}^{n_{si,au2}}) - K_{si,tes}(T_{si}^{n_{si,tes}} - T_{tes}^{n_{si,tes}}) \\
&\quad - K_{si,m}(T_{si}^{n_{si,m}} - T_m^{n_{si,m}}) \\
C_{tes}\frac{dT_{tes}}{dt} &= I^2R_{tes} - K_{au2,tes}(T_{tes}^{n_{au2,tes}} - T_{au2}^{n_{au2,tes}}) - K_{si,tes}(T_{tes}^{n_{si,tes}} - T_{si}^{n_{si,tes}}) - K_{tes,m}(T_{tes}^{n_{tes,m}} - T_m^{n_{tes,m}}) \\
C_m\frac{dT_m}{dt} &= K_{tes,m}(T_m^{n_{tes,m}} - T_{tes}^{n_{tes,m}}) - K_{si,m}(T_m^{n_{si,m}} - T_{si}^{n_{si,m}}) - K_{m,wb2}(T_m^{n_{m,wb2}} - T_{wb2}^{n_{m,wb2}}) \\
C_{wb2}\frac{dT_{wb2}}{dt} &= K_{m,wb2}(T_{wb2}^{n_{m,wb2}} - T_m^{n_{m,wb2}}) - K_{wb2,b}(T_{wb2}^{n_{wb2,b}} - T_b^{n_{wb2,b}})
\end{aligned}
$$

Differential equation → Matrices (time, frequency)

$$
\frac{d}{dt}\begin{bmatrix} \Delta I \\ \Delta T_{tes} \\ \Delta T_a \\ \Delta T_{au1} \\ \Delta T_g \\ \Delta T_{wb1} \\ \Delta T_{au2} \\ \Delta T_{si} \\ \Delta T_m \\ \Delta T_{wb2} \end{bmatrix}
=
\begin{bmatrix}
\frac{-R(1+\beta_I)}{L} & \frac{-\alpha V}{T_{tes}L} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(2+\beta_I)V}{C_{tes}} & \frac{-(G^{tes}_{au2,tes}+G^{tes}_{tes,m}+G^{tes}_{si,tes}-\frac{\alpha P}{T_{tes}})}{C_{tes}} & 0 & 0 & 0 & 0 & \frac{G^{au2}_{au2,tes}}{C_{tes}} & \frac{G^{si}_{si,tes}}{C_{tes}} & \frac{G^{m}_{tes,m}}{C_{tes}} & 0 \\
0 & 0 & \frac{-(G^a_{a,b}+G^a_{a,au1}+G^a_{a,g})}{C_a} & \frac{G^{au1}_{a,au1}}{C_a} & \frac{G^g_{a,g}}{C_a} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{G^a_{a,au1}}{C_{au1}} & \frac{-(G^{au1}_{a,au1}+G^{au1}_{au1,wb1})}{C_{au1}} & 0 & \frac{G^{wb1}_{au1,wb1}}{C_{au1}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{G^a_{a,g}}{C_g} & 0 & \frac{-(G^g_{a,g}+G^g_{g,si})}{C_g} & 0 & 0 & \frac{G^{si}_{g,si}}{C_g} & 0 & 0 \\
0 & 0 & 0 & \frac{G^{au1}_{au1,wb1}}{C_{wb1}} & 0 & \frac{-(G^{wb1}_{au1,wb1}+G^{wb1}_{wb1,au2})}{C_{wb1}} & \frac{G^{Au2}_{wb1,au2}}{C_{wb1}} & 0 & 0 & 0 \\
0 & \frac{G^{tes}_{au2,tes}}{C_{au2}} & 0 & 0 & 0 & \frac{G^{wb1}_{wb1,au2}}{C_{au2}} & \frac{-(G^{au2}_{au2,tes}+G^{au2}_{wb1,au2}+G^{au2}_{au2,si})}{C_{au2}} & \frac{G^{si}_{au2,si}}{C_{au2}} & 0 & 0 \\
0 & \frac{G^{tes}_{si,tes}}{C_{si}} & 0 & 0 & \frac{G^{si}_{g,si}}{C_{si}} & 0 & \frac{G^{au2}_{si,au2}}{C_{si}} & \frac{-(G^{si}_{g,si}+G^{si}_{si,au2}+G^{si}_{si,tes}+G^{si}_{si,m})}{C_{si}} & \frac{G^m_{si,m}}{C_{si}} & 0 \\
0 & \frac{G^{tes}_{tes,m}}{C_m} & 0 & 0 & 0 & 0 & 0 & \frac{G^{si}_{si,m}}{C_m} & \frac{-(G^m_{tes,m}+G^m_{si,m}+G^m_{m,wb2})}{C_m} & \frac{G^{wb2}_{m,wb2}}{C_m} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{G^m_{m,wb2}}{C_{wb2}} & \frac{-(G^{wb2}_{m,wb2}+G^{wb2}_{wb2,b})}{C_{wb2}}
\end{bmatrix}
\begin{bmatrix} \Delta I \\ \Delta T_{tes} \\ \Delta T_a + w(t) \\ \Delta T_{au1} \\ \Delta T_g \\ \Delta T_{wb1} \\ \Delta T_{au2} \\ \Delta T_{si} \\ \Delta T_m \\ \Delta T_{wb2} \end{bmatrix}
$$

$$
\begin{bmatrix}
R(1+\beta_I)+i\omega L & -1 & \frac{\alpha V}{T_c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
R_l & 1+i\omega R_l C_{cap} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(2+\beta_I)V & 0 & (G^{tes}_{au2,tes}+G^{tes}_{tes,m}+G^{tes}_{si,tes}-\frac{\alpha P}{T_{tes}})+i\omega C_{tes} & 0 & 0 & 0 & 0 & -G^{au2}_{au2,tes} & -G^{si}_{si,tes} & -G^m_{tes,m} & 0 \\
0 & 0 & 0 & (G^a_{a,b}+G^a_{a,au1}+G^a_{a,g})+i\omega C_a & -G^{au1}_{a,au1} & -G^g_{a,g} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -G^a_{a,au1} & (G^{au1}_{a,au1}+G^{au1}_{au1,wb1})+i\omega C_{au1} & 0 & -G^{wb1}_{au1,wb1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -G^a_{a,g} & 0 & (G^g_{a,g}+G^g_{g,si})+i\omega C_g & 0 & 0 & -G^{si}_{g,si} & 0 & 0 \\
0 & 0 & 0 & 0 & -G^{au1}_{au1,wb1} & 0 & (G^{wb1}_{au1,wb1}+G^{wb1}_{wb1,au2})+i\omega C_{wb1} & -G^{Au2}_{wb1,au2} & 0 & 0 & 0 \\
0 & 0 & -G^{tes}_{au2,tes} & 0 & 0 & 0 & -G^{wb1}_{wb1,au2} & (G^{au2}_{au2,tes}+G^{au2}_{wb1,au2}+G^{au2}_{au2,si})+i\omega C_{au2} & -G^{si}_{au2,si} & 0 & 0 \\
0 & 0 & -G^{tes}_{si,tes} & 0 & 0 & -G^{si}_{g,si} & 0 & -G^{au2}_{si,au2} & (G^{si}_{g,si}+G^{si}_{si,au2}+G^{si}_{si,tes}+G^{si}_{si,m})+i\omega C_{si} & -G^m_{si,m} & 0 \\
0 & 0 & -G^{tes}_{tes,m} & 0 & 0 & 0 & 0 & 0 & -G^{si}_{si,m} & (G^m_{tes,m}+G^m_{si,m}+G^m_{m,wb2})+i\omega C_m & -G^{wb2}_{m,wb2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -G^m_{m,wb2} & (G^{wb2}_{m,wb2}+G^{wb2}_{wb2,b})+i\omega C_{wb2}
\end{bmatrix}
\begin{bmatrix} \Delta I \\ \Delta V \\ \Delta T_{tes} \\ \Delta T_a \\ \Delta T_{au1} \\ \Delta T_g \\ \Delta T_{wb1} \\ \Delta T_{au2} \\ \Delta T_{si} \\ \Delta T_m \\ \Delta T_{wb2} \end{bmatrix}
=
\begin{bmatrix} e_{int} \\ e_{ext} \\ p_{au2,tes}+p_{si,tes}-p_{tes,m}-Ie_{int} \\ -p_{a,g}-p_{a,au1}-p_{a,b} \\ p_{a,au1}-p_{au1,wb1} \\ p_{a,g}-p_{g,si} \\ p_{au1,wb1}-p_{wb1,au2} \\ p_{wb1,au2}+p_{si,au2}-p_{au2,tes} \\ p_{g,si}-p_{si,tes}-p_{si,au2}-p_{si,m} \\ p_{tes,m}+p_{si,m}-p_{m,wb2} \\ p_{m,wb2}-p_{wb2,b} \end{bmatrix}
$$

One note about noise and resolution

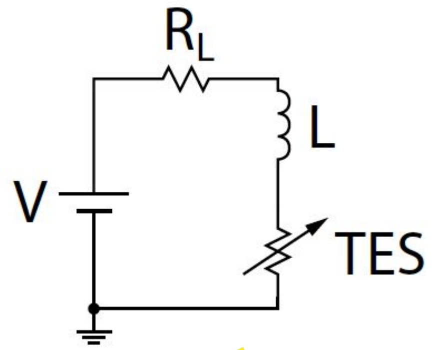
- With responsivity s_I, can derive theoretical resolution:
  - Model various noise
    - Johnson noise, thermal fluctuation noise, etc.
  - Convert to "Noise Equivalent Power"
  - $$\Delta E_{\mathrm{rms}} = \left(\sqrt{\int_0^\infty \frac{4df}{\mathrm{NEP}(f)^2}}\right)^{-1}$$
- Irwin's derivation shows

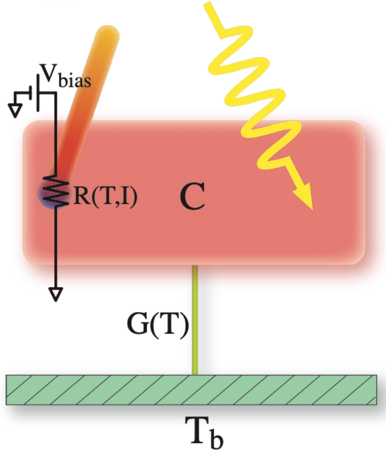
$$\Delta E_{\mathrm{rms}} = \sqrt{4k_{\mathrm{B}}T^2C\frac{1}{\alpha}\sqrt{\frac{n}{2}}}$$

- Will come back to this next class

Noise Modeling

- Characterize noises in an idealized TES system in four categories:
    - Internal Johnson Noise
    - External Johnson Noise
    - Thermal Fluctuation Noise
    - “Downstream” Amplifier noise
- General steps:
    - Calculate noise sources from each component
    - Derive “Noise Equivalent Power” (NEP) with responsivity of each component
    - Integrate NEP to get predicted resolution
        - This assumes “optimal filter” is used
        - Weighs each frequency component with signal-to-noise





$$\Delta E_{\mathrm{rms}} = \left(\sqrt{\int_0^\infty \frac{4df}{\mathrm{NEP}(f)^2}}\right)^{-1}$$

https://en.wikipedia.org/wiki/Johnson%E2%80%93Nyquist_noise

Johnson noise

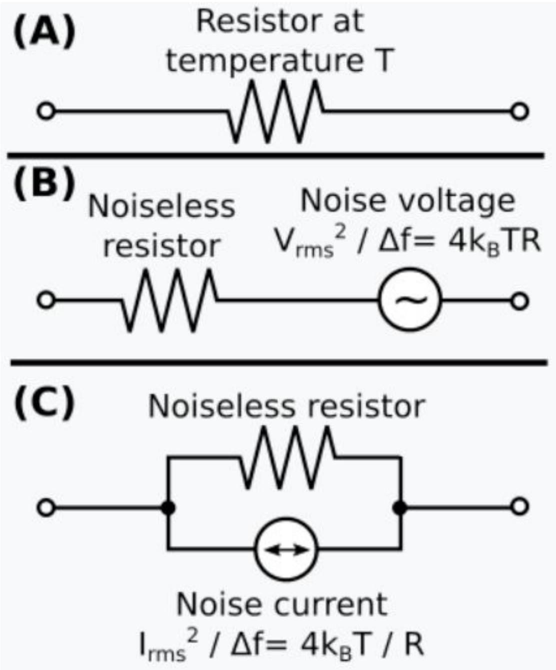
- Generated by thermal motions of charge carriers
- $$\sqrt{\overline{v_n^2}} = \sqrt{4k_B T R \Delta f}$$

$$\sqrt{\overline{i_n^2}} = \sqrt{\frac{4k_B T \Delta f}{R}}$$

- For a circuit with components at different temperatures

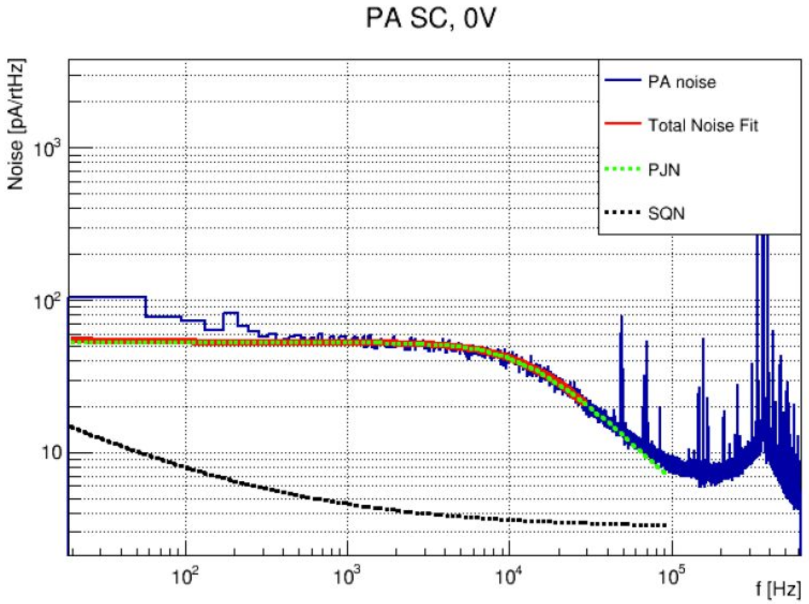
$$S_V = \bar{V}^2_{j,passive}(f) = \sum_{passive} S_V = 4k_b \sum_{passive} T_i R_i$$

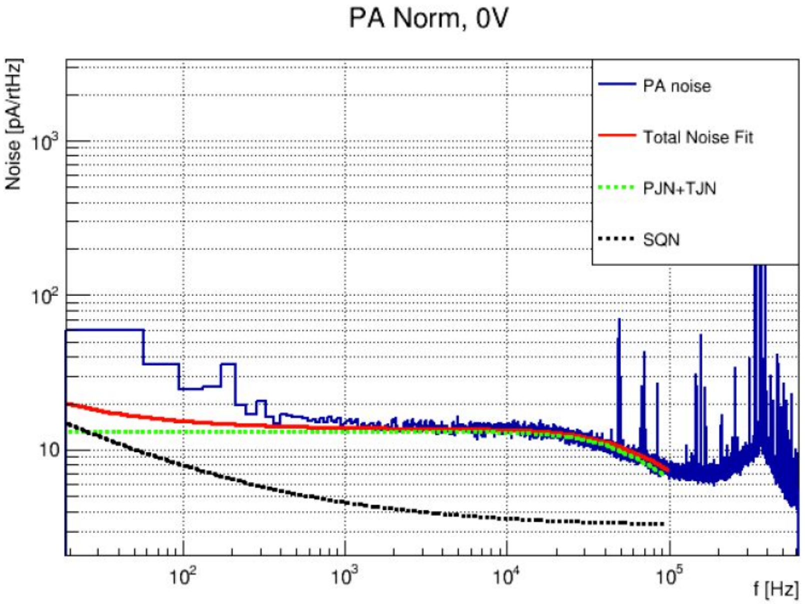
$$S_I = \bar{I}^2_{j,passive}(f) = \frac{4k_b}{|Z_{loop}|^2} \sum_{passive} T_i R_i$$



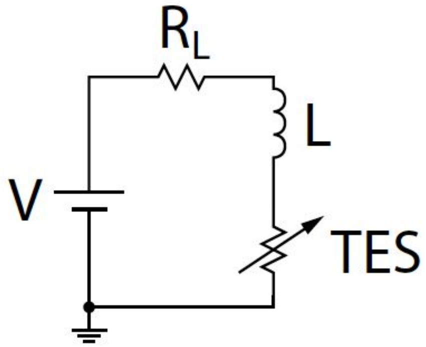
Johnson noise modeling

- Usually quantify noise in $\sqrt{\overline{i_n^2}}/\Delta f$
- Can direct model noise when TES is normal/superconducting
  - I.e. No electro-thermal feedback





Passive vs "active" Johnson Noise



- Johnson noise in non-TES part of the circuit wiggles current in the circuit
- Johnson noise in TES does that, plus heating up TES
  - Lead to changing of resistance of TES
- $$V_B + \delta V = I_0 R_{sh} + L\frac{dI}{dt} + I_0 R_0$$

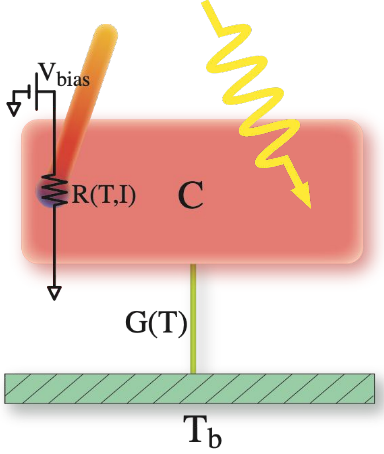
$$\begin{pmatrix} \delta I \\ \delta T \end{pmatrix} = M^{-1} \begin{pmatrix} \frac{\delta V}{L} \\ \frac{\delta P}{C} \end{pmatrix}$$

$$I_0 V_B = I_0^2 R_{sh} + \frac{d}{dt}\frac{LI^2}{2} + (I_0^2 R_0 - I_0 \delta V) \longrightarrow \delta P = -I_0 \delta V$$

$$\frac{\delta I}{\delta V} = \frac{M_{1,1}^{-1}}{L} - I_0 \frac{M_{1,2}^{-1}}{C} = \frac{1}{Z} + \frac{\mathscr{L}}{\tau L}$$

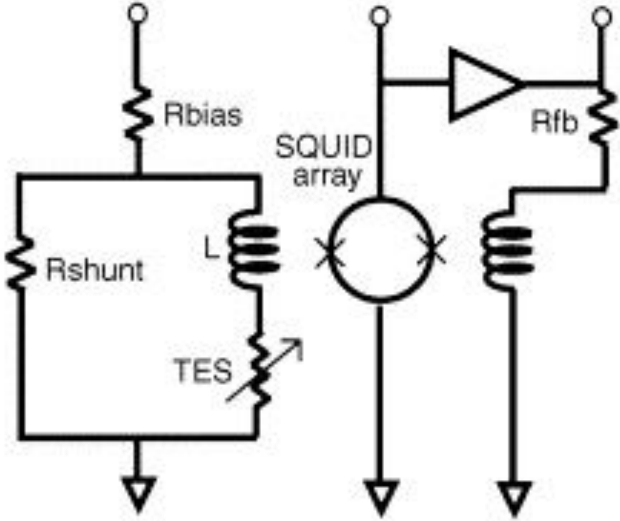
$$S_{TJN} = \left(\frac{\delta I}{\delta V}\right)^2 S_{V,TJN} = \frac{4k_B R_0 T_0}{(R_L + R_0(\beta+1))^2}\left(\frac{1 + i\omega\tau}{(1-\mathscr{L})(1+i\omega\tau_{el})(1+i\omega\tau_I) + \mathscr{L}(\beta+2)\frac{R_0\tau_{el}}{L}}\right)^2$$

Thermal Fluctuation Noise



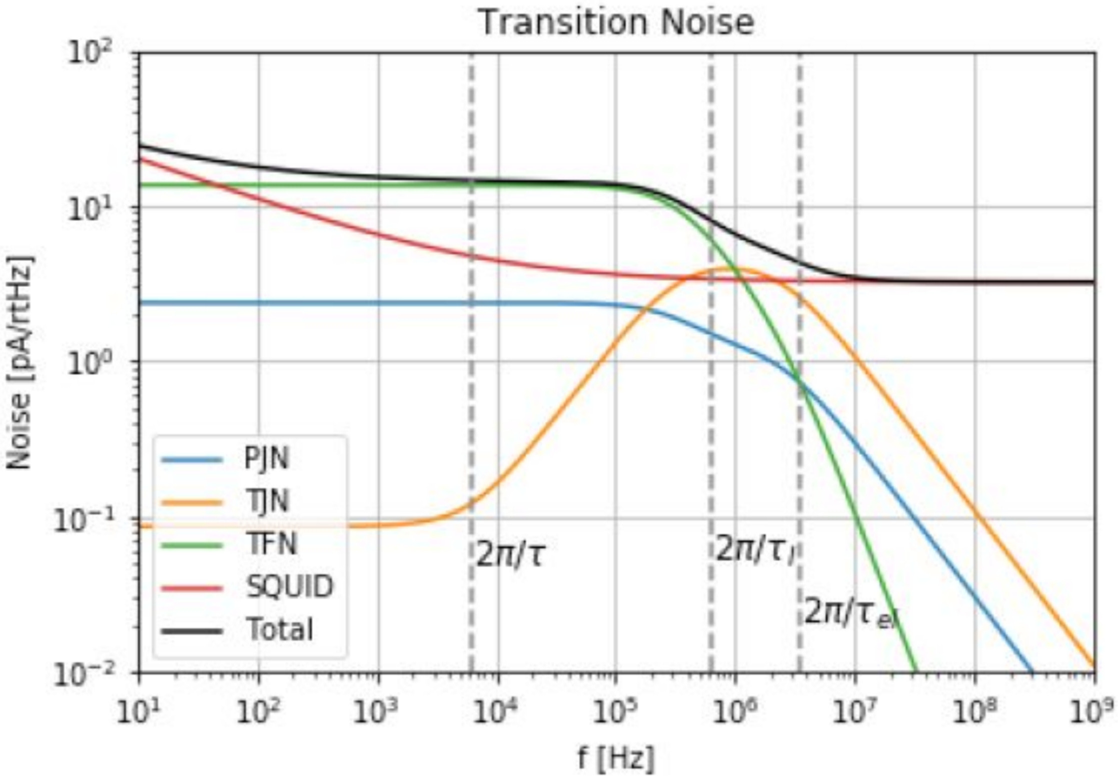
- TES, with heat capacity of C, is in thermal contact with Tb
- Heat conducting through G can fluctuate
  - → Thermal Fluctuation Noise
- $$S_{P_{\mathrm{TFN}}} = 4k_{\mathrm{B}}T_0^2G \times F(T_0, T_{\mathrm{bath}})\ ,$$
- $F(T_0, T_{bath})$ is to correct for the fact that TES and bath are not in thermal equilibrium for short time scale
  - In ideal case (Tb → 0, G → $nI_0^2R_0/T_0$), F → ½
- $$S_{I_{\mathrm{TFN}}}(\omega) = 4k_{\mathrm{B}}T_0^2G \times F(T_0, T_{\mathrm{bath}})|s_I(\omega)|^2$$

“Downstream” electronics noise



- TES usually read out with SQUID and a bunch of amplifiers.
- Can quantify the electronics noise in the input-coil referenced current noise
  - Convert all noises to the amount of current fluctuation through TES
- $$S_{P_{\mathrm{amp}}}(\omega) = \frac{S_{I_{\mathrm{amp}}}(\omega)}{|s_I(\omega)|^2}$$
- The total noise is the sum of all components

Noise modeling -- TES in transition



NEP and resolution

- Noise Equivalent Power $NEP(\omega) = \sqrt{S_P(\omega)}$.

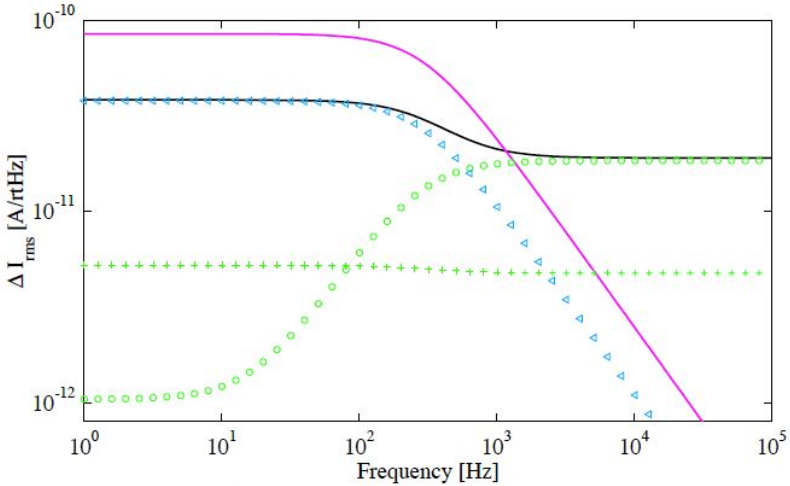
$$\Delta E_{\text{rms}} = \left(\sqrt{\int_0^\infty \frac{4df}{\text{NEP}(f)^2}} \right)^{-1}$$

- Derived with "optimal filter"
- Idealized TES, strong electro-thermal feedback, zero load resistance, with delta energy input

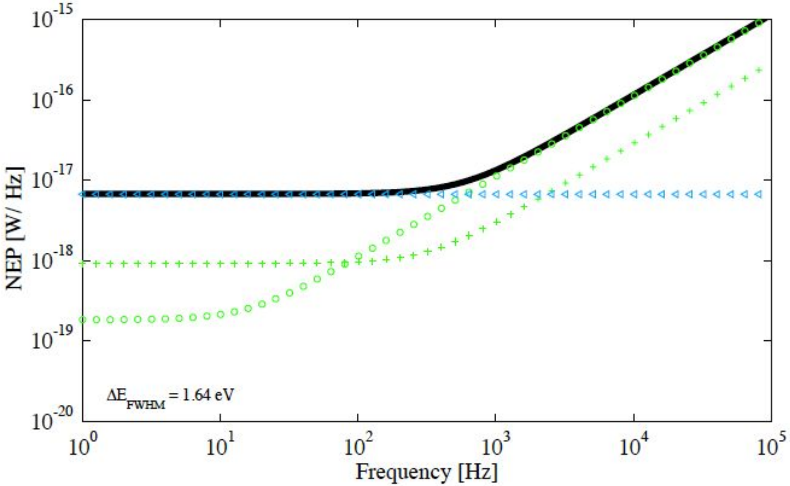
$$\Delta E_{\text{rms}} = \sqrt{4k_{\text{B}}T^2C\frac{1}{\alpha}\sqrt{\frac{n}{2}}}$$

- For macroscopic crystal detector, need to include phonon propagation time

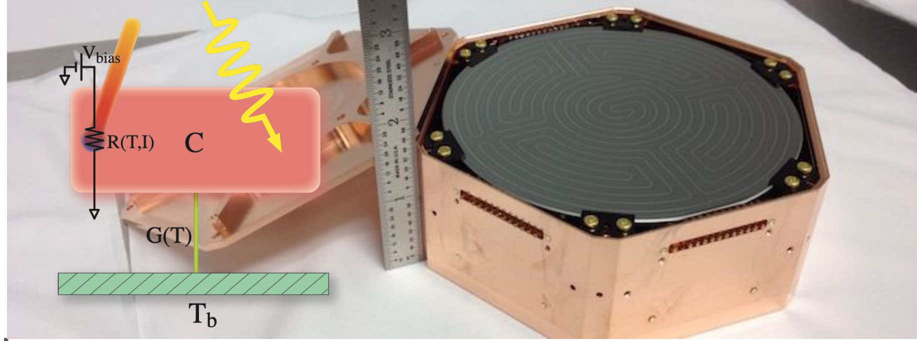
$$\Delta E_{rms} = \sqrt{2k_bT_c^2G\left(\tau_{pulse} + \frac{2\tau_{TES}}{n}\right)}$$

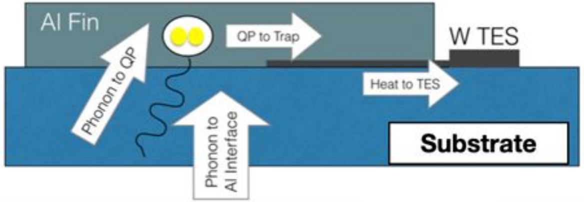


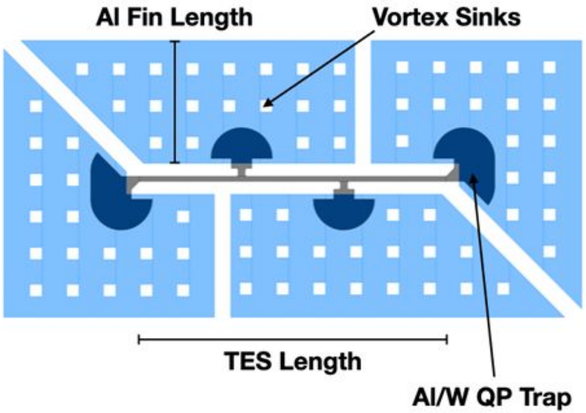
(a) Current noise in a simple calorimeter.



QET technology



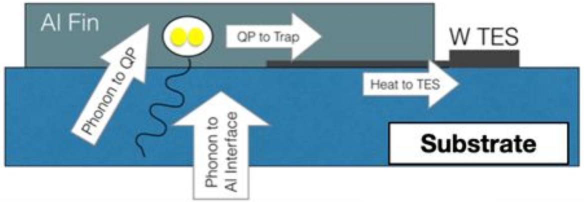


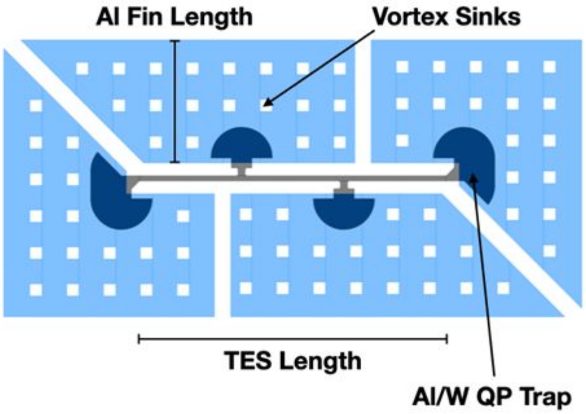


- Quasiparticle-trap-assisted Electrothermal-feedback Transition-edge sensors
- $$\Delta E_{\mathrm{rms}} = \sqrt{4k_B T^2 C \frac{1}{\alpha}\sqrt{\frac{n}{2}}}$$
- Minimizing C is important…
- SuperCDMS way -- Athermal phonon detector
  - Not include crystal heat capacity
  - Cool crystal strongly down to bath temperature
  - Use TES electron-phonon coupling as weak G
- Min TES volume while max phonon collection:
  - Aluminum fins
    - Superconductor, no heat capacity accounted

QET modelling

- Cost:
  - Al needs minimum 2Δ energy to excite
    $\Delta(T=0) = 1.764\, k_B T_c$
    - ~334 ueV, from BCS theory
    - Need to pick up phonons fast before they down-convert too much
  - Quasiparticle transportation has finite probability of funneling energy to TES
- Overall, captured as an energy efficiency





$$\Delta E_{rms} = \frac{1}{\epsilon}\sqrt{2k_b T_c^2 G\left(\tau_{pulse} + \frac{2\tau_{TES}}{n}\right)}$$

https://arxiv.org/pdf/2012.12430.pdf

Some optimizations can be performed