Physics 255F – Oscillations and Waves
Department of Physics
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Term Test – 26 October 2007

SOLUTIONS

[25] 1) Explain as succinctly as possible (about 3 or 4 sentences) the meaning *and* significance of each of the following, in the context of waves and oscillations:

See the text for basic answers; some comments:

i) phasor notation

use of $e^{i\omega t}$ to represent oscillations, by virtue of Euler's formula $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$ it simplifies the math, because derivatives become simpler and avoid trig identities; even better: it simply converts a differential equation to an algebraic equation to solve.

ii) the essential elements needed to make a simple harmonic oscillator (as discussed in class)

In class these were: (1) a displacement or equivalent (such as charge); (2) a linear restoring force (or equivalent, such as charge*voltage (technically 'e.m.f.', the electromotive force)); (3) a mass/inertia (or equivalent, like an inductance). The significance of the inertia is to carry the motion over the equilibrium point. NB: the restoring force must be linear. Any SHO needs to have these terms, but they needn't be a mass on a spring -- other things are analogous and will also make a SHO.

iii) relative phase ϕ between driving force and damped-oscillator response

See the textbook for the plot of ϕ vs. ω-drive; 'response' can mean displacement or velocity (or acceleration…), so you need to specify which one you're using. The phase relation between displacement and drive and the phase relation between displacement and velocity differ by $\pi/2$. Sketching the curve is good; the relationship for displacement is that for slow oscillations the displacement is basically in phase with the drive (very little inertia, dominated by restoring-force), and for fast oscillations the displacement is basically π out of phase (dominated by inertia, small amplitude and very little restoring force). For velocity, a main feature is that on resonce the driving force leads the velocity by $\pi/2$.

NB: the phase relationship ϕ of velocity vs. drive is NOT something you can control — you're NOT letting it oscillate and then driving at some relative frequency, you're driving the oscillator and it is responding, in some phase-relationship (common mistake).

iv) displacement-amplitude resonance of a driven damped simple harmonic oscillator

Resonance is a phenomenon, not a frequency: resonance is the strong enhancement of oscillation amplitude at a certain frequency, ω_r. (which is not ω'). Sketch is good. This resonant frequency is reduced due to damping, but not by as much as the frequency of a non-driven damped oscillator since it's still being driven, not oscillating only damped-naturally. This resonance-enhancement, compared to slow oscillation amplitude, is tied to the quality factor Q; it's important to using a quartz crystal in a watch to keep time -- the period of the special frequency associated with driven resonance is the reference of time. (Also bridge oscillations should not have strong resonances!)

v) normal mode

See textbook; this question was marked quite generously because the material was recent. In normal mode basis, the differential equations become uncoupled: each has a *single* independent variable, no cross-terms. Important due to exactly this; the modes are independent of one another (think normal = orthogonal, since the dot product is zero -- neither mode has in it any component of the other).

[25] 2) *Multiple choice:* For each item (i)–(iv) below, choose the answer A–E which best responds to the question. Answer questions in order, list your responses in your examination booklet (and keep a copy for yourself to compare with computer-marking of the question). Below that list, you may give *brief* explanations also — only if you think no answer is quite appropriate. Part marks may be given for certain incorrect choices A–E that have merit.

i) Which statements below describe a *critically damped simple harmonic oscillator*?

A. the oscillation frequency $\omega' \to \infty$ in the limit of critical damping

No, goes to zero...

B. the oscillator displacement never returns precisely to equilibrium

True, it never crosses the axis (part marks)

C. the oscillator displacement approaches equilibrium faster than any other solution

True, critical damping does better than any other in settling down (part marks)

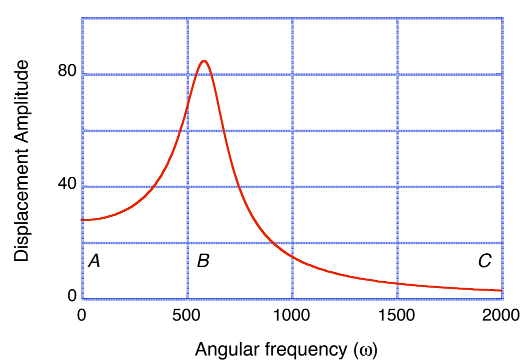
D. exactly two of the above are correct

True, (full marks)

E. all of A, B, C are correct

No, 'A' is not…

ii) Which of the following statements are true about the driven oscillator whose response is shown at right?



A. the peak of the displacement amplitude occurs at the natural frequency of the undamped simple harmonic oscillator

No, it's at $\omega_r < \omega_0$

B. the Q of the damped oscillator is roughly 82

No, it's the ratio over the zero-frequency amplitude -- about 2.8

C. the maximum power transfer occurs at the natural frequency of the undamped simple harmonic oscillator

Yes, that's where the velocity peaks, and the power is force * velocity

D. exactly two of the above are correct

No, just C

E. all of A, B, C are correct

No, just C…

iii) You'll recall from class that a steel ball on a spherical dish will oscillate, rolling back and forth. Likewise, a bead trapped without friction on a circle of wire (figure below-right) will oscillate back and forth due to gravity. If the circle of radius R now spins slowly at angular frequency Ω, what is the new frequency ω_0 of oscillation of the bead?

A. $\sqrt{\frac{g}{R} - \Omega^2}$

Yes. You can solve simply by finding the potential energy (gives restoring force). By inspection, it's about the gravity pulling down and the 'centrifugal force' pulling out (which becomes up), both must appear, but it's the net force (difference)

B. $\sqrt{\frac{mg}{\Omega}}$

wrong units, not even close…

C. $\sqrt{\left(\frac{g}{R}\right)^2 - \Omega^2}$

wrong units, but at least there's a difference (subtraction)

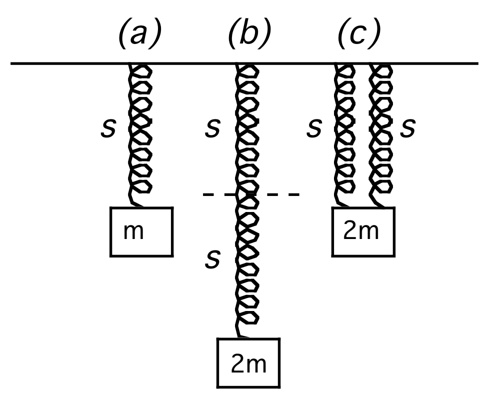
D. $\sqrt{\sqrt{\frac{g}{R}}\Omega}$

units OK, but weird formula-attempt…

E. none of the above is correct

No, A is OK

iv) The figure at right shows a mass on a spring (a). The mass and the spring are both doubled, in (b) and (c). Which statement below is most true about these?



A. all three have the same natural oscillation frequency, $\omega_a = \omega_b = \omega_c = \omega_0$.

No, (b) has twice the restoring force of (c) (two springs in a line together share one tension) but the same mass

B. the natural oscillation frequencies have the relation: $\omega_a > \omega_b > \omega_c$

No, (c) has double the restoring force but double the mass of (a) so must give same frequency

C. the natural oscillation frequencies have the relation: $\omega_a < \omega_b < \omega_c$

No, springs of (b) share a single tension T, but each stretches only half of the mass displacement so restoring force is less; also mass is higher -- both make the (b) frequency lower than (a)

D. the natural oscillation frequencies have the relation: $\omega_a < \omega_b = \omega_c$

No, as above the first relation is wrong (so is the second, actually)

E. none of the above is correct

Yes, bingo.

v) Two simple harmonic oscillations of close but different frequencies are added, as scalars. Which of the following statements are true?

A. a beat pattern can result; the beat frequency is the average of the two oscillator frequencies

No, the carrier has that frequency

B. a beat pattern can result; the beat frequency is the sum of the two oscillator frequencies

No, couldn't use it for tuning musical instruments then

C. a beat pattern can result; the beat frequency is half the difference of the two oscillator frequencies

No, the formula for the carrier and the envelope both have /2 BUT there are two beats per envelope period

D. a beat pattern can result; the beat frequency is the difference of the two oscillator frequencies

Yes, two beats per cycle of the envelope, and the envelope has the frequency of C. above

E. none of the above

No, D was OK…

[25] 3) *Pendulum*

a) Derive from forces the equation of motion for a pendulum, in terms of a small sideways displacement x. The pendulum has mass m on a string of length L. Show a solution for the differential equation for the natural oscillation frequency ω_0.

See text

b) Derive again the equation of motion for the pendulum in (a) but in terms of a small *angular* displacement θ.

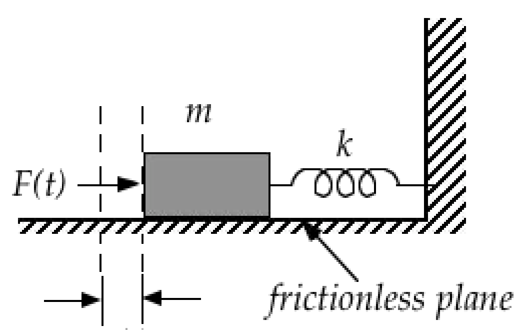
See problem set 1, which compared how linear and rotational oscillations both can be legitimate ways to understand the pendulum

c) For length L = 20 cm, mass m = 200 g, give ω_0. Find the full solution, for initial conditions $x(0)$ = 2 cm, $v(0) = -1.7$ cm s^{-1}.

$\sqrt{(g/L)}$ = 1.2 rad/sec; x(t) = A*sin(ωt + ϕ) → 2cm=x(0)=A*sin(ϕ) & -1.7cm s^{-1} = –A*ω*cos(ϕ); ratio gives –(2/1.7) = tan(ϕ) / ω, solve for ϕ then sub back into 1st to get A.

[25] 4) *Driven damped harmonic oscillators*

One design problem in the production of machine guns is that the periodic recoil of the bolt mechanism due to the gun firing must not excite a resonance in the system that would cause the gun to break. (The bolt is the part of a gun that puts a bullet in the chamber; its recoil automatically loads the next bullet). A simple model for a particular machine gun is shown at right. The recoil-force on the bolt (mass m = 0.4 kg) due to successive



firing of bullets is represented by a general periodic external force (in Newtons) of the form $F(t) = 25\sin(60t)$. The spring opposing the recoil of the bolt provides a linear restoring force, with a spring constant $k = 2000\ \mathrm{N\,m^{-1}}$. Initial conditions are $x(0) = 0$ m and $(dx/dt)\big|_{t=0} = 0\ \mathrm{m\,s^{-1}}$.

Using the model system as a guide:

i) Draw a force diagram, and write the *full* equation of motion for the mass. Show that the units of each term are the same (*i.e.,* verify that the equation is dimensionally consistent). Keep the forcing frequency in a general form ω for now.

ii) This system is characterized by a very high Q ($Q >> 1$). What terms can be neglected from the equations of motion? Why? Apply this approximation to the governing differential equation and obtain a simpler form. Find the natural frequency ω_o of the system in this approximation.

iii) Find $x(t)$ for the given initial conditions, still with arbitrary frequency ω. Show that your expression has units of length.

iv) Sketch the displacement amplitude response as a function of the frequency ω.

v) For the specified frequency in this system, what is the amplitude response? On your plot, identify the range of frequencies at which the gun is *least likely* to break.

See the attached PDF for a solution.

Note: there is a frictionless *slide*, but that does not mean that there is no friction: we discussed in class how power is dissipated into sound waves, air turbulence, spring heating (flexing metal back and forth), etc. These are relatively small; note that you're given that Q >>1 but not infinite! In the midterm I announced that you should consider the setup as realistically as possible.

The full equation of motion formally includes a realistic drag. Later, this small drag is neglected in comparison with other terms.

[100] TOTAL

PROBLEM 1

(3) a)  $mx'' + cx' + kx = F_0 \sin(\omega t)$ $\omega = 60\ s^{-1}$

REWRITE:

$F_S = kx$
$F_D = c\,dx/dt$

$$x'' + \frac{\omega_0}{Q}x' + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t)$$
$$[L/t^2]\quad [L/t^2]\quad [L/t^2] = [L/t^2]\ \checkmark$$

(2) b) THIS SYSTEM IS HIGH Q, SO GOV DE BECOMES (APPROXIMATELY):

$$x'' + \omega_0^2 x = \frac{F_0}{m}\sin(\omega t)$$

NOTE: THE DAMPING TERM IS 1/Q SMALLER THAN ALL OTHER TERMS

(2) c) $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{0.4}} = 70.7\ [s^{-1}]$

(8) d) SOLUTION TO UNDAMPED, FORCED CASE:

$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{F_0/m}{\omega^2-\omega_0^2}\cos(\omega t)$; $x(0) = 0$, $dx(0)/dt = 0 = V(0)$
(FORCED FROM REST)

$$x'(t) = -A\omega_0\sin(\omega_0 t) + B\omega_0\cos(\omega_0 t) - \frac{F_0/m\,\omega}{\omega^2-\omega_0^2}\sin(\omega t)$$

$x(0) = A\cdot 1 + 0 + \frac{F_0/m}{\omega^2-\omega_0^2} = 0$; $A = \frac{-F_0/m}{\omega^2-\omega_0^2}$

$$x'(0) = -0 + \omega_0 B\cdot 1 - \frac{F_0/m\,\omega}{\omega^2-\omega_0^2}\cdot 0 \Rightarrow B = 0$$

$x(t) = \frac{F_0/m}{\omega^2-\omega_0^2}\left(\cos(\omega t) - \cos(\omega_0 t)\right) \Rightarrow$ NOTE THAT $\cos(\omega t) - \cos(\omega_0 t)$
$= -2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$

SIMPLER FORM:

$$x(t) = \frac{F_0/m}{\omega^2-\omega_0^2}\,2\sin\left(\frac{\omega+\omega_0}{2}t\right)\sin\left(\frac{\omega-\omega_0}{2}t\right)$$

PLUGGING IN NUMBERS

$x(t) = (0.045)(2)\sin(65.4t)\sin(-5.4t)$; NOTE $\sin(-\alpha) = -\sin\alpha$

$\therefore$ $x(t) = -0.045(2)\sin(65.4t)\sin(5.4t)$

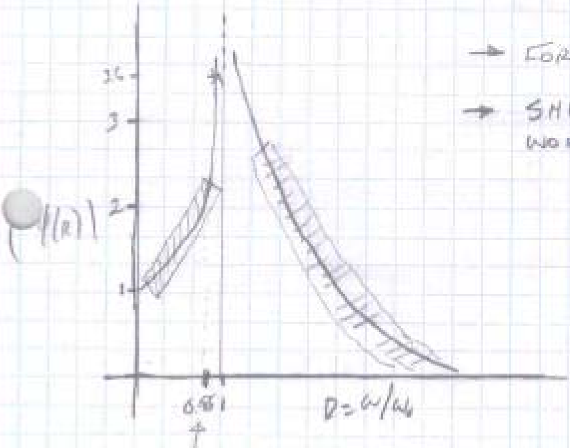
10 pts
(e,f,g)

DIMENSIONLESS FORM: LET $\omega/\omega_0 = R$

$$|\Psi| = \frac{|A|}{(F_0/k)} = \frac{1}{R}\frac{1}{(1/R - R)};$$

R	$\|\Psi(R)\|$
1	∞
2	3/4
0.5	4/3
0	$\to 1$
0.2	25/24
1.5	12/15
0.85	3.6

$$\left.\frac{\omega}{\omega_0}\right|_{\text{THIS SYSTEM}} = \frac{60}{70.7} = 0.85 \qquad (2\text{ pts})$$



→ FOR $R = 0.85$, $\Psi(R) = 3.6$

→ SHADED REGIONS: VIABLE WORKING CONDITIONS FOR GUN. (2 pts)

PLOT = 6 pts