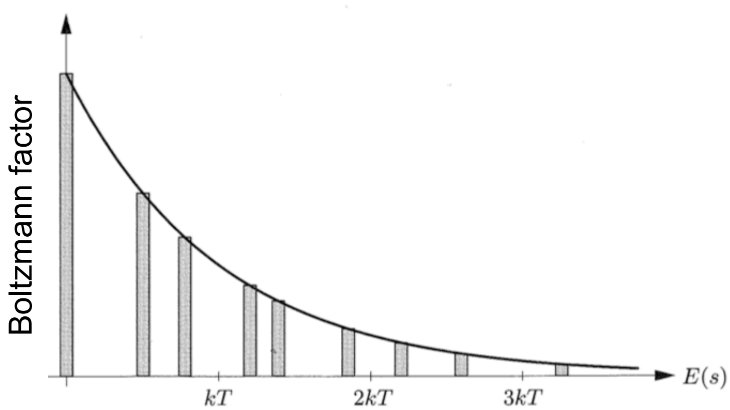
Example questions for Particles part of PHY293 Final Exam – 30 November 2009

Examples of Short Answer Part (typically worth 4 points each):
Please answer the following questions showing your reasoning as well as your final answer. To receive full marks for your answer, both your reasoning and your answer must be correct.

1. At low temperatures, the temperature dependence of the heat capacity of a metal can be described by: $C_v = aT + bT^3$.
   Briefly describe the origin of each of the terms in this equation and the model system we used to derive each term.

2. You have a 1000 cm^3 container of 10^{26} neon atoms in thermal equilibrium at room temperature and pressure. The mass of an atom of neon is 3.2×10^{-26} kg. Calculate the temperature range over which quantum statistics would apply to this system.

3. A Boltzmann factor ($e^{-E(s)/kT}$) is proportional to the probability of finding the system in a state, s, with a specific energy, E(s). The figure below shows a bar graph of the Boltzmann factors versus energy for a hypothetical system at temperature, T. Based on the information presented in this plot, what are the energy, E(s), and the Boltzmann factor for the system in its lowest energy (or ground state)? Using two labelled sketches, show how the distribution of Boltzmann factors will change when the temperature of this system is increased ($T_1 > T$) and when it is decreased ($T_2 < T$).



4. The Sackur-Tetrode Equation describes how the entropy of an ideal monatomic gas depends on its volume, total energy and number of particles. From this equation, write down an expression for the change in entropy, ΔS, when only the volume changes from V_i to V_f. For an ideal monatomic gas freely expanding into a vacuum, such that $V_f = 2 V_i$, calculate the change in entropy, ΔS, for this process. What is the change in total energy for this process? Briefly explain this result.

$$S = Nk\left[\ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]$$

Examples of Derivation Part (typically worth 20 points each):
Please answer the following questions showing all the mathematical steps that led to your final answer providing sufficient explanations to justify your answer.

5. For a diatomic molecule consisting of distinguishable atoms (such as CO or HCl), the energy for a particular rotational level is given by: $E(J) = J(J+1)\varepsilon$. The degeneracy of this energy level is given by: $(2J+1)$. Consider a single molecule for this question.

   a. Write down an expression for the rotational partition function of a diatomic molecule as a function of J.

   b. In the low-temperature limit, each term in the rotational partition function is much smaller than the one before it. By truncating the sum after the second term, determine the rotational partition function in the low-temperature limit.

   c. Calculate the average energy in the low-temperature limit. Simplify your answer by keeping only the leading order term.

   d. Using your simplified answer from part (c), calculate the heat capacity in the low-temperature limit.

6. Consider a particular crystalline solid consisting of N non-interacting atoms. The atoms are located at fixed locations in the crystal lattice and the nuclei of the atoms have spin one. Therefore each nucleus has three allowed independent spin states labelled by the quantum number m, where m = +1, 0, or -1. The energies of these states are equal to zero in the m = 0 state and equal to $\mu_N\varepsilon$ in the m = +1 and m = -1 states, where μ_N is the nuclear magneton and ε is the field strength.

   a. Calculate the partition function for this system.

   b. Determine the Helmholtz free energy, F.

   c. Calculate the total energy for the crystal. Simplify your answer to obtain the most compact expression.

   d. Show explicitly how your answers to parts (a) and (b) would change if the energies of the m = +1 and m = -1 states of the atomic nucleus were $+\mu_N\varepsilon$ and $-\mu_N\varepsilon$, respectively.

A list of formulae, perhaps useful and certainly in no particular order

$$k = 1.38 \times 10^{-23}\ \text{J/K} = 8.6 \times 10^{-5}\ \text{eV/K} \qquad N_A = 6.02 \times 10^{23}$$

$$h = 6.626 \times 10^{-34}\ \text{Js} \qquad c = 3 \times 10^{8}\ \text{m/s}$$

$$e = 1.6 \times 10^{19}\ \text{C} \qquad R = 8.31\ \text{J/mol/K}$$

$$m_p = 1.6 \times 10^{-27}\ \text{kg} \qquad m_e = 9.1 \times 10^{-31}\ \text{kg}$$

Physics Formulae:

$$PV = NkT = nRT \qquad \ln N! \approx N \ln N - N$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \qquad N! \approx N^N e^{-N}\sqrt{2\pi N}$$

$$T^{-1} = \left(\frac{\partial S}{\partial U}\right)_{N,V} \qquad S = Nk\left[\ln\left(\frac{V}{Nv_Q}\right) + \frac{5}{2}\right]$$

$$dU = TdS - PdV + \mu dN \qquad v_Q = \left(\frac{h^2}{2\pi mkT}\right)^{3/2}$$

$$dF = -SdT - PdV + \mu dN \qquad F = U - TS$$

$$\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} \qquad \beta = \frac{1}{kT}$$

$$\lambda_{deBroglie} = h/p \qquad E_{K.E.} = mv^2/2 = p^2/2m$$

$$\bar{n}_{Planck} = \frac{1}{e^{hf/kT} - 1} \qquad u(\epsilon) = \frac{8\pi}{(hc)^3}\frac{\epsilon^3}{e^{\epsilon/kT} - 1}$$

Stefan's law: power per unit area $= \sigma T^4$ where $\sigma = \frac{2\pi^5 k^4}{15h^3c^2} = 5.67 \times 10^{-8}\ \text{W/m}^2\text{K}^4$
Wien's law: peak photon energy of spectrum (in eV); $\epsilon = 2.82kT$
$Z_{total} = (Z_1)^N$ for distinguishable particles
$Z_{total} = \frac{(Z_1)^N}{N!}$ for indistinguishable particles
Average value of parameter X: $\bar{X} = \sum_s X(s)P(s)$

Math formulae:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi} \qquad \int_{-\infty}^{\infty} x^2 e^{-x^2}dx = \sqrt{\pi}/2 \qquad \int_0^{\infty}\frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}$$

$$\int_0^{\infty}\frac{x}{e^x - 1}dx = \frac{\pi^2}{6} \qquad \int_0^{\infty}\frac{x}{e^x + 1}dx = \frac{\pi^2}{12} \qquad \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

For $x \ll 1$, $e^x \approx 1+x$, $\ln(1+x) \approx x$, $\sin x \approx x$, $\cos x \approx 1-x^2/2$, $\sinh x \approx x$, $\cosh x \approx 1+x^2/2$, $\tanh x \approx x$. For $x < 1$, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.