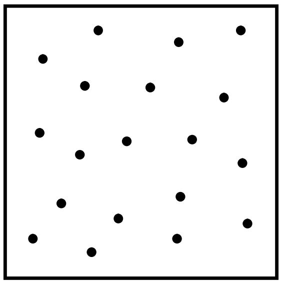
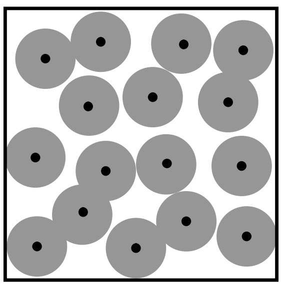
Review

- Quantum gas versus (normal or) ideal gas:



normal

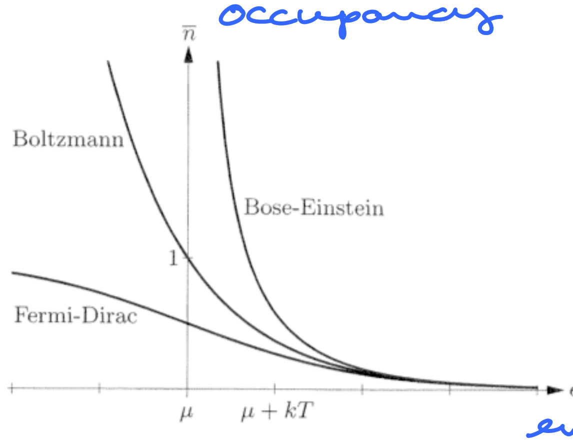
$\frac{V}{N} \gg v_Q$



quantum

$\frac{V}{N} \approx v_Q$

occupancy



energy

quantum statistics

$Z_1 \gg N$ does not hold

Boltzmann statistics

$Z_1 \gg N$

Review

Degenerate Fermi gas:

- Conduction electrons in metal - treat as free particles

  non-interacting

- Defined Fermi energy, ε_F, equal to $\mu(T=0)$

  $$v_Q = \left(\frac{h}{\sqrt{2\pi m k T}}\right)^3$$

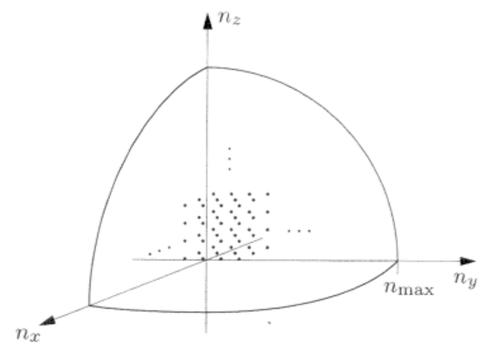
- Degenerate gas
  - Almost all states below ε_F are occupied and all above are unoccupied

Degenerate Fermi Gas

- Wavefunctions for free electron in metal block
  - Represented by standing wave wavefunctions

$$\varepsilon = \frac{|\vec{p}|^2}{2m} = \frac{h^2}{8mL^2}\left(n_x^2 + n_y^2 + n_z^2\right) \qquad \vec{n} \rightarrow n_x, n_y, n_z$$

  - Each "lattice point" (n_x, n_y, n_z) is pair of electron states
    - One for each spin orientation
  - Surface of sphere => radius n_{max}



Number of Occupied States

- Energy of state on surface of sphere $\varepsilon_F = \dfrac{h^2 n_{max}^2}{8mL^2}$

number of occupied states

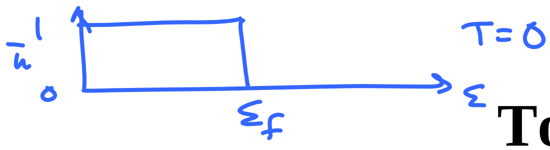
$$N = 2 \times \frac{1}{8} \times \frac{4}{3}\pi n_{max}^3$$

$$= \frac{\pi n_{max}^3}{3} \rightarrow n_{max} = \sqrt[3]{\frac{3N}{\pi}}$$

$$\varepsilon_F = \frac{h^2}{8mL^2}\left(\frac{3N}{\pi}\right)^{2/3}$$

$$= \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$$

→ does not depend on size of metal block



Total Energy

$$U = 2\sum_{nx}\sum_{ny}\sum_{nz}\varepsilon(\vec{n}) = 2\int\int\int \varepsilon(\vec{n})\,dn_x\,dn_y\,dn_z$$

$$U = 2\int_0^{n_{max}} n^2 \varepsilon(n)\,dn \underbrace{\int_0^{\pi/2} d\phi \int_0^{\pi/2} \sin\theta\,d\theta}_{\pi/2}$$

$$= \pi \int_0^{n_{max}} n^2 \frac{h^2 n^2}{8mL^2}\,dn$$

$$= \frac{\pi h^2}{8mL^2}\int_0^{n_{max}} n^4\,dn$$

$$= \frac{\pi h^2 n_{max}^5}{8mL^2 \cdot 5} = \frac{\pi h^2}{8 \cdot 5 m L^2}\left(\frac{3N}{\pi}\right)^{5/3} = \underbrace{\frac{h^2}{8mL^2}\left(\frac{3N}{\pi}\right)^{2/3}}_{\varepsilon_f} \cdot \frac{3N}{5}$$

$$U = \frac{3}{5}N\varepsilon_f \quad \text{at } T=0$$

Comparing Energies

- Fermi energy for conduction electrons in typical metal
  - A few eV
- Thermal energy of a particle at room temperature
  - $kT \approx 1/40$ eV
- Same as comparing quantum volume and volume per particle

$$\underbrace{\frac{V}{N}}_{\propto \frac{1}{\varepsilon_f^{3/2}}} \ll \underbrace{v_Q}_{\propto \frac{1}{(kT)^{3/2}}}$$

same as $\varepsilon_f \gg kT$

if met $T \simeq 0$ approx. is ok

→ can term degenerate fermi gas to describe system

$\varepsilon_f \approx kT$ $\qquad k \sim 8 \times 10^{-5}\ \text{eV/K}$

for $\varepsilon_f \sim 1\ \text{eV}$

$T_f \approx 11{,}600\text{K}$

fermi temperature

Degeneracy Pressure

- Using $P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$ → thermodynamic identity

$$P = -\frac{\partial}{\partial V}\left[\frac{3}{5}N \cdot \frac{h^2}{8m}\left(\frac{3N}{\pi}\right)^{2/3} V^{-2/3}\right]$$

$$= -\frac{3}{5}N\frac{h^2}{8m}\left(\frac{3N}{\pi}\right)^{2/3} \cdot -\frac{2}{3}V^{-5/3}$$

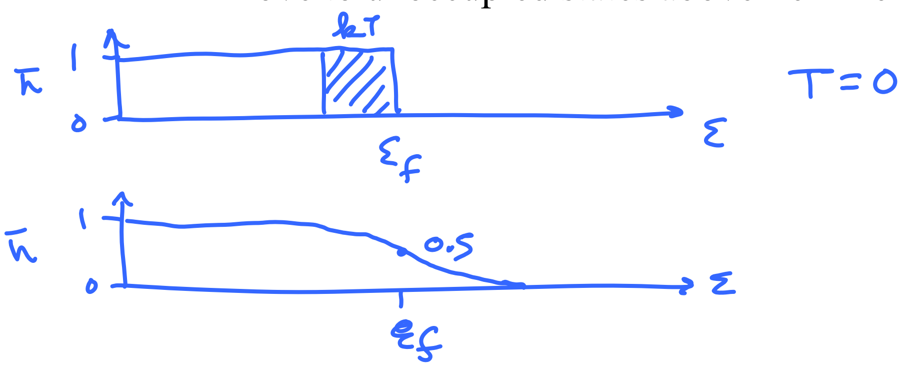
$$= \frac{2}{5}\frac{N}{V}\varepsilon_f$$

$$P = \frac{2}{3}\frac{U}{V} \quad \text{at } T=0$$

Temperatures above Zero K

- At T=0, cannot determine heat capacity
  - Need small non-zero temperatures to do this

- Increase temperature from zero, all particles typically gain thermal energy of *kT*
  - Only electrons within ~*kT* of Fermi energy can acquire
  - Move to unoccupied states above Fermi energy



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- Additional energy gained related to temperature
  - Number of affected electrons proportional to kT and N

Degenerate Fermi Gas Above 0 K

Dimensional analysis approach: see text for further details

Additional energy = (number of affected electrons) x (energy acquired by each)

$$\propto (NkT)(kT)$$
$$\propto N(kT)^2$$

→ divide by ε_f → constant of system

→ proportionality constant $\pi^2/4$

↳ from Sommerfeld Expansion

$$U = \underbrace{\frac{3}{5} N \varepsilon_f}_{\text{from } T=0} + \underbrace{\frac{\pi^2}{4} \frac{N(kT)^2}{\varepsilon_f}}_{\text{additional for } T>0}$$

need to be in low T limit

$T \ll \varepsilon_f / k$

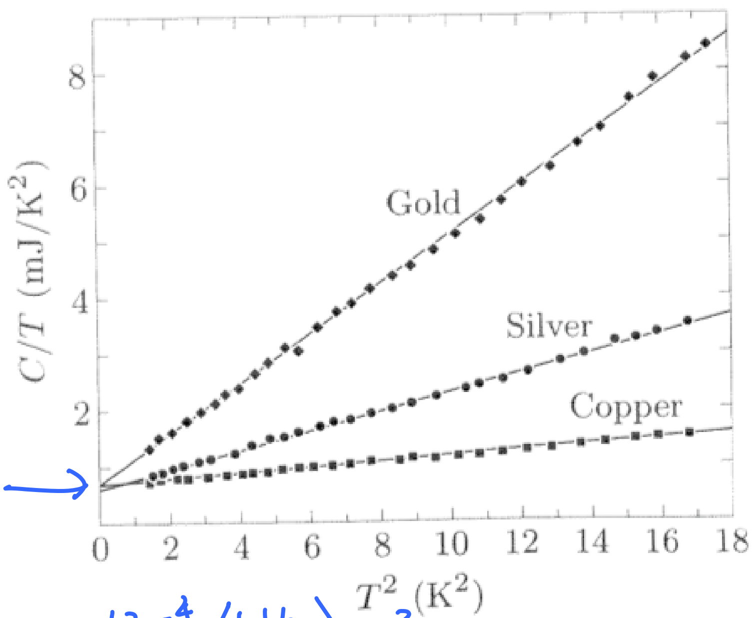
room T meets this requirement

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2}{4} \frac{N}{\varepsilon_f} 2(k^2 T)$$

$$= \underbrace{\frac{\pi^2 N k^2}{2 \varepsilon_f} T}_{\text{linear term in } C_V}$$

Debye Solid - Experimental Results

$\frac{C}{T}$ vs T^2



$\gamma = \frac{\pi^2 N k^2}{2 \varepsilon_f}$

$$C = \underbrace{\gamma T}_{\text{conduction electrons}} + \underbrace{\frac{12\pi^4}{5}\left(\frac{Nk}{\Theta_D^3}\right)T^3}_{\text{Debye theory}}$$

Distribution Functions Used

Distribution functions describe the average number of particles in a state $\bar{n} = \sum_n n P(n)$

Quantum statistics for cases when $Z_1 >> N$ condition is not met:

- Fermi-Dirac Distribution (fermions)

$$\bar{n}_{FD} = \frac{1}{e^{(\varepsilon-\mu)/kT} + 1}$$

- Bose-Einstein Distribution (bosons)

$$\bar{n}_{BE} = \frac{1}{e^{(\varepsilon-\mu)/kT} - 1}$$

Relation to Planck Distribution

- Bose-Einstein Distribution for photons and phonons becomes the Planck Distribution

$$\bar{n}_{BE} = \frac{1}{e^{(\varepsilon-\mu)/kT}-1} \longrightarrow \bar{n}_{Pl} = \frac{1}{e^{hf/kT}-1} = \frac{1}{e^{\varepsilon/kT}-1}$$

- Because μ=0 for photons and phonons

$\mu = \left(\frac{\partial f}{\partial N}\right)_{T,V}$ at eq. $= 0$

f has to be minimum at equilibrium

photons not conserved

NOTE: Not responsible for density of states or Sommerfeld Expansion in Sec 7.3