PHY255F SOLUTION SET midterm 2003

1) See textbook; significance is essential. Answers do not need to be as long as these, but these are quite good.

i) a dynamical system that executes sinusoidal oscillations, ie $A\sin(\omega t+\phi)$; obeys the equation of motion $m\ddot{x} + ax = 0$

Many systems execute SHO in the limit of small oscillations, eg. molecular vibrations, stop signs in the wind, a single atom in a crystal lattice. This simple mathematical model can help explain eg heat radiation, sound waves, molecular spectra etc.

ii) a factor characterizing the relative decay of a damped SHO. $\frac{Q}{2\pi}$ is the number of cycles a damped SHO executes in having its energy decay to $\frac{1}{e}$ of its initial value. Q also characterizes the velocity-resonance of a driven damped SHO $Q = \frac{\omega_0}{\Delta\omega}$. So, a high-quality (Q) oscillator will oscillate for very many cycles and is characterized by a very precise frequency.

iii) phasor notation $e^{i\omega t}$

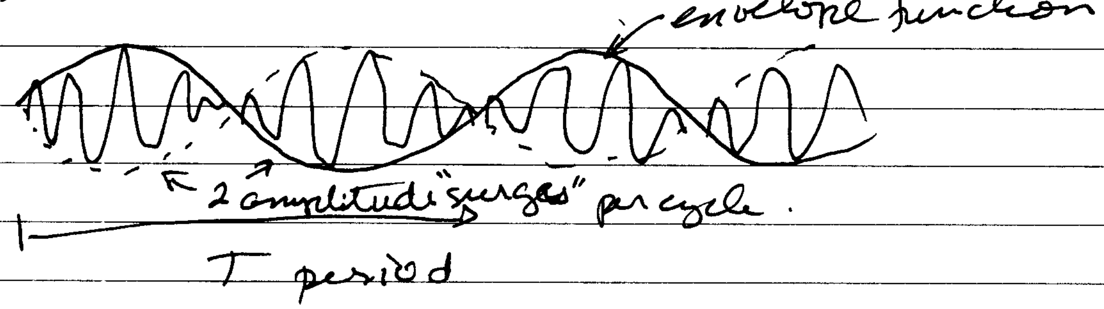
when we take the Re[ ] part, gives
$\cos\omega t$ since $e^{i\omega t} = \cos\omega t + i\sin\omega t$

Significant because it greatly simplifies manipulation of linear ODE's, like the one for the SHO. After all calculations one must still take the Re[ ] part for a solution (is basis of vector formalism for SHOs)

iv) Lissajous figure : an x-y plot in which two independent oscillation amplitudes $x_1(t)$, $x_2(t)$ are plotted against each other parametrically (ie "t") $x_1(t)$ $x_2(t)$ need not represent oscillations that are physically in perpendicular directions but they should be independent.

Significant because it quickly and graphically shows changes in relative phase and amplitude for the relationship between the two oscillators.

v) beat frequency : the frequency at which the net amplitude envelope rises and falls. Equal to $\omega_2 - \omega_1$ where ω_1 and ω_2 are frequencies of superimposed oscillators. Note that the envelope is at $\frac{\omega_2 - \omega_1}{2}$ but there are two amplitude peaks per cycle, thus $\omega_2 - \omega_1$ for beats



Significant because it can, for instance, be used to tune two musical instruments to the same frequency — one listens for the beats and makes them slower until they vanish when tuned.

2) i) B must stay real valued for small r

ii) A oops - this was already in Q 1(iv)!

iii) F R serves as damping, due to resistive heating (Joule heating) by current that passes through

E is typical - all current must flow through R

A is also true (from PS #3): any voltage across C or L will also be through R, so some current will be dissipated in R

part marks: E, A

D: has no inertial component, no $\ddot{x}$ 2nd deriv. (L $\Leftrightarrow$ mass)

B: has no damping (R $\Leftrightarrow$ damping)

C: has no restoring force (C $\Leftrightarrow$ spring)

iv) $0.01 = e^{-15\,\text{sec}/\tau}$

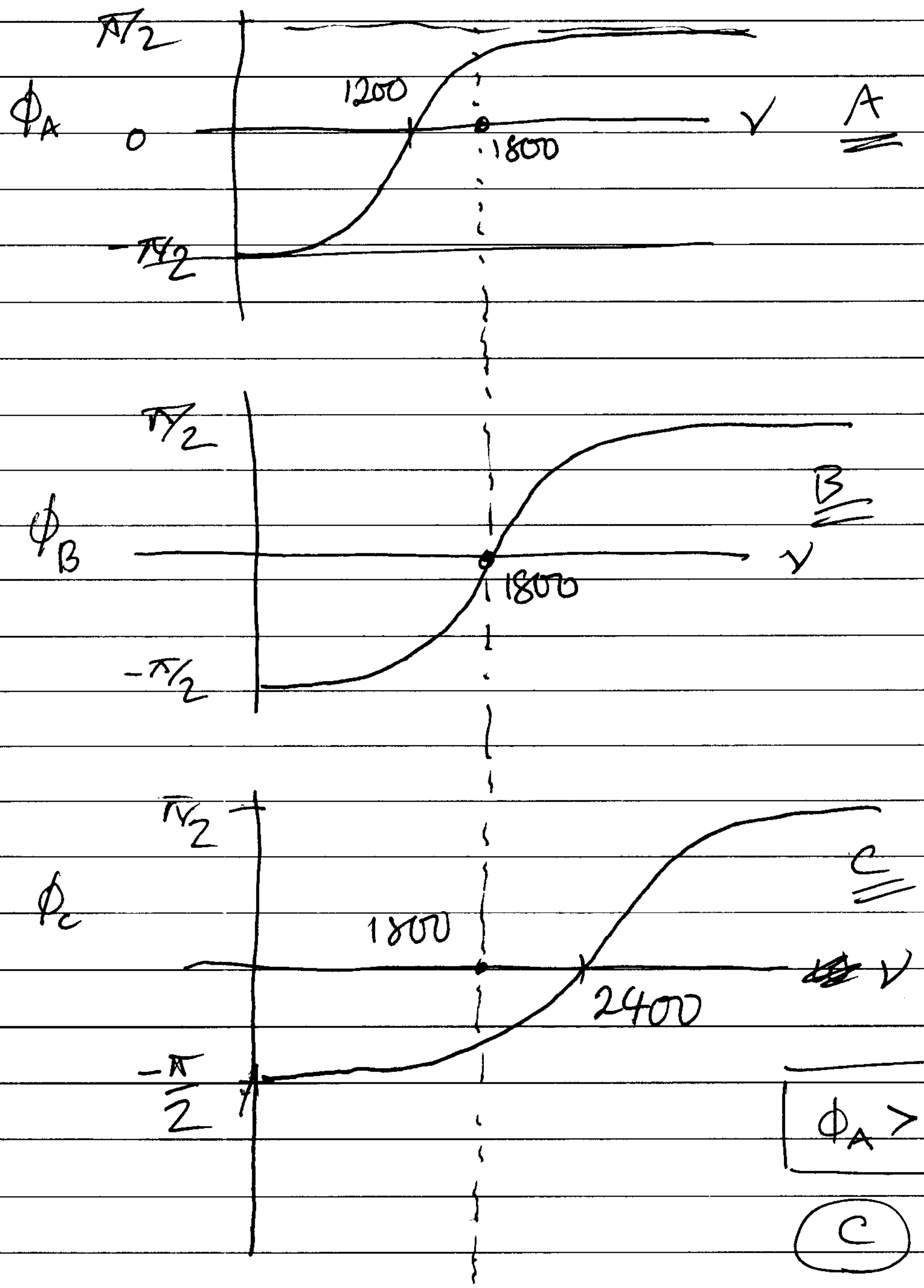
$$\tau = -\frac{15\,\text{sec}}{\ln(0.01)} = 3. \text{ seconds}$$

Thus oscillator ~~amplitude~~ energy (not amplitude) drops to $\frac{1}{e}$ of initial in time τ

so $$\frac{Q}{2\pi} = \tau \Rightarrow Q = 2\pi\tau$$

D

v) a little tricky: is at 1800, it's ABOVE resonance for A and below resonance for C



$\phi_A > 0$

$\phi_B = 0$

$\phi_C < 0$

$$\boxed{\phi_A > \overset{=0}{\phi_B} > \phi_C}$$

C

phase is zero whether velocity or amplitude

3) (i) $m\ddot{x} + r\dot{x} + sx = F_0 \cos\omega t$

(ii) In phasor notation with complex-valued amplitude $\tilde{A}$ (vector representation):

~~note~~ tilde notation indicates $\mathbb{C}$-valued (a reminder)

$$\tilde{x}(t) = \tilde{A}(\omega)\, e^{i\omega t}$$

(ie $\tilde{A}$ can depend ~~of~~ on ω, to give resonance, etc)

and use

$$F_0 \cos\omega t \rightarrow F_0\, e^{i\omega t}$$

take $\mathrm{Re}[\;]$ part at end

NOTE! : $\mathrm{Re}[\tilde{A} e^{i\omega t}] \neq \mathrm{Re}[\tilde{A}]\,\mathrm{Re}[e^{i\omega t}]$

$$m\tilde{A}(i\omega)^2 e^{i\omega t} + r\tilde{A}(i\omega)e^{i\omega t} + s\tilde{A}e^{i\omega t} = F_0 e^{i\omega t}$$

$$\tilde{A}(-\omega^2 m + ir\omega + s) = F_0$$

$$\tilde{A}(\omega) = \frac{F_0}{-\omega^2 m + ir\omega + s}$$

$$= \frac{F_0}{i\omega r + (s - \omega^2 m)} \times \frac{-i}{-i}$$

$$= \frac{-iF_0}{\omega r + i(s - \omega^2 m)}$$

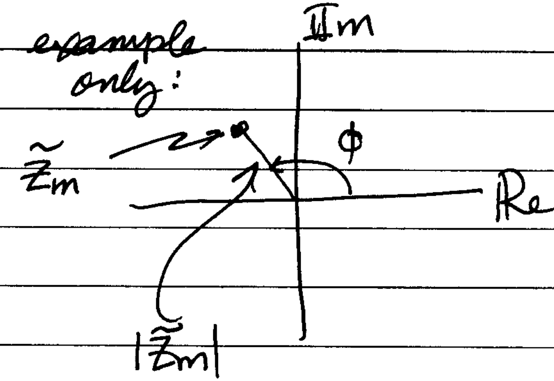
$$= \frac{F_0 e^{-i\pi/2}}{\omega[r + i(\omega m - s/\omega)]} \qquad -i = e^{-i\pi/2}$$

$$= \frac{F_0 e^{-i\pi/2}}{\omega \tilde{Z}_m}$$

but $\tilde{Z}_m = |\tilde{Z}_m|\, e^{i\phi}$

$$|\tilde{Z}_m| = \sqrt{\mathrm{Re}[\tilde{Z}_m]^2 + \mathrm{Im}[\tilde{Z}_m]^2}$$
$$= \sqrt{r^2 + (\omega m - s/\omega)^2}$$

and $\tan\phi = \dfrac{\mathrm{Im}[\tilde{Z}_m]}{\mathrm{Re}[\tilde{Z}_m]}$



but $\mathrm{Im}[\tilde{Z}_m] = (\omega m - \frac{s}{\omega})$

$\mathrm{Re}[\tilde{Z}_m] = r$

$$\tan\phi = \frac{\omega m - s/\omega}{r}$$

ie $\phi = \tan^{-1}\left[\dfrac{\omega m - s/\omega}{r}\right]$

$$\tilde{A}(\omega) = \frac{F_0 e^{-i\pi/2}}{\omega\sqrt{r^2 + (\omega m - s/\omega)^2}}\, e^{-i\phi}$$

$$\boxed{\tilde{x}(t) = \frac{F_0}{\omega\sqrt{r^2 + (\omega m - s/\omega)^2}}\, e^{i(\omega t - \phi - \pi/2)}}$$

and we take the Re part

so $\Rightarrow \cos(\omega t - \phi - \pi/2)$

$= \sin(\omega t - \phi)$

iii) $\underbrace{m\ddot{x}}_{\text{inertia force}} + \underbrace{r\dot{x}}_{\text{damping force}} + \underbrace{sx}_{\text{restoring force}} = \underbrace{F_0 \cos\omega t}_{\text{driving force}}$

The equation of motion is a relationship among forces - but which matters most?

notice : we know $x(t) = Ae^{i\omega t}$

(velocity) $v(t) = i\omega Ae^{i\omega t} = i\omega x(t)$

(acceleration) $a(t) = -\omega^2 Ae^{i\omega t} = -\omega^2 x(t)$

So <u>always</u>

$$x : v : a = 1 : \omega : \omega^2$$

This means that as $\omega \to 0$ $x(t)$ is far more important (larger) than $v(t)$ or $a(t)$, since $\omega \to 0$ and $\omega^2 \to 0$.

similarly, as $\omega \to 0$, $a(t)$ is far larger than $v(t)$ or $x(t)$

So our equation of motion

$$m\ddot{x}(t) + r\dot{x}(t) + sx(t) = F_0 \cos\omega t$$

as $\omega \to 0$

$$\Rightarrow \cancel{-\omega^2 m x(t)} + \cancel{i\omega r x(t)} + s x(t) = F_0 \cos\omega t$$

$$s x(t) = F_0 \cos\omega t$$

$$\boxed{\omega \to 0 \quad \Rightarrow \quad x(t) = \frac{F_0}{s}\cos\omega t}$$

comparison with spring constant

One moves so slowly, as $\omega \to 0$ (and the period $T = \frac{2\pi}{\omega} \to \infty$) that ~~acceleration~~ inertia and drag can be made as small as we want, essentially vanishing. All we do with the driving force is slowly load up the spring and then unload it — the spring balances all the force.

As $\omega \to \infty$

$$-\omega^2 m x(t) + \underbrace{\cancel{i\omega r x(t)}}_{\text{smaller by } \frac{1}{\omega}} + \underbrace{\cancel{s x(t)}}_{\text{smaller by } \frac{1}{\omega^2}} = F_0 \cos\omega t$$

$$-\omega^2 m x(t) = F_0 \cos\omega t$$

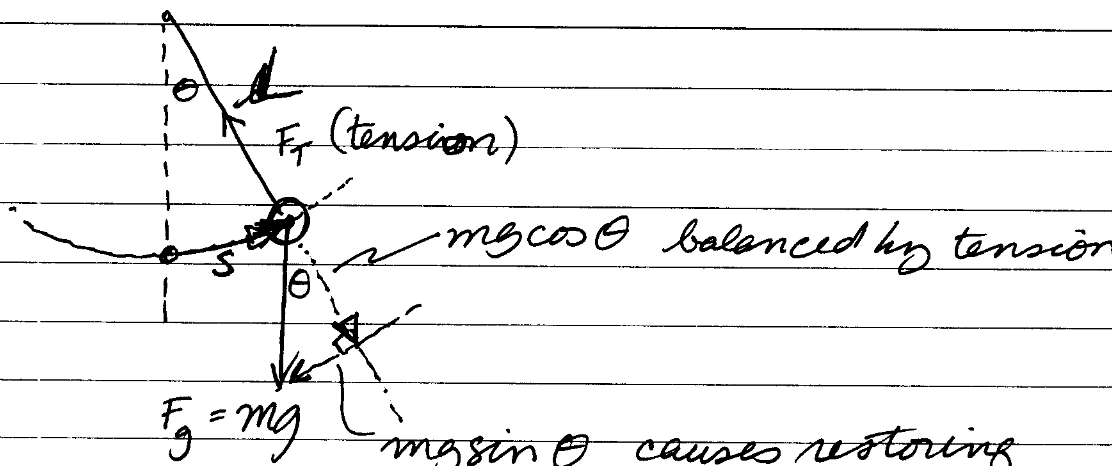
$$\boxed{\omega \to \infty \Rightarrow \quad x(t) = -\frac{F_0}{\omega^2 m}\cos\omega t}$$

comparison with inertia constant (which is what <u>mass</u> is)

As one moves very quickly, the force is used for the increasing rapid changes of direction — everything is dominated by inertia. The amplitude of ***all*** driven damped SHOs drops off as $\frac{1}{\omega^2}$ as ω gets very large.

[NB: a much shorter answer is worth full marks, but I give a full explanation for what is happening here, especially since I've several times tried to emphasize that physics is determined by relationships.]

4. i)



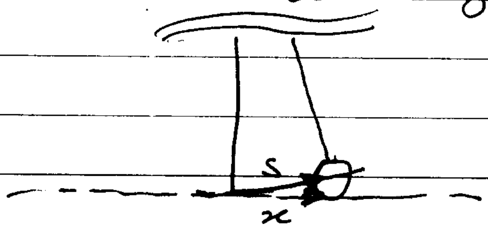
arc length $s = l\theta$

~~that~~ acceleration of mass $\ddot{s}$

$ma = F$

$m\ddot{s} = -mg\sin\theta$ (restoring force acts opposite to displacement s)

for small angles



$s \cong x$

$\sin\theta \cong \theta \cong \frac{x}{l}$

thus

$m\ddot{x} = -m\frac{g}{l}x$

$$\boxed{\ddot{x} + \frac{g}{l}x = 0} \qquad \omega_0^2 = \frac{g}{l}$$

ii) with damping force, just need to keep track of force directions:

$$m\ddot{x} + r\dot{x} + \frac{mg}{L}x = 0$$

iii) $m = 0.1\ kg$, $L = 1\ m$, $r = 0.626\ kg\,s^{-1}$ } $\dfrac{r^2}{4m^2} - \dfrac{s}{m} = 0$

The system is <u>critically</u> damped.

For critical damping, the form of solutions is:

$$x(t) = (A + Bt)\,e^{-rt/2m}$$

$$\frac{r}{2m} = \frac{0.626\ kg\,s^{-1}}{2(0.1)\,kg} = 3.13\,s^{-1}$$

$$x(t) = (A + Bt)\,e^{-3.13t}$$

then

$$v(t) = Be^{-3.13t} + (A + Bt)(-3.13)\,e^{-3.13t}$$

$1\ cm = 0.01\ m$

$-5\ cm/s^{-1} = -0.05\ ms^{-1}$

$\cancel{0.01\ m = Ae^{-3.13\cdot(0)} = A}$

$\cancel{-0.05\ ms^{-1} =}$

$$0.01\ m = x(0) = (A + B\cdot 0)\,e^{-3.13\cdot(0)}$$
$$= A$$

$$A = 0.01\ m$$

$$-0.05\ ms^{-1} = v(0) = Be^{-3.13(0)} + (A + B\cdot(0))(-3.13)e^{-3.13}$$
$$= B + (A)(-3.13)\cdot 1$$
$$= -3.13\,(0.01m) + B$$
$$B = -0.05\,m\,s^{-1} + 0.03135\,\overset{-1}{m}$$
$$= -0.0187\ ms^{-1}$$

$$\boxed{\begin{aligned} x(t) &= (0.01\,m - 0.0187\,ms^{-1}\cdot t)\,e^{-3.13t} \\ \text{or} \quad &= (1.0\,cm - 1.87\,cm\,s^{-1}\cdot t)\,e^{-3.13t} \end{aligned}}$$

(CAN BE CM but must show units clearly)