OVERVIEW OF ACCELERATORS

- START WITH BEAM FOCUSSING $\mathcal{L}$
- DEFINE $\mathcal{L}$ AND ITS IMPORTANCE / LIMITATIONS
- DISCUSS ACCELERATION TECHNIQUES E
- APPLY THESE CONCEPTS TO CURRENT ACCELERATOR COMPLEXS AND POSSIBLE FUTURE ACCELERATORS

A USEFUL REFERENCE:

AN INTRODUCTION TO THE PHYSICS OF HIGH ENERGY ACCELERATORS

Edwards and Syphers.

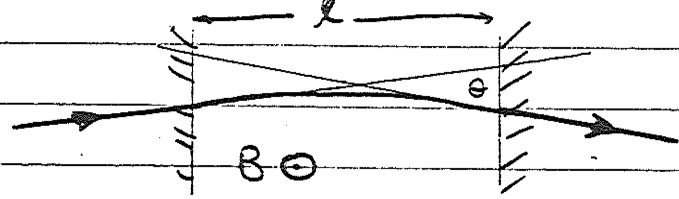
In the library
~~(after next week!)~~

BASIC ACCELERATOR FORMALISM

START WITH CIRCULAR ACCELERATOR STRUCTURES

- GUIDE FIELD : DIPOLES
- FOCUSSING FIELD : QUADRUPOLES
- ACCELERATING FIELD : RF CAVITIES

DIPOLE



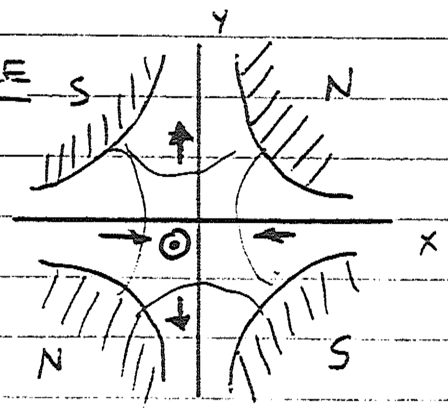
$$\theta = l/\rho$$

$$\rho = \frac{P}{0.3\,B}$$

P in GeV/
B in Tesl
ρ in m

MAGNETIC RIGIDITY $\equiv (B\rho)$

QUADRUPOLE



⊙ Beam

$\Phi = Kxy$ ideal pole tip

$B_x = Ky$ (focussing)

$B_y = -Kx$ (defocussing)

$K \simeq 12\,T/m$ (iron magnets @ CESR 10 GeV e^-)

$\simeq 75\,T/m$ (superconducting @ Tevatron 1 TeV p

223 T/m (LHC main quads)

(1)

WEAK FOCUSSING

IF :
$$B_y = \frac{B_o}{r^n} \qquad 0 \geq n \geq 1$$

THEN A SINGLE MAGNETIC STRUCTURE COULD PROVIDE BOTH FOCUSSING AND BENDING

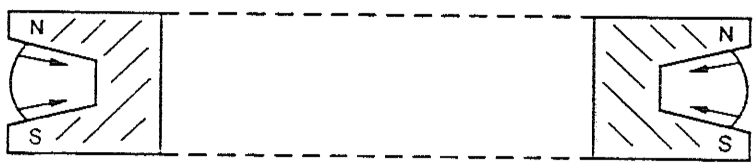
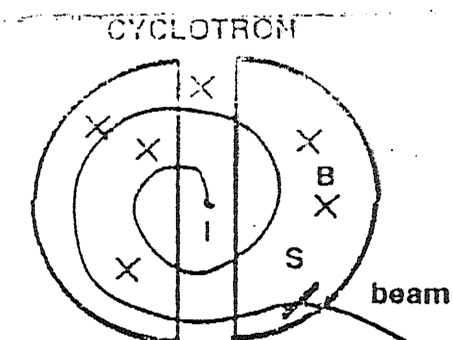


Figure 3.3. Cross section of weak focusing circular accelerator.

THIS WAS THE BASIS OF EARLY CYCLOTRONS



r.f. applied to gap between "dee"s

I = ion source

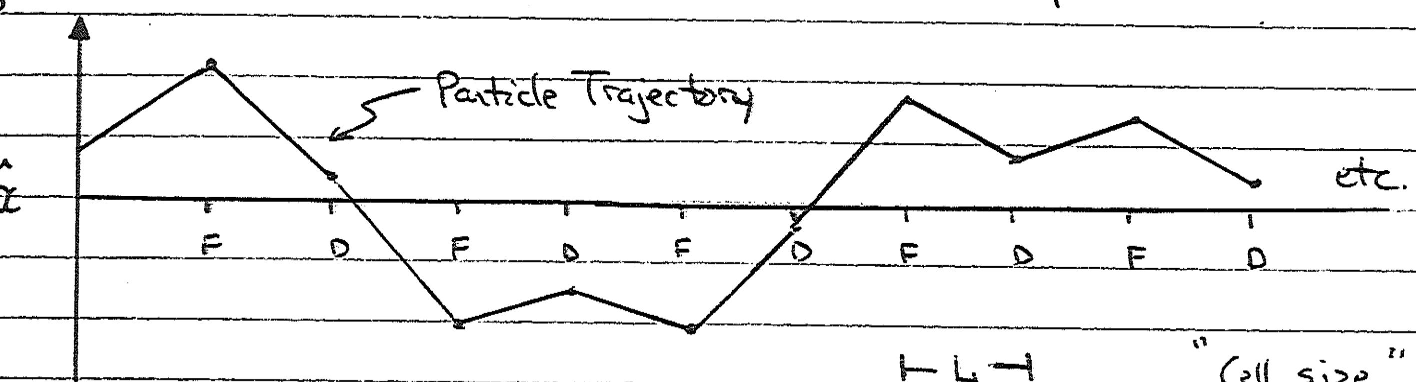
S = septum (electrostatic wires,or magnet)

STRONG FOCUSSING

Recall from optics that net focussing can be achieved if we alternate a series of Focussing and Defocussing elements such that:

$$\langle |x,y| \rangle_D < \langle |x,y| \rangle_F$$

FODO series



Transverse kick in a quadrupole magnet is:

$$\Delta p_x = e K l x \qquad \text{(recall field strength is } Kx\text{)}$$

$$\theta_x \equiv x' = \frac{\Delta p_x}{p} \simeq \frac{K l x}{(B\rho)}$$

From optics $\quad x' \simeq \dfrac{x}{F} \qquad$ **F ≡ FOCAL LENGTH**

$$\Rightarrow \quad F \sim \left(\frac{K l}{(B\rho)}\right)^{-1} \qquad F \propto p$$

Practical Considerations

$F \simeq L$ and $l \sim \dfrac{L}{10}$ DICTATES K & p

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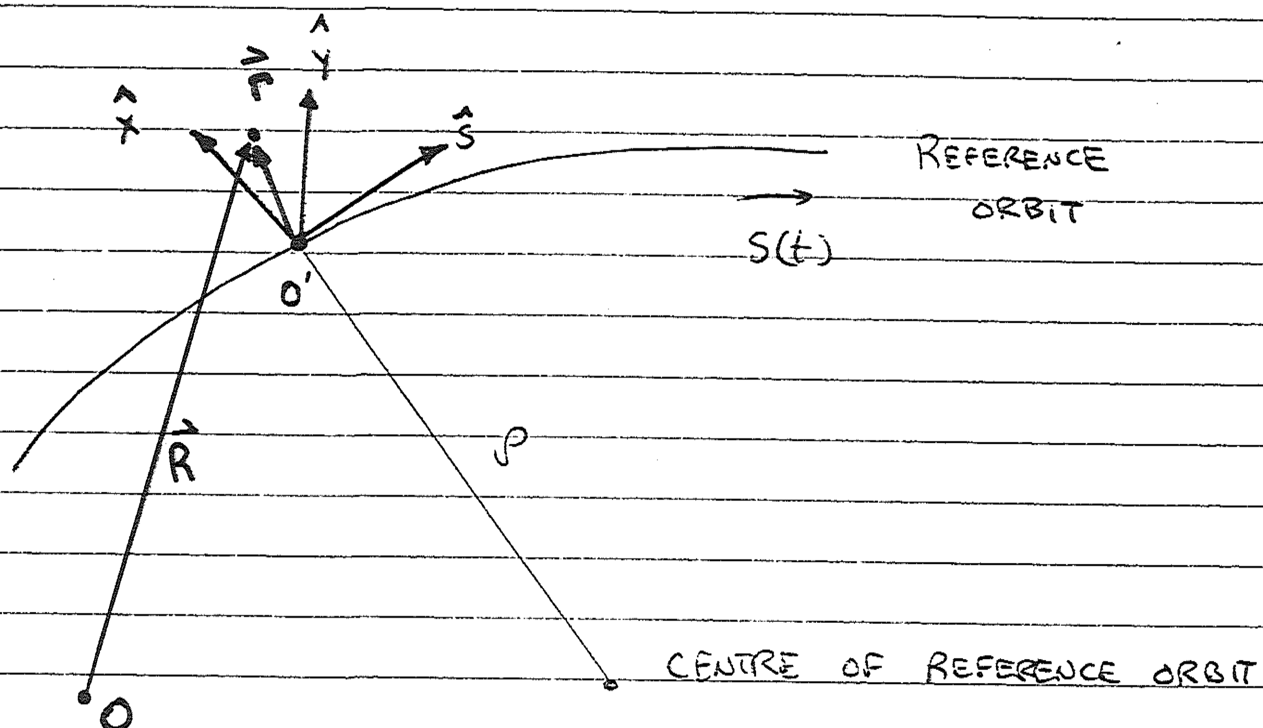
COORDINATE SYSTEM OF CHOICE

IF ACCELERATOR IS TO "WORK" PARTICLES MUST EXECUTE STABLE OSCILLATIONS AROUND ORBIT.

ESTABLISH CONCEPT OF A REFERENCE ORBIT



WANT TO FIND THE EQUATIONS OF MOTION DESCRIBING DEVIATIONS FROM A REFERENCE ORBIT. COORDINATE SYSTEM MOVES AROUND THE RING WITH PARTICLE SO THAT IT IS ALWAYS IN xy PLANE.

$$u \equiv \frac{d}{dt} s(t) \qquad \vec{u} = u\,\hat{s} \qquad \text{VELOCITY OF } O'$$

IN THE MOVING SYSTEM THE POSITION OF A PARTICLE IS

$$\vec{r} = x\hat{x} + y\hat{y}$$

SOME USEFUL RELATIONS :

$$\vec{v}_{LAB} = \frac{d\vec{R}}{dt} = u\hat{s} + \frac{d\vec{r}}{dt} + (\vec{\omega} \times \vec{r})$$ ROTATING COORDINATE SYSTE

$$\vec{\omega} = -\frac{u}{\rho}\hat{y}$$ ($\because$ $\hat{x}$ // centre of rotatio

$$\frac{d\vec{r}}{dt} = \dot{x}\hat{x} + \dot{y}\hat{y}$$ (in rotating frame)

$$\Rightarrow \quad \vec{v}_{LAB} = \dot{x}\hat{x} + \dot{y}\hat{y} + u\left(1 + \frac{x}{\rho}\right)\hat{s}$$

ALSO NOTE :

$$\frac{d\hat{x}}{dt} = \frac{u}{\rho}\hat{s} \qquad \frac{d\hat{s}}{dt} = -\frac{u}{\rho}\hat{x} \qquad \frac{d\hat{y}}{dt} = 0$$

E RCISE SHOW :

(4) $$\vec{a}_{LAB} = \left(\ddot{x} - \frac{u^2}{\rho}x\right)\hat{x} + \ddot{y}\hat{y} + \frac{2u\dot{x}}{\rho}\hat{s}$$

EQUATION OF MOTION OF CHARGED PARTICLE IN $\vec{B}$

$$\vec{F} = \frac{d}{dt}\vec{P} = e\left(\vec{v}\times\vec{B}\right) \qquad (1)$$

USING: $\frac{d}{dt} = v\frac{d}{dR}$ $\quad \vec{v} = v\frac{d\vec{R}}{dR}$ $\quad \vec{p} = p\frac{d\vec{R}}{dR}$

$\frac{d\vec{R}}{dR}$ = unit vec tan to trajec

substitute in (1)

$$\cancel{v}\frac{d}{dR}\left(p\frac{d\vec{R}}{dR}\right) = e\cancel{v}\left(\frac{d\vec{R}}{dR}\times\vec{B}\right)$$

$$\frac{d^2\vec{R}}{dR^2} + \underbrace{\frac{dP}{dR}\frac{d\vec{R}}{dR}}_{\text{vanishes why?}} = e\left(\frac{d\vec{R}}{dR}\times\vec{B}\right)$$

TO "SIMPLIFY" THIS NOTE:

$$\frac{d\vec{R}}{dR} = \frac{\dot{\vec{R}}}{\dot{R}}$$

$$\text{\&} \quad \frac{d^2\vec{R}}{dR^2} = \frac{1}{\dot{R}} \frac{d}{dt}\left(\frac{\dot{\vec{R}}}{\dot{R}}\right)$$

$$\Rightarrow \quad \dot{R}^2 \frac{d^2\vec{R}}{dR^2} = \ddot{\vec{R}} - \frac{1}{2} \frac{\dot{\vec{R}}}{(\dot{R})^2} \underbrace{\frac{d}{dt}(\dot{R}^2)}_{\to\ 0}$$

why?

THUS EQUATION OF MOTION BECOMES:

$$\ddot{\vec{R}} = \frac{e}{p} \dot{R} \left(\dot{\vec{R}} \times \vec{B} \right) \qquad (3)$$

Trivial for non-relativistic particl

Now write in terms of spatial derivatives (w.r.t. s)

viz $\dot{y} = \frac{dy}{ds}\frac{ds}{dt} = y' u$

$$\ddot{y} = y'' u^2 + \dot{u} y'$$

Then write (3) in terms of two components

$\hat{x}$ & $\hat{y}$ ($\hat{s}$ less interesting)

$$\text{I} \quad y'' + \frac{\dot{u}}{u^2} y' = \frac{e}{p} \frac{v}{u} \left(\left(1 + \frac{x}{\rho}\right) B_x - x' B_s \right)$$

$$\text{II} \quad x'' + \frac{\dot{u}}{u^2} x' - \frac{1}{\rho} \left(1 + \frac{x}{\rho}\right) = \frac{e}{p} \frac{v}{u} \left(y' B_s - \left(1 + \frac{x}{\rho}\right) B_y \right)$$

<u>NOW APPLY THIS TO REFERENCE TRAJECTORY:</u>

$x = y = 0$ also $x'' = x' = y'' = y' = 0$; $v = u$

Only term which survives:

$$\frac{1}{\rho} = \frac{e}{p} B_y \qquad \text{MOTION in a DIPOLE}$$

For more complicated situations we will need:

$$\frac{v}{u} \equiv \frac{|\vec{v}_{lab}|}{u} = \left[x'^2 + y'^2 + \left(1 + \frac{x}{\rho}\right)^2 \right]^{1/2} \simeq \left(1 + \frac{x}{\rho}\right)$$

(Check for typical x', y' late

Also $\frac{\dot{u}}{u} \ll 1 \Rightarrow \dot{u} x'$ is second order

$\Rightarrow$ drop.

NOW USE SYMMETRY OF $\vec{B}$ (DIPOLE OR QUADRUPOLE) TO SIMPLY THIS TO "REALISTIC" SITUATION.

$$\vec{B} = B_x \hat{x} + B_y \hat{y}$$

FOR SMALL DISPLACEMENTS TAYLOR EXPAND

$$B_x = B_x^{\circ} + y \frac{\partial B_x}{\partial y} + x \frac{\partial B_x}{\partial x}$$

But $B_x(-x) = B_x(x) \Rightarrow \frac{\partial B_x}{\partial x} = 0$

$$B_x(y) = -B_x(-y) \Rightarrow B_x^{\circ} = 0$$

SIMILARLY $B_x = y \frac{dB_x}{dy}$

$$B_y = B_y^{\circ} + x \left.\frac{\partial B_y}{\partial x}\right|_{\circ}$$

Finally $\vec{\nabla} \times \vec{B} = 0$ (Maxwell's Equation!)

So we write

$$B_x = y \frac{\partial B_x}{\partial y} = y \left.\frac{\partial B_y}{\partial x}\right|_{\circ}$$

IF WE DEFINE: $K \equiv \frac{e}{p} \left. \frac{\partial B_y}{\partial x} \right|_0$

Then $\frac{e}{p} B_y = \frac{1}{\rho} + Kx$

AND (I) ; (II) REDUCE TO:

(I') $y'' - Ky = 0$

(II') $x'' + \left(K + \frac{1}{\rho^2}\right) x = 0$

A PRETTY FAMILIAR SET OF LINEAR DIFFERENTIAL EQN'

TWO SOLUTIONS: PERIODIC IN s.

CONSIDER 3 "USEFUL" SITUATIONS

(α) DIPOLE: $K = 0$ $\frac{1}{\rho^2}$ small

FREE/DRIFT SPACE: $K = 0$ $\frac{1}{\rho^2} = 0$

(β) QUADRUPOLE: $K \neq 0$ but constant $\frac{1}{\rho^2} \to 0$

(α) $x(s) = x_0 + s\,x_0'$ $x_0 \equiv x(s=0)$

$x'(s) = x_0'$ $x_0' \equiv x'(s=0)$

Can write this as a matrix

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

<u>TRANSFER MATRIX</u>

(β)

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} \cos(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sin(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) & \cos(\sqrt{K}s) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

FOCUSSIN

$\Rightarrow \left(K + \frac{1}{\rho^2} \text{ as necessary}\right)$ $K > 0$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = \begin{pmatrix} \cosh(\sqrt{K}s) & \frac{1}{\sqrt{K}}\sinh(\sqrt{K}s) \\ \sqrt{K}\sinh(\sqrt{K}s) & \cosh(\sqrt{K}s) \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

DEFOCUS

Solution becomes a ‘transfer matrix’ multiplication around whole circumference of the collider.

Multiplying dipole, free-space or quadrupole transfer matrices.

Solve numberically for stable orbits.

PHASE AMPLITUDE FORMALISM

UNTIL NOW WE HAVE BEEN CONSIDERING TRAJECTORY OF A SINGLE PARTICLE IN A BEAM. NOW CONSIDER BEAM ENVELOPE WHICH CONTAINS ALL PARTICLES.

HILL'S EQUATION $\quad y'' + K(s)\, y = 0$

$$K(s) = K(s+C)$$

PERIODIC BOUNDARY CONDITIONS

LOOK FOR SOLUTIONS: $\quad y(s) = a(s)\cos(\phi(s) + \delta)$.

$$y(0) \equiv y_0 \qquad y'(0) \equiv y_0'$$

$$\Rightarrow \quad y(s) = u(s)\, y_0 + v(s)\, y_0' \qquad \begin{array}{ll} u(0)=1 & v(0)=0 \\ u'(0)=0 & v'(0)=1 \end{array}$$

And $\quad u'' + Ku = 0 \;; \quad v'' + Kv = 0$

Ansatz: $\quad \begin{array}{l} u = w(s)\cos(\phi(s)) \\ v = w(s)\sin(\phi(s)) \end{array} \Rightarrow \begin{array}{ll} w(0)=1 & \phi(0)=0 \\ w'(0)=0 & \phi'(0)=1 \end{array}$

And now $\quad w'' + Kw = \frac{1}{w^3} \;; \quad w^2\phi' = 1 \qquad (1)$

(Can you prove this?)

SOLUTIONS TO THE EQUATIONS ARE MOST EASILY CAST IN TERMS OF THE

COURANT - SNYDER PARAMETERS

$$\beta(s) = w^2(s)$$

$$\alpha(s) = -\frac{1}{2}\beta'(s)$$

$$\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}$$

SOME USEFUL RELATIONSHIPS:

$$① \quad \phi'(s) = \frac{1}{\beta(s)} \Rightarrow \phi(s) = \int_0^s \frac{ds}{\beta(s)}$$

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4$$

(Just substitute in CS pars and recover Ⓣ)

VERIFY THAT THE FOLLOWING ARE SOLUTIONS:

$$y(s) = \sqrt{\beta(s)}\left[y_0 \cos(\phi(s)) + y_0' \sin(\phi(s))\right]$$

$$y'(s) = \frac{1}{\sqrt{\beta(s)}}\left[y_0\left(-\alpha(s)\cos\phi(s) - \sin\phi(s)\right) + y_0'\left(\cos\phi(s) - \alpha\sin\phi(s)\right)\right]$$

THE COURANT-SNYDER INVARIANT

$$\beta y'^2 + 2\alpha y y' + \gamma y^2 = y_0^2 + y_0'^2$$

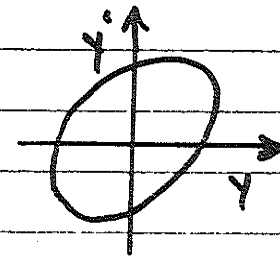
(Exercise)

THIS IS AN ELLIPSE IN (y, y') SPACE WITH AREA:

$$\boxed{\epsilon = \frac{1}{\pi}\left(y_0^2 + y_0'^2\right)}$$

NLY IF:

$\beta\gamma - \alpha^2 = 1$



THUS CAN RE-CAST "FINAL" SOLUTION IN TERMS OF THIS INVARIANT

$$\boxed{y(s) = \sqrt{\epsilon\,\beta(s)}\ \cos\left(\phi(s) + \delta\right)}$$

arbitrary phase!

$\beta(s)$ IS KNOWN AS THE BETATRON FUNCTION AND CAN BE INTERPRETED AS THE BEAM ENVELOPE AROUND THE MACHINE

BETATRON OSCILLATIONS OCCUR IN BOTH PLANES

$\beta_x(s)$, ϵ_x ; $\beta_y(s)$, ϵ_y INDEPENDENT TO 1ST ORDER

BETATRON NUMBER

RECALL $$\phi(s) = \int_0^s \frac{d(s)}{\beta(s)}$$

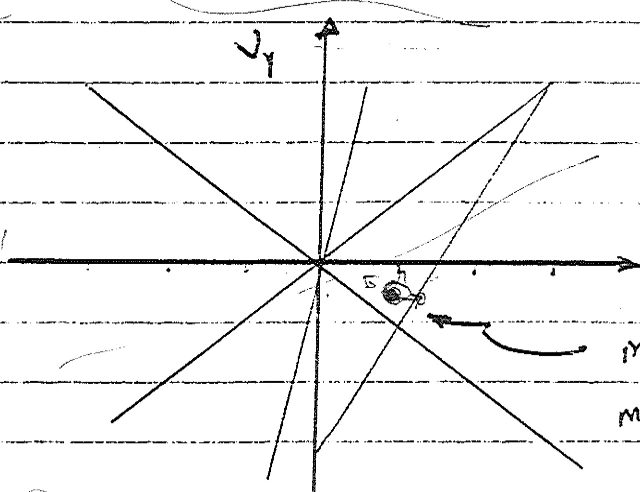
$\Rightarrow$ TOTAL PHASE ADVANCE IN ONE REVOLUTION

$$2\pi\nu = \int_{s_0}^{s_0+C} \frac{d(s)}{\beta(s)}$$

$$\nu = \frac{1}{2\pi} \oint \frac{d(s)}{\beta(s)}$$

IF ν = INTEGER PARTICLES PASS THRU SAME PART OF ACCELERATOR ON EVERY REVOLUTION ($\sim$ 50 kHz). IF THE IS AN IMPERFECTION IN ANY ELEMENT THIS IS A DISASTER (OR MORE POLITELY RESONANCE)

In GENERAL AVOID: $p\nu_x + q\nu_y$ = Integer



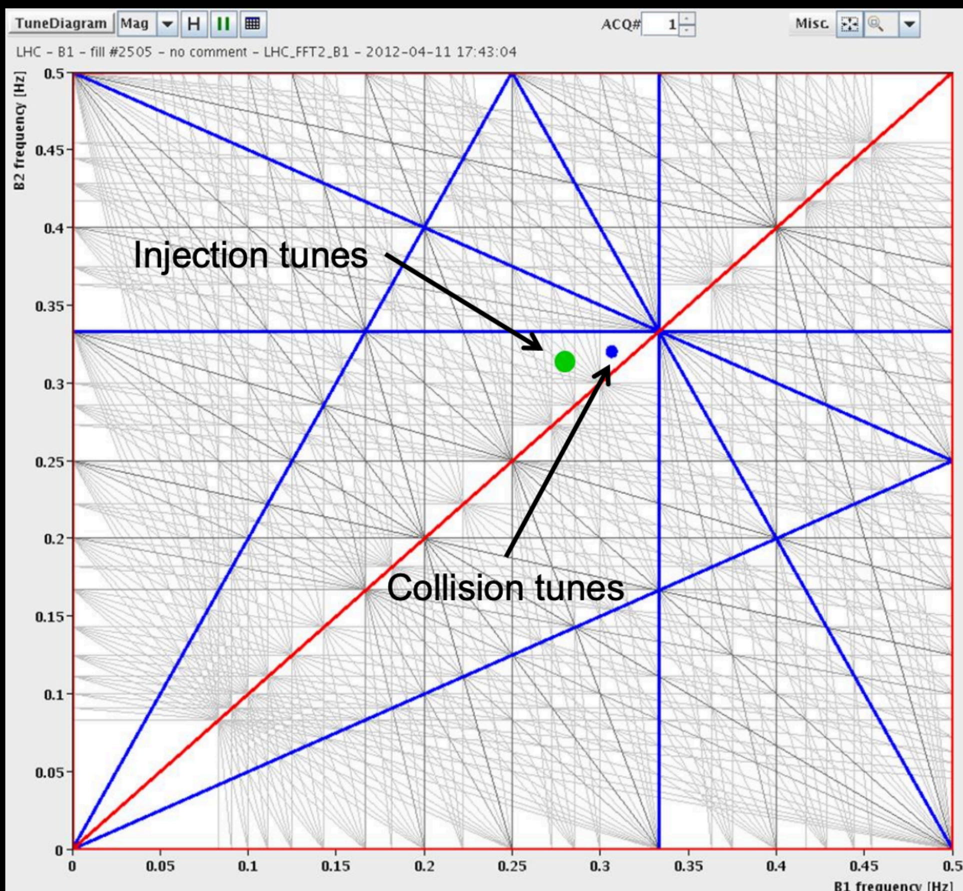
FNAL Tevatron $\nu = 19.4$

$$= \frac{2\pi\,(1000\text{ m})}{19.4} \simeq 320\text{ m}$$

MACHINE OPERATING POINT MUST BE CAREFULLY CHOSEN.

LHC: $p=59$ $q=64$ EVEN SMALLER OPERATING MARG

LHC Tune Diagram



PERTURBATIONS IN ACCELERATOR STRUCTURES

- IN PRACTICE MAGNETS HAVE A FINITE LENGTH (a few m → 10 m) AND THEY ARE "STRAIGHT" ⇒ THIER AXIS OF SYMMETRY DOES NOT FOLLOW IDEAL REFERENCE ORBIT (CIRCLE).

- FURTHER MAGNETS CAN BE DISPLACED FROM IDEAL LOCATION (c.f. Tevatron in 1994)

- THIS GIVES RISE TO A PERTURBATION TO WHAT WE HAVE BEEN DISCUSSING UNTIL NOW

RECALL $\Delta\theta = \frac{eB}{p}\Delta s = \frac{\Delta s}{\rho}$

was l, the magnet length last time we saw this.

IF AN ERROR ΔB EXISTS IT GIVES RISE TO A CHANGE OF SLOPE:

$$\delta x' = \frac{\Delta B}{B}\Delta\theta = \frac{\Delta B}{B}\frac{\Delta s}{\rho}$$

STILL HAVE A CLOSED ORBIT (OTHERWISE THIS IS A FUTILE ACADEMIC EXERCISE) BUT WITH A SLOPE DISCONTINUITY AT S_0 (LOCATION AROUND THE RING OF IMPERFECTION)

THUS LOOK FOR A CLOSED ORBIT :

$$x(s_0) = \sqrt{\beta \epsilon}\, \cos(\phi_0)$$

$$x(s_0 + C) = \sqrt{\beta \epsilon}\, \cos(\phi_0 + 2\pi \nu_x)$$

CLOSES IF $\phi_0 = -\pi\nu_x$

↑ sign for convenience see below.

$$\Rightarrow \quad x'(s_0) = \sqrt{\frac{\epsilon}{\beta}}\, \sin\phi_0 + \frac{1}{2}\frac{\beta'}{\beta}\, x(s_0)$$

$$x'(s_0 + C) = \sqrt{\frac{\epsilon}{\beta}}\, \sin(\phi_0 + 2\pi\nu_x) + \frac{1}{2}\frac{\beta'}{\beta}\, x(s_0 + C)$$

SINCE CONT. AMPLITUDE SECOND TERMS CANCEL IN DIFFERENCE

THUS SLOPE DISCONTINUITY:

$$\delta x' = x'(s_0 + C) - x'(s_0)$$

$$= \sqrt{\frac{\epsilon}{\beta}} \left[\sin(\phi_0) - \sin(\phi_0 + 2\pi\nu_x) \right]$$

$$= -2\sqrt{\frac{\epsilon}{\beta(s_0)}}\, \sin(\pi\nu_x) = \frac{\Delta B}{B}\frac{\Delta s}{\rho}$$

THIS GIVES US A CONSTRAINT ON $\sqrt{\epsilon\beta(s_0)}$

$\Rightarrow$ $\sqrt{\epsilon}$ EMITTANCE CAN BE EXPRESSED IN TERMS OF THE PERTURBATION.

GIVING:

$$X(s) = \underbrace{\frac{\sqrt{\beta(s_0)\,\beta(s)}}{2 \sin \pi \nu_x}} \frac{\Delta B}{B} \frac{\Delta s}{\rho} \cos\left\{ \int_{s_0}^{s} \frac{ds'}{\beta(s')} - \pi \nu_x \right\}$$

NB. ν_x = INTEGER $\Rightarrow$ AMPLITUDE BLOWS UP!

THIS IS βETATRON FUNCTION FOR A ~~CLOSED~~ PERTURBED CLOSED ORBIT.

IN PRACTICE THE EMITTANCE WILL BE A (QUADRATURE ?) SUM OF THE "INJECTION EMITTANCE" AND ADDITIONS FROM:

i) IMPERFECTIONS IN FOCUSSING STRUCTURE

ii) FINITE MOMENTUM BITE OF BEAM (SEE PROBLEM 1c))

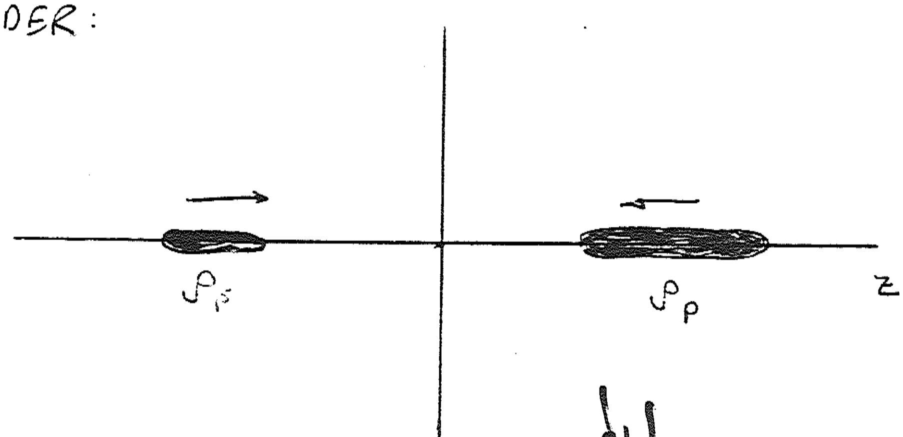
iii) OTHER THINGS ...

LUMINOSITY : $\mathcal{L}$

HAVE ALREADY SEEN THAT :

$$\text{INTERACTION RATE} \equiv \mathcal{L} \cdot \sigma$$

CONSIDER :



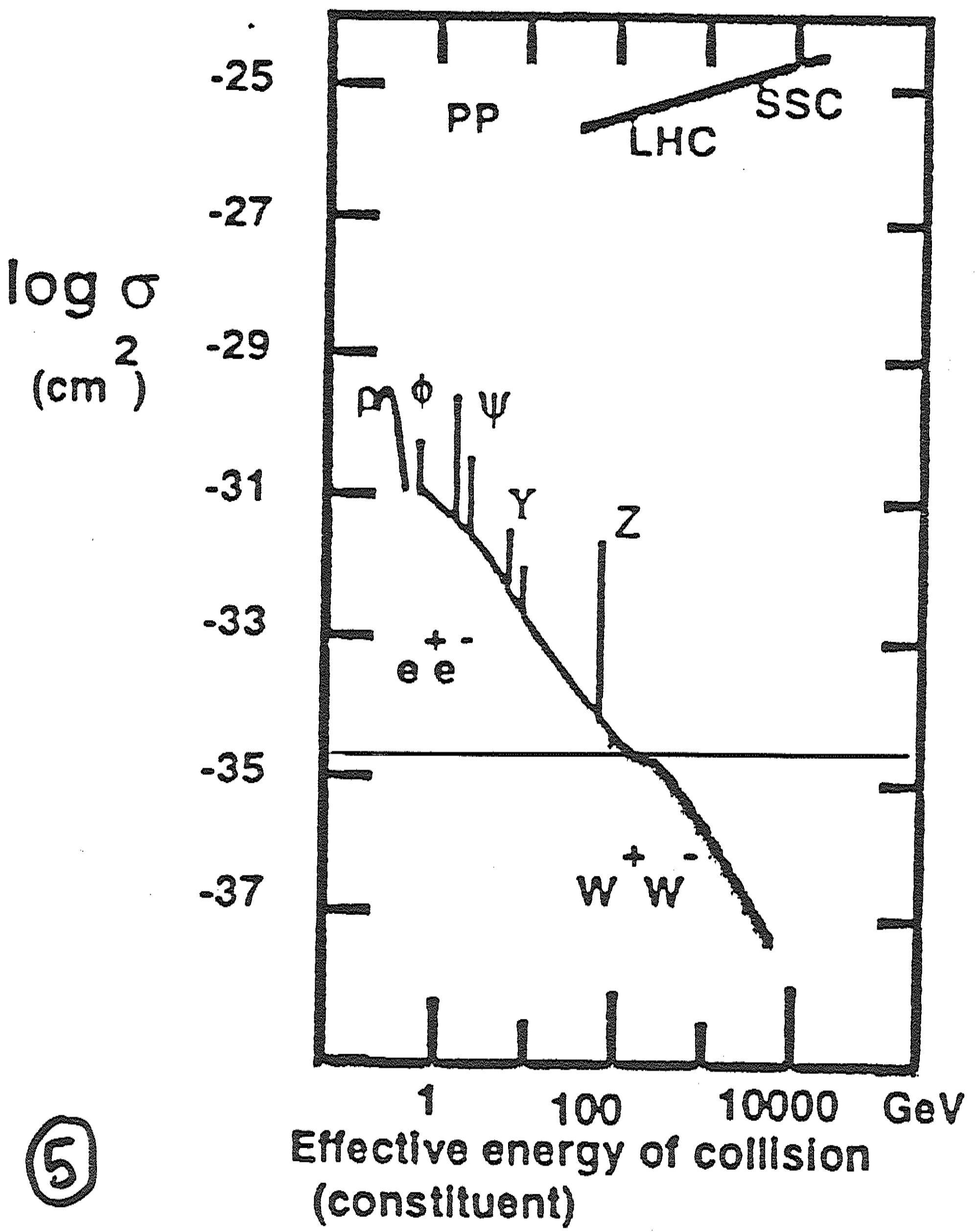
$$dn = \overbrace{n f \cdot \rho_{\bar{p}} \, |\vec{v}_{\bar{p}} - \vec{v}_p| \, (\rho_p \, d^3x) \, dt}^{d\mathcal{L}} \cdot \sigma$$

number of bunches

frequency of collisions ; related to # of bunches

$$\rho_{\bar{p}} = \rho_{\bar{p}}\,(x, y, z - ct) \qquad \sigma_{x\bar{p}}, \sigma_{y\bar{p}}, \sigma_{z\bar{p}}$$

$$\rho_p = \rho_p\,(x, y, z + ct) \qquad \underbrace{\sigma_{xp}, \sigma_{yp}, \sigma_{zp}}_{\text{Assuming charge distributions are Gaussian.}}$$



→ $e^+e^- \to W^+W^-$
~ 20 - 50 pb
at threshold

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INTEGRATE $d^3x \; dt$ ← over sufficient time to have bunches pass thru each other!

$$\mathcal{L} = \frac{nf \; N_{\bar{p}} N_p}{2\pi \left[\left(\sigma_{x\bar{p}}^2 + \sigma_{xp}^2 \right) \cdot \left(\sigma_{y\bar{p}}^2 + \sigma_{yp}^2 \right) \right]^{1/2}}$$

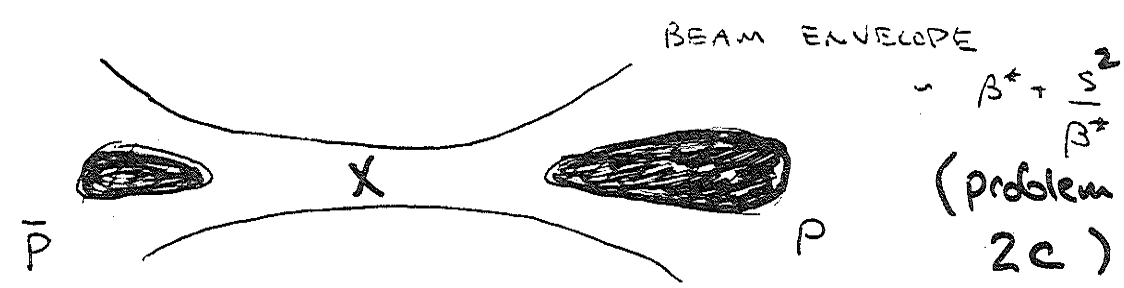
$$\rightarrow \quad nf \frac{N_{\bar{p}} N_p}{4\pi \sigma^2}$$

assuming $\sigma_{x\bar{p}} = \sigma_{xp} = \sigma_{yp} = \sigma_{y\bar{p}}$

(ie Tevatron)

TO FIRST ORDER THE LENGTH OF THE BUNCHES DOESN'T APPEAR SINCE ALL OF BOTH BUNCHES EVENTUALLY MEET.

THIS NEED NOT BE TRUE IF BETATRON WAVE LENGTH COMPARABLE TO BUNCH LENGTH :



THIS CAN BE A $\mathcal{L}$ LIMITATION IF YOU TRY TO SQUEEZE β^* TOO MUCH.

CAN WE RELATE THIS BACK TO EMITTANCE ?

(RECALL $\epsilon_y = \frac{1}{\pi}(y_0^2 + y_0'^2)$)

$$y = \sqrt{\epsilon_y \beta_y(s)} \cos(\phi(s) + \delta)$$

$$\phi(s) = \int \frac{ds'}{\beta_y(s)}$$

AT ANY ϕ THE SIZE OF THE BEAM ENVELOPE IS :

$$\sigma_y(s) = \sqrt{\epsilon_y \beta_y(s)}$$

ASSUMING TRAJECTORIES ARE POPULATE GAUSSIANLY

DESIGN MACHINE OPTICS SUCH THAT :

$\beta(s_0) = \beta^*$ is A MINIMUM AT COLLISION POINT

$$\Rightarrow \quad \sigma_{int} = \sqrt{\epsilon \beta^*}$$

or $$\mathcal{L} = n f N_{\bar{p}} N_p \Big/ 4\pi \sqrt{\epsilon_x \epsilon_y \beta_x^* \beta_y^*}$$

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$$\Rightarrow \quad \sigma_{int} = \sqrt{\epsilon \beta^*}$$

or $$\mathcal{L} = n f N_{\bar{p}} N_p / 4\pi \sqrt{\epsilon_x \epsilon_y \beta_x^* \beta_y^*}$$

SOME OBSERVATIONS ON $\mathcal{L}$

IN A CIRCULAR COLLIDER : $f \simeq \beta c / 2\pi R$ FIXED AS $\beta \to 1$

1) Want n AS LARGE AS POSSIBLE:

$n = 6$ Tevatron Run 1 $\Rightarrow \Delta t = 3.5\,\mu s$
$n \simeq 150$ Run 2 $= 132\,ns$

$n = 4$-8 LEP 1 $\Rightarrow \Delta t = 5$-$11\,\mu s$
$n = 8 \times (3$-$4)$ LEP 2

$n \simeq 2500$ for LHC $\Delta t = 25\,ns$

2) Want N_1, N_2 as large as possible subject to:

- Charge density too high $\Rightarrow$ repulsion and beam blow up ($\beta < 1$!)
- Limited by source of beam particles (cf $\bar{p}$ sources and cooling rates)
- LHC all protons ($\sim 1.5\,E11$ protons)

3) Want beam area as small as possible but:

- β^* small $\Rightarrow$ $\beta(s)$ large nearby - acceptance
- ϵ grows with machine imperfections and collisions!

TYPICALLY : $N_1 \cong N_2 = 10^{11}$ per bunch $p\bar{p}$

$n = 2500$ $\beta^* = 30\,\mathrm{cm}$ $n = 6$; $\epsilon \simeq 10^{-6}$ rad·cm ; $\beta^* \simeq 100\,\mathrm{cm}$

$f \approx 10\,\mathrm{kHz}$ $R = 1\,\mathrm{Km} \Rightarrow f \simeq 100\,\mathrm{kHz}$ ($Sp\bar{p}S$ Tevatron)

$\Rightarrow \mathcal{L} =$ few $\times 10^{31} \frac{1}{\mathrm{cm}^2 \cdot \mathrm{s}}$ (at peak performance)

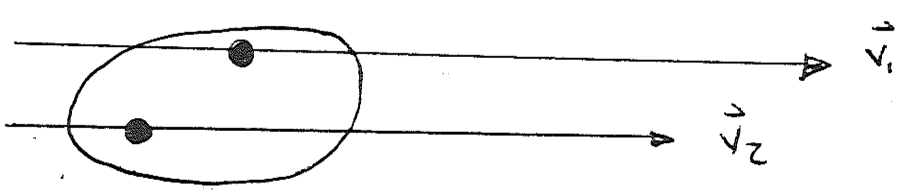
$\mathcal{L} \to 10^{34}/\mathrm{cm}^2\cdot\mathrm{s}$

This can never compete with "FIXED TARGET" $\mathcal{L}$ OF $\simeq 10^{37}$ SIMPLY BECAUSE IT IS IMPOSSIBLE TO PACK BUNCHES AS DENSELY AS SOLID MATTER

→ WE WILL SEE SERIOUS DISADVANTAGE (E of M Energy) OF FIXED TARGET LATER)

LHC still can compete with FT for $\mathcal{L}$

CONSIDER THE ELECTROMAGNETIC STABILITY OF A BUNCH OF PARTICLES



$$\vec{F}_{12} = e_1\left(\vec{E}_2 + \frac{\vec{v}_1}{c} \times \vec{B}_2\right)$$ IN LAB FRAME

FROM RELATIVITY / E & M

$$B_2^{\perp} = -\gamma_2 \frac{v_2}{c} E_2^{\perp'} \qquad \because \; E_2^{\perp} = \gamma_2 E_2^{\perp'}$$

in lab — in rest frame of "2"

$$\Rightarrow \quad F_{12}^{\perp} = e_1 \gamma_2 (1 - \beta_1 \beta_2) E_2^{\perp'}$$

$$\to 0 \quad \text{as} \quad \beta \to 1$$

$\Rightarrow$ Relativistic beams are stable against transverse forces due to space charge

(but slower ones aren't!)

WHAT HAPPENS TO COLLIDING BUNCHES?

$$\beta_1 \to -\beta_1 \quad !$$

$$\Rightarrow \quad F_{12}^{\perp} = e \gamma_2 (1 + \beta_1 \beta_2) E_2^{\perp'}$$

THUS COLLIDING BEAMS SEE AN IMPORTANT DEFOCUSSING EFFECT AS THEY PASS THROUGH ON-COMING BUNCHES.

MACHINE TUNE

THE "TUNE" IS DEFINED AS THE NUMBER OF BETATRON OSCILLATIONS PER REVOLUTION

$$\nu \equiv \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

THE TUNE SHIFT CAN THEN BE USEFULLY DEFINE AS THE ACCUMULATED SHIFT IN PHASE PER ORBIT

$$\Delta\nu_x \equiv \frac{1}{4\pi} \oint ds\ \beta_x(s)\ K(s)$$

Analogous to $K = \frac{1}{B\rho}\frac{\partial B}{\partial x}$

but usually due to imperfections & OTHER BUNCHES!

$$\Delta\nu_x^{\bar{p}} = \frac{r_{\bar{p}}\ \beta_{x\bar{p}}\ N_p}{2\pi\ \gamma_{\bar{p}}\ \sigma_{xp}\ (\sigma_{xp} + \sigma_{yp})} \qquad (1)$$

$$\Delta\nu_x^{p} = \frac{r_p\ \beta_{xp}\ N_{\bar{p}}}{2\pi\ \gamma_p\ \sigma_{x\bar{p}}\ (\sigma_{x\bar{p}} + \sigma_{y\bar{p}})} \qquad (2)$$

Can re-write

$$\mathcal{L} \sim \Delta\nu^{\bar{p}} \, \Delta\nu^{p}$$

Empirically these are limited to about

$$\Delta\nu^{p \text{ or } \bar{p}} \lesssim 0.005 \qquad \text{Tevatron} \checkmark$$

$$\Delta\nu^{e^- \text{ or } e^+} \lesssim 0.05 \qquad \text{LEP} \checkmark$$

OTHERWISE BEAM INSTABILITIES SET IN (ie. YOU GET TOO CLOSE TO INTEGER RESONANCES) AND BEAM LOSS OCCURS

SO FROM (1) YOU CAN SEE THAT $\Delta\nu^{\bar{p}} \sim N_p$

$\Rightarrow$ IT DOESN'T PAY TO INCREASE N_p TO IMPROVE $\mathcal{L}$ $\rightarrow$ BETTER TO MATCH N_p AND $N_{\bar{p}}$ OR $\Delta\nu_p$ and $\Delta\nu_{\bar{p}}$ AND INCREASE THEM AS CLOSE TO PHYSICAL LIMIT AS POSSIBLE. **IN A BALANCED WAY.**

So why colliding beams?

Luminosity advantage to fixed target but

$$E_{com} \equiv \sqrt{s} = P_{1\mu} \cdot P_2^{\mu}$$

$$s = m_{projectile}^2 + M_{target}^2 + 2 M_{target} E_{proj}$$

$$\sqrt{s}_{FT} \simeq \sqrt{2ME} \qquad E >> M, m$$

Colliders have luminosity disadvantage but

$$\sqrt{s} = \sqrt{(E_1 + E_2)^2 - (\vec{P}_1 + \vec{P}_2)^2}$$

$$= \left\{ m_1^2 + m_2^2 + 2(E_1 E_2 - \vec{P}_1 \cdot \vec{P}_2) \right\}^{1/2}$$

$$p \simeq E \quad \vec{P}_1 = -\vec{P}_2$$

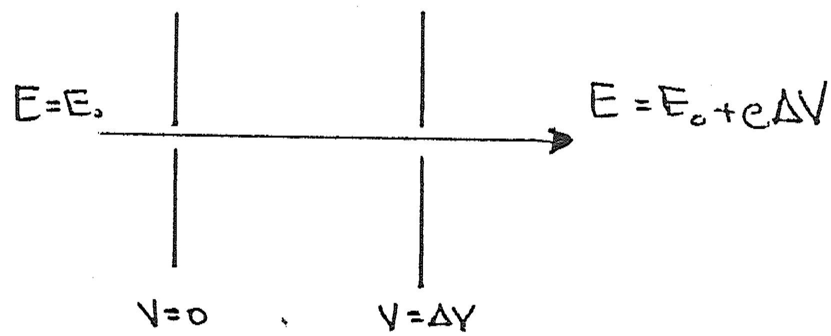
$$\sqrt{s}_{collider} \simeq 2E \qquad E_1 = E_2$$

eg. E = 1 TeV ; M = 1 GeV (proton)

$$\underline{\sqrt{s}_{FT} = 43 \text{ GeV}} \quad ; \quad \underline{\sqrt{s}_{collider} = 2000 \text{ GeV}}$$

THE BASICS OF ACCELERATION

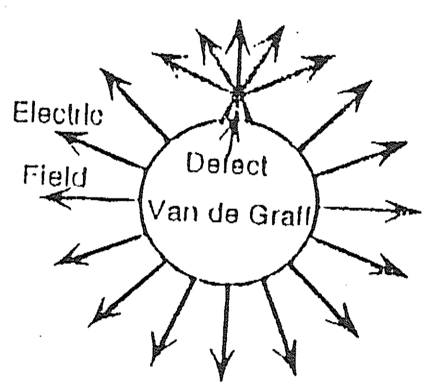
EARLIEST ACCELERATORS USED STATIC BETWEEN TWO ELECTRODES:



NB: Breakdown in dry air occurs @ 3 MV/m

→ When ionised electrons receive enough energy between collisions to do further ionisation

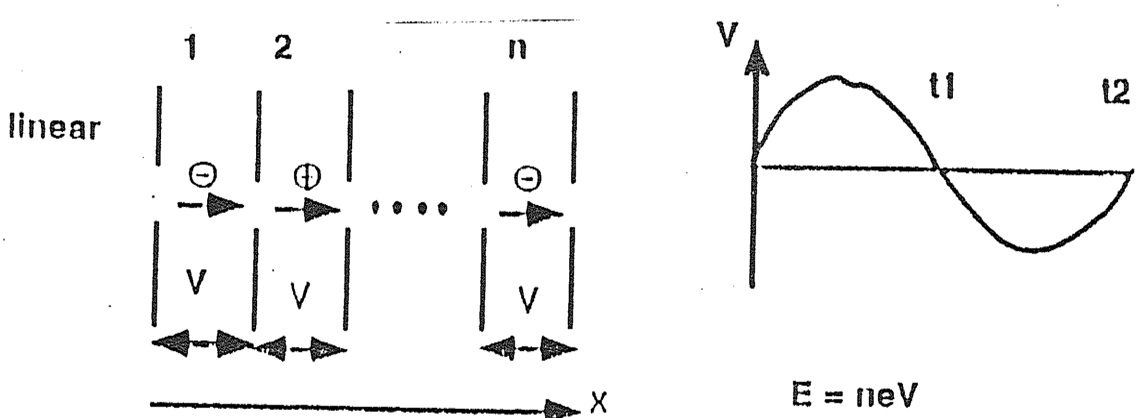
ANOTHER LIMITATION IS SURFACE DEFECTS IN CAVITY WALLS



MUCH HIGHER FIELDS AROUND AN IMPERFECTION.

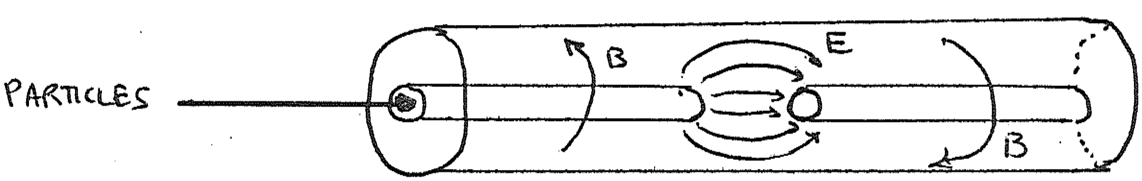
TO AVOID BREAK DOWN MUST GO TO AN RF (RADIO FREQUENCY) STRUCTURE

AC FIELDS MEAN THAT FIELD IS NOT PRESENT "LONG ENOUGH" FOR BREAKDOWN TO OCCUR:



E = neV
(n = number of stage

IN PRACTICE THESE ALTERNATING POTENTIALS ARE OBTAINED FROM A LONGITUDINAL ELECTRIC FIELD IN A RESONANT CAVITY



ORBITAL LENGTH & R.F. FREQUENCY

FOR SIMPLICITY ASSUME 1 RF CAVITY PER ORBIT
THE R.F. WILL BE TUNED SO THAT A REFEREN
PARTICLE ARRIVES "IN TIME" WITH THE RF
FIELD :

$$2\pi\,\omega_{rf} = \nu_{ORBIT} = \frac{1}{\tau_{ORBIT}}$$

$$(h\omega_0)$$

$$\tau_{ORBIT} = L_{PART} / V_{PART}$$

CONSIDER THE TWO "MOST LIKELY" CAUSES OF DEVIATION FROM ARRIVAL AT IDEAL TIME:

1) PARTICLE OFF AXIS $\Rightarrow$ $L \neq L_{IDEAL}$

2) PARTICLE OFF MOMENTUM $\Rightarrow$ $V \neq V_{IDEAL}$

$\Rightarrow$ LEADS TO AN OUT OF TIME ARRIVAL:

$$\frac{\Delta T}{T} = \frac{\Delta L}{L} - \frac{\Delta V}{V}$$

FASTER $\Rightarrow$ SOONER
LONGER $\Rightarrow$ LATER.

BUT WE CAN EXPRESS BOTH OF THESE TERMS IN TERMS OF MOMENTUM IMPERFECTION:

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2}\left(\frac{\Delta p}{p}\right)$$

(Recall $p = \gamma m v$
$= \frac{mv}{\sqrt{1-v^2}}$

ALSO MIGHT EXPECT THAT (AS $v \to c$) AN OFF AXIS PARTICLE MIGHT TAKE A LONGER ORBIT THUS DEFINE:

$$\frac{\Delta L}{L} = \frac{1}{\gamma_T^2}\left(\frac{\Delta p}{p}\right)$$

THUS CONCLUDE:

$$\frac{\Delta \tau}{\tau} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_T^2}\right)\frac{\Delta p}{p} \qquad (1)$$

Thus CHANGES IN ORBITAL FREQUENCY (COMING FROM CHANGES IN RF FREQUENCY) CAN BE RELATED TO MOMENTUM SPREAD IN THE BEAM (Δp) OFTEN REFERRED TO AS:

LONGITUDINAL EMITTANCE

The RF Voltage across the accelerating structure has the form:

$$V = V_0 \sin(\omega_{rf} t)$$

$$\omega_{rf} = h\,\omega_0$$

h: IS THE MACHINE <u>HARMONIC NUMBER</u>

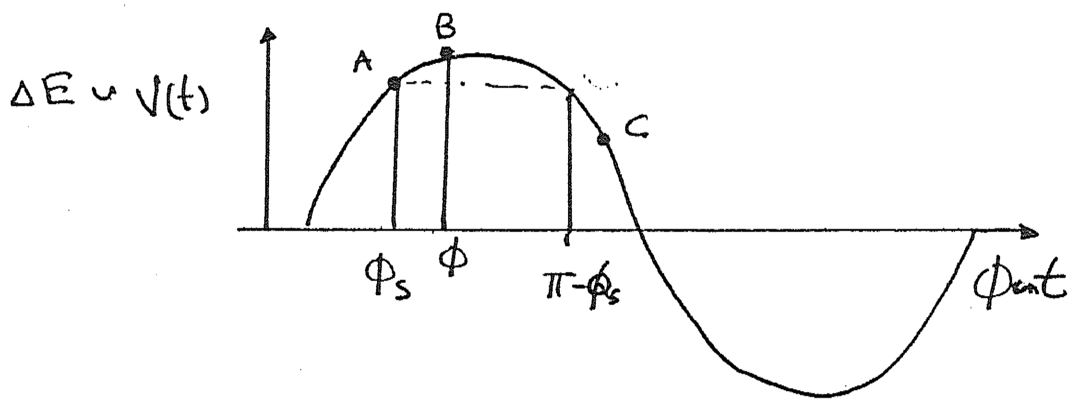
ω_0: IS THE BASE RF FREQUENCY AND THIS MUST CHANGE TO MATCH THE PARTICLES' REVOLUTION FREQUENCY AS THEY ARE ACCELERATED.

c.f. $h = 4620$ in CERN SPS (1113 at FNAL mainRing)

$\Rightarrow$ 4620 places around the ring where a particle can arrive "in time" with the R.F. field

NB: THIS MEANS $Sp\bar{p}S$ COULD ACCOMODATE 4620 bunches although in practice no more than 4 were ever used

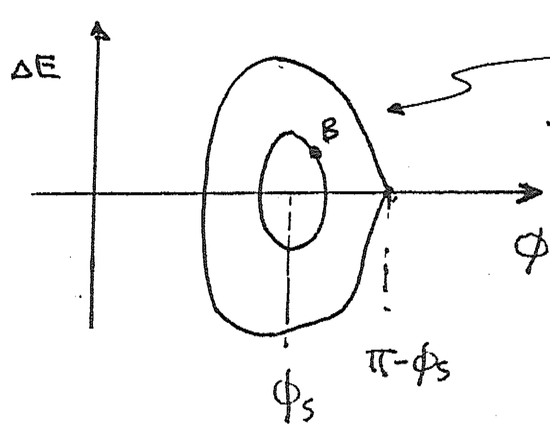
SO WHAT DOES IT MEAN TO ARRIVE "IN TIM WITH THE RF ?



$$E_s = V_0 \sin \phi_s \qquad \text{(REFERENCE PARTICLE)}$$

$$\Delta E = V_0 (\sin\phi - \sin\phi_s)$$

BELOW THE <u>TRANSITION ENERGY</u> A PARTICLE AT "B" GETS <u>MORE</u> ENERGY AND ARRIVES AT $\phi < \phi_s$ IN THE NEXT REVOLUTION. THUS IN THE <u>NEXT</u> REVOLUTION IT WILL GET <u>LESS</u> ENERGY AND THE MOTION WILL BE STABLE. IN ΔE SPACE THE STABLE REGION IS AN ELLIPSE



LIMIT OF STABLE MOTION CALLED A <u>SEPARATRIX</u>

"SPACE IN $\Delta E, \phi$ CALLE AN RF BUCKET"

FOR PARTICLE C $(\phi_c > \pi - \phi_s)$ ACQUIRES LESS ENERGY THAN REFERENCE PARTICLE. THIS ONLY CAUSES IT TO LAG FURTHER BEHIND ⇒ LOST FROM ACCELERATED BUNCH

NOTICE THAT THIS ARGUMENT ONLY HOLDS IF EXTRA ENERGY ⇒ SHORTER ORBIT TIME

IF EXTRA ENERGY ⇒ LONGER ORBIT TIME THEN STABLE POINT IS ON "OTHER SIDE" OF SINUSOID ⇒ <u>ABOVE TRANSITION</u>

FROM 1 TRANSITION ENERGY OCCURS AT:

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_T^2} \qquad (2)$$

γ_T^2 IS A PROPERTY OF THE MACHINE (PRINCIPALLY MACHINE RADIUS AND SIZE OF BETATRON OSCILLATIONS)

LEFT AS AN EXERCISE FOR THE STUDENTS ... 2024.

CAN WE DO THIS IN A MORE QUANTITATIVE WAY

REFERENCE PARTICLE, p_0, FOLLOWS CENTRAL ORBIT, $2\pi R_0$

$$\Rightarrow \quad t_{revol} = 2\pi R_0 / \beta c \qquad (\beta = v/c\ !)$$

$$\omega_{RF} = h\,\omega_0 = h\left(\frac{\beta c}{R_0}\right)$$

A CHANGE IN ENERGY, ΔE, CHANGES REVOLUTION:

$$\frac{\Delta\beta}{\beta} = \frac{1}{\gamma^2\beta^2}\frac{\Delta E}{E} \qquad \frac{\Delta p}{p} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

$$\Rightarrow \quad \Delta R \equiv \bar{\eta}\,\frac{\Delta p}{p}$$

(LIKE η in 2 BUT T WAS " $\Delta R/R$)

$$\Rightarrow \quad \Delta t_r = \frac{2\pi R_0}{\beta^3 c}\left(\bar{\eta}/R_0 - \frac{1}{\gamma^2}\right)\frac{\Delta E}{E}$$

(HERE WE CAN DEFINE $\gamma_T^2 = R_0/\bar{\eta}$)

RECALL THAT BELOW TRANSITION $\Delta t_r < 0$

ABOVE TRANSITION $\Delta t_r > 0$

NOW CONSIDER THE CHANGE IN RF FREQUENCY $\Delta\phi$ NECESSARY TO "KEEP UP WITH" A CHANGE IN ENERGY ΔE :

$$\Delta\phi = \omega_{RF}\,\Delta t_r$$

$$= \frac{2\pi h}{\beta^2}\left(\frac{1}{\gamma_T^2} - \frac{1}{\gamma^2}\right)\frac{\Delta E}{E}$$

$$\equiv \Phi\,\frac{\Delta E}{E}$$

$\Phi \rightarrow$ CONSTANT AS $\gamma \rightarrow 0$ ABOVE γ_T .

NOW CONSIDER THE ACCELERATING KICK EACH TURN:

$$\Delta E = eV_0\left(\sin\phi - \sin\phi_s\right) \qquad (3)$$

IN REAL ACCELERATORS THERE IS ONLY A SMALL KICK PER TURN SO AFTER Δn ($\Delta n \ggg 1$) WE ACCUMULATE

$$\frac{\Delta\phi}{\Delta n} = \Phi\,\frac{\Delta E}{E}$$

$$\rightarrow \quad \frac{d\phi}{dn} = \Phi\,\frac{\Delta E}{E} \qquad (4)$$

HOWEVER THE RELATIVE CHANGE IN ENERGY AFTER THE $n^{th.}$ TURN IS JUST GIVEN (3) SO:

$$\frac{d}{dn}\left(\frac{\Delta E}{E}\right) = \overbrace{\frac{eV_0}{E}\left(\sin\phi - \sin\phi_s\right)}^{(3)} \qquad (5)$$

COMBINING (4) AND (5) WE GET:

$$\frac{d^2}{dn^2}(\phi) = \frac{eV_0}{E}\,\Phi\,\left(\sin\phi - \sin\phi_s\right) \qquad (6)$$

AGAIN ASSUME THAT $\phi = \phi_s + \Delta\phi \qquad \Delta\phi \ll 1$

$$\Rightarrow \quad \sin\phi - \sin\phi_s = \Delta\phi\,\cos\phi_s \qquad (7)$$

$$\Rightarrow \quad \frac{d^2}{dn^2}(\Delta\phi) + \left\{-\frac{eV_0}{E}\,\Phi\,\cos\phi_s\right\}\Delta\phi = 0$$

THIS WILL LEAD TO STABLE OSCILLATIONS IF

$$\boxed{\Phi\,\cos\phi_s < 0}$$

THIS CONDITION CAN ONLY BE MET IF RF PHASE CHANGES FROM

$$\phi_s \rightarrow \pi - \phi_s$$

AS TRANSITION IS CROSSED.

THIS OPERATION RESULTS IN THE LOSS OF PARTICLES HENCE IT IS PREFERABLE TO HAVE A SERIES OF MACHINES OF VARYING ENERGIES (WITH DIFFERENT RADII) SO THAT "ALL" ARE INJECTED INTO ABOVE TRANSITION.

THIS IS ONE OF MANY REASONS FOR HAVING A SERIES OF ACCELERATORS (OR BOOSTERS) IN AN ACCELERATOR COMPLEX.

OTHERS ARE:

- CURRENT LIMITATIONS IN ONE MACHINE (FILL EACH BUNCH IN "FINAL" MACHINE WITH FULL COMPLEMENT OF PREVIOUS MACHINE)
- USE "SMALLER" MACHINES TO DO OTHER THINGS (e.g. MAKE ANTI-PROTONS FIXED TARGET PHYSICS etc.)

SYNCHROTRON OSCILLATIONS

→ THESE ARE THE OSCILLATIONS IN ENERGY FROM AHEAD TO BEHIND THE REFERENCE PARTICLE

$$n_{sync} = 2\pi \sqrt{\frac{E}{eV_0 \left| \Phi \cos\phi_s \right|}} \quad \text{turns}$$

" several hundred turns in a proton machine

$$f_{sync.} = 0 \longrightarrow 1000\ \text{Hz}$$

↑
at transition

Quite slow for anything that happens near the speed of light.

PHASE SPACE FOR STABLE OSCILLATIONS

TAKE THE DIFFERENTIAL EQUATION OF MOTION (6) MULTIPLY BY $d\phi/dn$ AND INTEGRATE OVER dn:

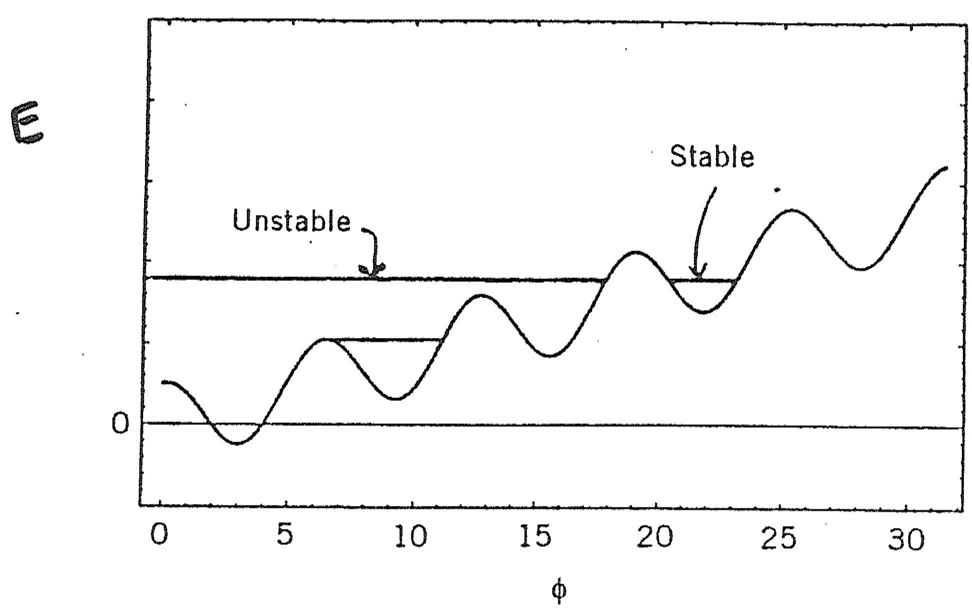
$$\int \frac{d^2\phi}{dn^2}\frac{d\phi}{dn}\,dn = \frac{eV_0}{E}\Phi \int (\sin\phi - \sin\phi_s)\frac{d\phi}{dn}\,dn$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 = -\frac{eV_0}{E}\Phi\left(\cos\phi + \phi\sin\phi_s\right) + \text{constant.}$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{d\phi}{dn}\right)^2 + \frac{eV_0}{E}\Phi\left(\cos\phi + \phi\sin\phi_s\right) = \text{constant} \qquad (8)$$

THIS IS OF THE FORM:

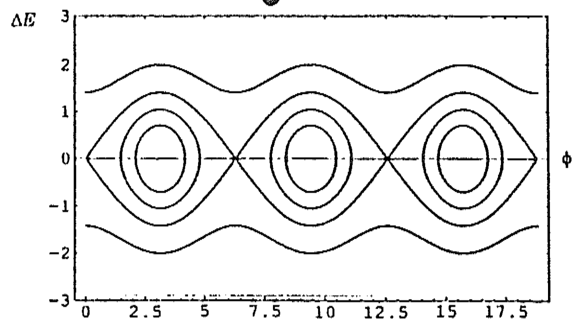
KINETIC ENERGY + POTENTIAL ENERGY = TOTAL ENERGY



(13)

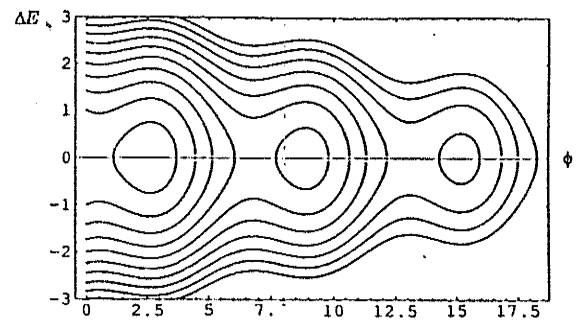
THE SOLUTIONS OF (8) CAN BE OBTAINED NUMERICALLY AND LOOK LIKE:

$\eta > 0$



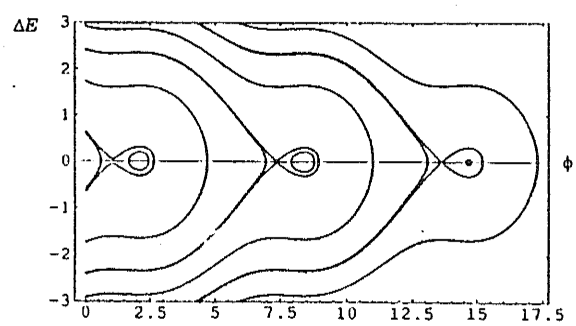
(a)

$\phi_s = \pi$



(b)

$\phi_s = \frac{5\pi}{6}$



(c)

$\phi_s = \frac{2\pi}{3}$

SEE PROBLEM FOR HOW LARGE ΔE AND $\Delta\phi$ CAN BECOME TO STILL PRESERVE A LONGITUDINALLY STABLE BEAM.